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OSCILLATION CRITERIA FOR NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH DAMPING TERM

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Abstract. Some new criteria for the oscillation of all solutions of second order differential equations of the form

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right) + p(t)\varphi\left(|x|^{\alpha-2}x, r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right) + q(t)\,|x|^{\alpha-2}\,x = 0,$$

and the more general equation

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right) + p(t)\varphi\left(g(x), r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right) + q(t)g(x) = 0,$$

are established. our results generalize and extend some known oscillation criterain in the literature.

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1. Introduction. We are concerned with the oscillation of solutions of second order differential equations with damping of the following form

$$(E_1)\quad \frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(|x|^{\alpha-2}x, r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)$$
$$+q(t)\,|x|^{\alpha-2}\,x=0,$$

and the more general equation

$$(E_2)\quad \frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(g(x), r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)$$
$$+q(t)g(x)=0,$$

where $r\in C[[t_0,\infty),\mathbf{R}^+]$, $p\in C[[t_0,\infty),[0,\infty)]$, $q\in C[[t_0,\infty),\mathbf{R}]$, $\psi\in C[\mathbf{R},\mathbf{R}^+]$ and $g\in C^1[\mathbf{R},\mathbf{R}]$ such that $xg(x)>0$ for $x\neq 0$ and $\frac{d}{dx}g(x)>0$ for $x\neq 0$. φ is defined and continuous on $\mathbf{R}\times\mathbf{R}-\{0\}$ with $u\varphi(u,v)>0$ for $uv\neq 0$ and $\varphi(\lambda u,\lambda v)=\lambda\varphi(u,v)$ for $0<\lambda<\infty$ and $(u,v)\in\mathbf{R}\times\mathbf{R}-\{0\}$.

By the oscillation of equation $(E_1)\,[(E_2)]$, we mean a function $x\in C^1([T_x,\infty),\mathbf{R})$ for some $T_x\geq t_0$, which has the property that $r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\in C^1([T_x,\infty),\mathbf{R})$ and satisfies equation $(E_1)\,[(E_2)]$ on $[T_x,\infty)$.

A solution of equation $(E_1)\,[(E_2)]$ is called oscillatory if it has arbitrarly large zeros otherwise, it is called nonoscillatory. Finally, equation $(E_1)\,[(E_2)]$ is called oscillatory if all its solutions are oscillatory.

In Section 2 we provide sufficient conditions for the oscillation of all solutions of (E_1). Several particular cases of (E_1) have been discussed in the literature. To cite a few examples, the differential equation

$$(E_3)\quad \frac{d}{dt}\left(r(t)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)\,|x|^{\alpha-1}\,x=0$$

has been studied by Hsu and yeh [2] Kusano and Naito [4] Kusano, Yoshida [5], Li and Yeh [6], [7], [8], [9], [10] and Lian, Yeh and Li [11]. A more general equation than (E_3)

$$(E_4)\quad \frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)\,|x|^{\alpha-1}\,x=0$$

has been considered by Ayanlar and Tiryaki [1] and Wu et al. [16]. Our results include, as special cases, known oscillation theorems for (E_3) and (E_4). In particular, we extend and improve the results obtained in [12], [16] and [14].

In Section 3 we will establish some oscillation criteria for equation (E_2). Several particular cases of (E_2) have been discussed in the literature. The differential equation

$$(E_5) \qquad \frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0,$$

established by Manojlovic [13]. Wong and Agarwal [15] considered a special case of this equation as

$$(E_6) \qquad \frac{d}{dt}\left(r(t)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0.$$

Our results in this section generalize and improve Manojlovic [13].

2. Oscillation results for (E_1). In order to discuss our main results, we need the following well-known inequality which is due to Hardy et al. [3, Theorem 41].

Lemma 1. *If X and Y are nonnegative, then*

$$X^\lambda+(\lambda-1)Y^\lambda-\lambda XY^{\lambda-1}\geq 0, \quad \lambda>1,$$

where equality holds if and only if $X=Y$.

