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A UNIFIED GROUP-THEORETIC METHOD ON IMPROPER PARTIAL SEMI-BILATERAL GENERATING FUNCTIONS

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Abstract. A unifed group-theoretic method of obtaining more general class of generating functions from a given class of improper partial semi-bilateral generating functions involving Laguerre and Gegenbauer polynomials are discussed.

1. Introduction and preliminaries. The usual generating relation involving one special function is called linear or unilateral generating relation. By the term bilateral generating function, we mean a function $G(x, z, w)$ which can be expanded in powers of w in the following form:

$$G(x, z, w) = \sum_{n=0}^{\infty} a_n f_n(x) g_n(z) w^n,$$

where a_n's are arbitrary, that is independent of x, z and $f_n(x)$, $g_n(z)$ are two different special functions. In particular, when two special functions are same, that is $f_n \equiv g_n$, we call the generating relation as bilinear generating relation.

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Unlike the usual bilateral or bilinear generating relations ([1], and [2]), we shall introduce the concept of partial semi-bilateral and improper partial semi-bilateral generating relations involving some special functions.

Definition 1.1. *By the term partial semi-bilateral generating relation for two classical polynomials, we mean the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n S_\alpha^{(m+n)}(x)T_p^{(m+n)}(z) \tag{1.1}$$

where the coefficients a_n's are quite arbitrary and $S_\alpha^{(m+n)}(x)$, $T_p^{(m+n)}(z)$ are two particular special functions of order α, p and of parameter $(m+n)$.

Definition 1.2. *By the term improper partial semi-bilateral generating relation for two classical polynomials, we mean the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n S_\alpha^{(m+n)}(x)T_p^{(k+n)}(z) \tag{1.2}$$

where the coefficients a_n's are quite arbitrary and $S_\alpha^{(m+n)}(x)$, $T_p^{(k+n)}(z)$ are two particular special functions of order α, p and of parameters $(m+n)$, $(k+n)$, respectively.

The object of this paper is to suggest a unified group-theoretic method for obtaining a more general class of generating relations from a given class of improper partial semi-bilateral or partial semi-bilinear generating relations involving some special functions, when suitable one-parameter continuous transformation group can be constructed for those special functions.

The present unified group-theoretic method was originated from our previous work [2] where we have derived a more general class of generating relations from a given class of improper partial-quasi bilateral generating relations. We have also given some indications for deriving more general class of improper and proper quasi-bilateral generating relations in some of my recent works [3] and [4].

2. Main results. *The Unified Group-theoretic method:* Let us consider the following improper partial semi-bilateral generating relation involving two particular special functions of the form:

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n S_\alpha^{(m+n)}(x)T_p^{(k+n)}(z), \tag{2.1}$$

where the coefficients a_n's are quite arbitrary, $S_n^{(m+n)}(x)$, $T_p^{(k+n)}(z)$ are two particular special functions of order α, p and of parameters $(m+n)$, $(k+n)$, respectively.

Now we shall find out two one-parameter continuous transformation groups generated by the operators

$$R_1 = \phi_1(x,y)\frac{\partial}{\partial x} + \phi_2(x,y)\frac{\partial}{\partial y} + \phi_3(x,y)$$

and

$$R_2 = \psi_1(z,t)\frac{\partial}{\partial z} + \psi_2(z,t)\frac{\partial}{\partial t} + \psi_3(z,t),$$

such that

$$R_1\left[S_\alpha^{(m+n)}(x)y^{m+n}\right] = \rho_{m+n}^{(1)} S_\alpha^{(m+n+1)}(x)y^{m+n+1}$$

and

$$R_2\left[T_p^{(k+n)}(z)t^{k+n}\right] = \rho_{k+n}^{(2)} T_p^{(k+n+1)}(z)t^{t+n+1},$$

where

$$\exp(wR_1)f(x,y) = \lambda_1(x,y)f(g_1(x,y),h_1(x,y))$$

and

$$\exp(\nu R_2)f(z,t) = \lambda_2(z,t)f(g_2(z,t),h_2(z,t)).$$

Multiplying both sides of (2.1) by $y^m t^k$, we get

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n \left(S_\alpha^{(m+n)}(x)y^m\right)\left(T_p^{(k+n)}(z)t^k\right). \tag{2.2}$$

Next we replace w by $wvyz$ in (2.2)

$$y^m t^k G(x,z,wvyz) = \sum_{n=0}^{\infty} a_n (wv)^n \left(S_\alpha^{(m+n)}(x)y^{m+n}\right)\left(T_p^{(k+n)}(z)t^{k+n}\right). \tag{2.3}$$

We now operate both sides of (2.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.3) reduces to

$$(h_1(x,y))^m (h_2(z,t))^t \lambda_1(x,y)\lambda_2(z,t)G(g_1(x,y),g_2(z,t),wvh_1(x,y)h_2(z,t))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!} S_\alpha^{(m+n)}(x)y^{m+n}\right)\left(\frac{(vR_2)^r}{r!} T_p^{(k+n)}(z)t^{k+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}\nu^{n+r}}{s!r!} \left(\rho^{(1)}_{m+n}\rho^{(1)}_{m+n+1}\cdots\rho^{(1)}_{m+n+s-1} S_\alpha^{(m+n+s)}(x) y^{m+n+s}\right)$$

$$\cdot \left(\rho^{(2)}_{k+n}\rho^{(2)}_{k+n-1}\cdots\rho^{(2)}_{k+n+r-1} T_p^{(k+n+r)}(z) t^{k+n+r}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{s-1}\rho^{(1)}_{m+n+i}\prod_{j=0}^{r-1}\rho^{(2)}_{k+n+j} a_n \frac{w^{n+s}\nu^{n+r}}{s!r!}\left(S_\alpha^{(m+n+s)}(x) y^{m+n+s}\right)$$

$$\cdot \left(T_p^{(k+n+r)}(z) t^{(k+p+r)}\right).$$

Now putting $y = t = 1$ in the above relation, we get

$$(h_1(x,1))^m \left(h_2(z,1)\right)^k \lambda_1(x,1)\lambda_2(z,1) G(g_1(x,1), g_2(z,1); w\nu h_1(x,1)h_2(z,1)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{s-1}\rho^{(1)}_{m+n+i}\prod_{j=0}^{r-1}\rho^{(2)}_{k+n+j} a_n \frac{w^{n+s}\nu^{n+r}}{s!r!}\left(S_\alpha^{(m+n+s)}(x)\right)\left(T_p^{(k+n+r)}(z)\right).$$

Thus we state the following general theorem which we propose to discuss for the said unification:

Theorem 2.1. *If there exists a bilateral geneating relation of the form*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n S_\alpha^{(m+n)}(x) T_p^{(k+n)}(z)$$

then

$$(h_1(x,1))^m (h_2(z,1))^k \lambda_1(x,1)\lambda_2(z,1) G(g_1(x,1), g_2(z,1), wv h_1(x,1)h_2(z,1))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{s-1}\rho^{(1)}_{m+n+i}\prod_{j=0}^{r-1}\rho^{(2)}_{k+n+j} a_n \frac{w^{n+s}\nu^{n+r}}{s!r!}\left(S_\alpha^{(m+n+s)}(x)\right)\left(T_p^{(k+n+r)}(z)\right),$$

where the coefficients a_n's are arbitrary, $S_\alpha^{(m+n)}(x)$, $T_p^{(k+n)}(z)$ are two particular special functions and R_1, R_2 are two generators of two one-parameter continuous transformation groups such that

$$R_1\left[S_\alpha^{(m+n)}(x)y^{m+n}\right] = \rho_{m+n}^{(1)} S_\alpha^{(m+n+1)}(x)y^{m+n+1}$$

and

$$R_2\left[T_p^{(k+n)}(z)t^{k+n}\right] = \rho_{k+n}^{(2)} T_p^{(k+n+1)}(z)t^{k+n+1}$$

and also

$$\exp(wR_1)f(x,y) = \lambda_1(x,y)f(g_1(x,y),h_1(x,y))$$

and

$$\exp(\nu R_2)f(z,t) = \lambda_2(z,t)f(g_2(z,t),h_2(z,t)).$$

Particular cases: It may be of interest to point out that for $k = m$, the above Theorem 2.1 becomes nice general class of generating functions from a given class of partial semi-bilateral generating functions, which need not be derived independently. Thus we state in the following form the result involving two particular special functions for partial semi-bilateral generating functions.

Theorem 2.2. *If there exists a bilateral generating relation of the form:*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n S_\alpha^{(m+n)}(x) T_p^{(m+n)}(z)$$

then

$$(h_1(x,1))(h_2(z,1))^m \lambda_1(x,1)\lambda_2(z,1)G(g_1(x,1),g_2(z,1),wvh_1(x,1)h_2(z,1))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{r-1}\rho_{m+n+i}^{(1)}\prod_{j=0}^{t-1}\rho_{m+n+j}^{(2)} a_n \frac{w^{n+s}\nu^{n+r}}{s!r!}\left(S_\alpha^{(m+n+s)}(x)\right)\left(T_p^{(m+n+r)}(z)\right),$$

where the coefficients a_n's are arbitrary, $S_\alpha^{(m+n)}(x)$, $T_p^{(m+n)}(z)$ are two particular special functions and R_1, R_2 are two generators of two one-parameter continuous transformation groups such that