Theorem 1. *Suppose that*

$$(1) \qquad \varphi(1,z)\geq z \quad \textit{for all } z\neq 0,$$

$$(2) \qquad 0<\psi(x)\leq\gamma \quad \textit{for all } x,$$

and there exist differentiable functions

$$k\ ,\rho:[t_0,\infty)\to(0,\infty),$$

and the continuous function

$$H:D\equiv\{(t,s):t\geq s\geq t_0\}\to\mathbf{R} \quad \textit{and} \quad h:D_0\equiv\{(t,s):t>s\geq t_0\}\to\mathbf{R},$$

where H has a continuous nonpositive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \quad for \quad t\geq t_0, \quad H(t,s)>0 \quad for \quad t>s\geq t_0,$$

and

$$h(t,s)=-\frac{\partial}{\partial s}\left(H(t,s)k(s)\right)-\left(\frac{\frac{d}{ds}\rho(s)}{\rho(s)}-p(s)\right)H(t,s)k(s) \quad for\ all \quad (t,s)\in D_0.$$

Then equation (E_1) is oscillatory if

$$\text{(3)} \qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)k(s)q(s)-\frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^{\alpha}}{\alpha^{\alpha}(H(t,s)k(s))^{\alpha-1}}\right]ds=\infty.$$

Proof. On the contrary we assume that (E_1) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t)>0$ for all $t\in[t_0,\infty)$. We define the function $\omega(t)$ as

$$\omega(t)=\rho(t)\frac{r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}}{|x|^{\alpha-2}x} \quad \text{for} \quad t\geq t_0.$$

Thus,

$$\frac{d}{dt}\omega(t)=\frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t)+\rho(t)\frac{\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)}{|x|^{\alpha-2}x}-(\alpha-1)\frac{\rho(t)r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha}}{|x|^{\alpha}}.$$

This and equation (E_1) imply

$$\frac{d}{dt}\omega(t)\leq\frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t)-\rho(t)[q(t)+p(t)\varphi(1,\frac{\omega(t)}{\rho(t)})]-(\alpha-1)[\gamma\rho(t)r(t)]^{\frac{-1}{\alpha-1}}\left|\omega(t)\right|^{\frac{\alpha}{\alpha-1}}.$$

From (1) we obtain

$$\frac{d}{dt}\omega(t)\leq\frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t)-\rho(t)q(t)-p(t)\omega(t)-(\alpha-1)[\gamma\rho(t)r(t)]^{\frac{-1}{\alpha-1}}\left|\omega(t)\right|^{\frac{\alpha}{\alpha-1}}.$$

Multiply the above inequality by $H(t,s)k(s)$ and integrate from T to t we obtain

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le \int_T^t H(t,s)k(s)\left(\frac{\frac{d}{ds}\rho(s)}{\rho(s)} - p(s)\right)\omega(s)ds$$

$$-\int_T^t H(t,s)k(s)\frac{d}{ds}\omega(s)ds - (\alpha-1)\int_T^t H(t,s)k(s)[\gamma\rho(s)r(s)]^{\frac{-1}{\alpha-1}}|\omega(s)|^{\frac{\alpha}{\alpha-1}}\,ds.$$

Since

$$-\int_T^t H(t,s)k(s)\frac{d}{ds}\omega(s)ds = H(t,T)k(T)\omega(T) + \int_T^t \frac{\partial}{\partial s}(H(t,s)k(s))\omega(s)ds,$$

the previous inequality becomes

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T) - \int_T^t h(t,s)\omega(s)ds$$

$$-(\alpha-1)\int_T^t H(t,s)k(s)[\gamma\rho(s)r(s)]^{-1/(\alpha-1)}|\omega(s)|^{\alpha/(\alpha-1)}\,ds.$$

Hence we have

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T) + \int_T^t |h(t,s)|\,|\omega(s)|\,ds$$

$$-(\alpha-1)\int_T^t H(t,s)k(s)[\gamma\rho(s)r(s)]^{-1/(\alpha-1)}|\omega(s)|^{\alpha/(\alpha-1)}\,ds. \tag{4}$$

Define

$$X = [\gamma\rho(s)r(s)]^{1/\alpha}\frac{h(t,s)}{\alpha\,(H(t,s)k(s))^{\alpha/(\alpha-1)}},$$

$$Y = \left(H(t,s)k(s)[\gamma\rho(s)r(s)]^{-1/(\alpha-1)}|\omega(s)|^{\alpha/(\alpha-1)}\right)^{1/\alpha}.$$