$$R_1\left[S_\alpha^{(m+n)}(x)y^{m+n}\right] = \rho_{m+n}^{(1)} S_\alpha^{(m+n+1)}(x)y^{m+n+1}$$

and

$$R_2\left[T_p^{(m+n)}(z)t^{m+n}\right]=\rho_{m+n}^{(2)}T_p^{(m+n+1)}(z)t^{m+n+1}$$

and also

$$\exp(wR_1)f(x,y)=\lambda_1(x,y)f(g_1(x,y),h_1(x,y))$$

and

$$\exp(\nu R_2)f(z,t)=\lambda_2(z,t)f(g_2(z,t),h_2(z,t)).$$

Remark. In a similar manner, some new results on partial semi-bilinear as well as on improper partial semi-bilinear generating functions can also be derived by adopting the said unified group-theoretic method in a suitable manner.

3. Applications. We shall now state the generating relations derived directly from Theorem 2.1. for Laguerre and Gegenbauer polynomials instead of $S_\alpha^{(m+n)}(x)$ and $T_p^{(k+n)}(z)$. The following are some of the generating relations given in the form of applications:

Application 3.1. For $S_\alpha^{(m+n)}(x)=L_\alpha^{(m+n)}(x)$ and $T_p^{(k+n)}(z)=L_p^{(k+n)}(z)$, where $L_\alpha^{(m+n)}(x)$ and $L_p^{(k+n)}(z)$ are two Laguerre polynomials of order α, p and of parameters $(m+n)$, $(k+n)$, we see that

$$R_1=y\frac{\partial}{\partial x}-y;\qquad R_2=t\frac{\partial}{\partial z}-t;$$

$$\rho_{m+n}^{(1)}=(-1);\qquad \rho_{k+n}^{(2)}=(-1);$$

$$\lambda_1(x,y)=\exp(-wy);\qquad \lambda_2(z,t)=\exp(-\nu t);$$

$$g_1(x,y)=x+wy;\qquad g_2(z,t)=z+\nu t;$$

$$h_1(x,y)=y;\qquad h_1(z,t)=t.$$

Thus from Theorem 2.1, it follows that

$$\exp(-w)\exp(-\nu)G(x+w,z+v,wv)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(-1)^r a_n\frac{w^{n+s}v^{k+s}}{s!r!}\left(L_\alpha^{(m+n+s)}(x)\right)\left(L_p^{(k+n+r)}(z)\right),$$

or

$$\exp(-w-\nu)G(x+w,z+v,wv)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+r}a_n\frac{w^{n+s}v^{k+r}}{s!r!}\left(L_\alpha^{(m+n+s)}(x)\right)\left(L_p^{(k+n+r)}(z)\right)$$

whereas

$$G(x,z,w)==\sum_{n=0}^{\infty}a_nw^nL_\alpha^{(m+n)}(x)L_p^{(k+n)}(z).$$

Application 3.2. For $S_\alpha^{(m+n)}(x)=C_\alpha^{(m+n)}(x)$ and $T_p^{(k+n)}(z)=C_p^{(k+n)}(z)$ where $C_\alpha^{(m+n)}(x)$ and $C_p^{(k+n)}(z)$ are two Gegebauer polynomials of order α, p and of parameters $(m+n)$, $(k+n)$, we see that

$$R_1=xy\frac{\partial}{\partial x}+2y^2\frac{\partial}{\partial y}+\alpha y; \qquad R_2=zt\frac{\partial}{\partial z}+2t^2\frac{\partial}{\partial t}+pt;$$

$$\rho_{m+n}^{(1)}=2(m+n); \qquad \rho_{k+n}^{(2)}=2(k+n);$$

$$\lambda_1(x,y)=(1-2wy)^{-\frac{\alpha}{2}}; \qquad \lambda_2(z,t)=(1-2vt)^{-\frac{p}{2}};$$

$$g_1(x,y)=\frac{x}{\sqrt{1-2wy}}; \qquad g_2(z,t)=\frac{2}{\sqrt{1-2vt}};$$

$$h_1(x,y)=\frac{y}{1-2wy}; \qquad h_2(z,t)=\frac{t}{1-2vt}.$$

Thus from Theorem 2.1, it follows that

$$(1-2w)^{-\alpha-\frac{m}{2}}(1-2v)^{-k-\frac{p}{2}}G\left(\frac{x}{\sqrt{1-2w}},\frac{z}{\sqrt{1-2v}},\frac{wv}{(1-2w)(1-2v)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{s-1}(2^s(m+n+1))\prod_{j=0}^{r-1}(2^r(k+n+j))$$

$$a_n\frac{w^{n+s}v^{k+r}}{s!r!}\left(C_\alpha^{(m+n+s)}(x)\right)\left(C_p^{(k+n+r)}(z)\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}2^{s+r}a_n\frac{w^{n+s}v^{k+s}(m+n)_s(k+n)_r}{s!r!}\left(C_\alpha^{(m+n+s)}(x)\right)\left(C_p^{(k+n+r)}(z)\right)$$

wherever

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n C_{\alpha}^{(m+n)}(x)C_p^{(k+n)}(z)$$

and

$$|v|<\frac{1}{2},\qquad |w|<\frac{1}{2}.$$

Application 3.3. For $S_{\alpha}^{(m+n)}(x)=L_p^{(m+n)}(x)$ and $T_p^{(k+n)}(z)=C_p^{(k+n)}(z)$, where $L_{\alpha}^{(m+n)}(x)$ and $C_p^{(k+n)}(z)$ are Laguerre and Gegenbauer polynomials of order α, p and of parameters $(m+n)$ and $(k+n)$, we see that

$$R_1=y\frac{\partial}{\partial x}-y;\qquad R_2=zt\frac{\partial}{\partial z}+2t^2\frac{\partial}{\partial t}+pt;$$

$$\rho_{m+n}^{(1)}=(-1);\qquad \rho_{k+n}^{(2)}=2(k+n);$$

$$\lambda_1(x,y)=\exp(-wy);\qquad \lambda_2(z,t)=(1-2vt)^{-\frac{p}{2}};$$

$$g_1(x,y)=x+wy;\qquad g_2(z,t)=\frac{z}{\sqrt{1-2vt}};$$

$$h_1(x,y)=y;\qquad h_2(z,t)=\frac{t}{1-2vt}.$$

Thus from Theorem 2,1, it follows that

$$(1-2v)^{-\frac{p}{2}-k}\exp(-w)G\left(x+w,\frac{z}{\sqrt{1-2v}},\frac{wv}{1-2v}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\prod_{j=0}^{r-1}(2^r(k+n+j))a_n\frac{w^{n+s}v^{k+r}}{s!r!}\left(L_{\alpha}^{(m+n+s)}(x)\right)\left(C_p^{(k+n+r)}(z)\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s 2^r a_n\frac{w^{n+s}v^{k+r}(k+n)_r}{s!r!}\left(L_{\alpha}^{(m+n+s)}(x)\right)\left(C_p^{(k+n+r)}(z)\right)$$

whenever $G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n L_{\alpha}^{(m+n)}(x)C_p^{(k+n)}(z)$ and $|v|<\frac{1}{2}$.

Application 3.4. For $S_{\alpha}^{(m+n)}(x) = C_{\alpha}^{(m+n)}(x)$ and $T_p^{(k+n)}(z) = L_p^{(k+n)}(z)$, where $C_{\alpha}^{(m+n)}(x)$ and $L_p^{(k+n)}(z)$ are Gegenbauer and Laguerre polynomials of order α, p and of parameters $(m+n)$, $(k+n)$, we see that

$$R_1 = xy\frac{\partial}{\partial x} + 2y^2\frac{\partial}{\partial y} + \alpha y; \qquad R_2 = t\frac{\partial}{\partial z} - t;$$

$$\rho_{m+n}^{(1)} = 2(m+n); \qquad \rho_{k+n}^{(2)} = (-1);$$

$$\lambda_1(x,y) = (1-2wy)^{-\frac{\alpha}{2}}; \qquad \lambda_2(z,t) = \exp(-vt);$$

$$g_1(x,y) = \frac{x}{\sqrt{1-2wy}}; \qquad g_2(z,t) = z + vt;$$

$$h_1(x,y) = \frac{y}{1-2wy}; \qquad h_2(z,t) = t.$$

Thus from Theorem 2.1, it follows that

$$\exp(-v)(1-2w)^{-\alpha-\frac{m}{2}} G\left(\frac{x}{\sqrt{1-2w}}, z+v, \frac{wv}{1-2w}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\prod_{i=0}^{s-1}(2^s(m+n+i))(-1)^r a_n \frac{w^{n+s}v^{k+r}}{s!r!}\left(C_{\alpha}^{(m+n+s)}(x)\right)\left(L_p^{(k+n+r)}(z)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r.2^s a_n \frac{w^{n+s}v^{k+r}(m+n)_s}{s!r!}\left(C_{\alpha}^{(m+n+s)}(x)\right)\left(L_p^{(k+n+s)}(z)\right)$$

whenever

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n C_{\alpha}^{(m+n)}(x) L_p^{(k+n)}(z) \quad \text{and} \quad |w| < \frac{1}{2}.$$

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