Since $\alpha > 1$, by Lemma 1 we obtain

$$|h(t,s)|\,|\omega(s)| - \frac{(\alpha-1)H(t,s)k(s)}{[\gamma\rho(s)r(s)]^{1/(\alpha-1)}}|\omega(s)|^{\alpha/(\alpha-1)} \le \frac{\gamma\rho(s)r(s)\,|h(t,s)|^\alpha}{\alpha^\alpha\,(H(t,s)k(s))^{\alpha-1}},$$

for all $t > s \geq T$. Moreover, by (4), we also have for every $t \geq T$,

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \leq H(t,T)k(T)\omega(T) + \int_T^t \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}ds,$$

or

$$(5)\ \int_T^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds \leq H(t,T)k(T)\omega(T)$$
$$\leq H(t,T)k(T)\left|\omega(T)\right|$$
$$\leq H(t,t_0)k(T)\left|\omega(T)\right|.$$

We use the above inequality for $T = T_0$ to obtain

$$\int_{T_0}^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds \leq H(t,t_0)k(T_0)\left|\omega(T_0)\right|.$$

Therefore,

$$\begin{aligned}&\int_{t_0}^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds\\ &= \int_{t_0}^{T_0} \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds\\ &\quad + \int_{T_0}^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha}\right]ds.\end{aligned}$$

Hence for every $t \geq t_0$ we have

$$\int_{t_0}^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds$$

$$\begin{aligned}&\leq \ H(t,t_0)\int_{t_0}^{T_0} k(s)\rho(s)\left|q(s)\right|ds + H(t,t_0)k(T_0)\left|\omega(T_0)\right|\\ &= \ H(t,t_0)\left\{\int_{t_0}^{T_0} k(s)\rho(s)\left|q(s)\right|ds + k(T_0)\left|\omega(T_0)\right|\right\}.\end{aligned}$$

This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \left[H(t,s)\rho(s)k(s)q(s) - \frac{\gamma\rho(s)r(s)\left|h(t,s)\right|^\alpha}{\alpha^\alpha\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds$$

$$\leq \left\{ \int_{t_0}^{T_0} k(s)\rho(s) |q(s)| \, ds + k(T_0) |\omega(T_0)| \right\} < \infty$$

which contradicts the assumption (3). This completes the proof. □

Corollary 1. *If the condition* (3) *is replaced by the condition*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds = \infty,$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s) |h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} ds < \infty,$$

then the conclusion of Theorem 1 *remains valid.*

Theorem 2. *Suppose that* (1) *and* (2) *hold, and let the functions* H, h, ρ *and* k *be the same as in Theorem* 1. *Moreover, assume that*

$$0 < \inf_{s \geq t_0} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty, \tag{6}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s) |h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} < \infty, \tag{7}$$

hold. If there exists a function $\Omega \in C([t_0,\infty),\mathbf{R})$ *such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \frac{\Omega_{+}^{\alpha/(\alpha-1)}(s)}{(k(s)\rho(s)r(s))^{1/(\alpha-1)}} = \infty, \tag{8}$$

and for every $T \geq t_0$,

(9)

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\gamma\rho(s)r(s) |h(t,s)|^{\alpha}}{\alpha^{\alpha} (H(t,s)k(s))^{\alpha-1}} \right] ds \geq \Omega(T),$$

where $\Omega_{+}(t) = \max\{\Omega(t), 0\}$ *for* $t \geq t_0$, *then equation* (E_1) *is oscillatory.*

Proof. On the contrary we assume that (E_1) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_0, \infty)$. Defining $\omega(t)$ as in the proof of Theorem 1, we obtain (4) and (5). Then, for $t > T \geq t_0$ we have

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)k(s)\rho(s)q(s) - \frac{\gamma\rho(s)r(s) |h(t,s)|^{\alpha}}{\alpha^{\alpha} (H(t,s)k(s))^{\alpha-1}} \right] ds \leq k(T)\omega(T).$$

Thus, by (9) we have

$$\Omega(T) \leq k(T)\omega(T) \qquad \text{for all} \quad T \geq T_0, \tag{10}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\rho(s)k(s)q(s)ds \geq \Omega(T_0). \tag{11}$$

Let

$$F(t) = \frac{1}{H(t,T_0)} \int_{T_0}^{t} |h(t,s)|\, |\omega(s)|\, ds,$$

$$G(t) = \frac{(\alpha-1)}{H(t,T_0)} \int_{T_0}^{t} H(t,s)k(s)[\rho(s)r(s)]^{\frac{-1}{\alpha-1}}\, |\omega(s)|^{\frac{\alpha}{\alpha-1}}\, ds,$$

for $t > T_0$. Then by (4) and (11), we get that

$$\begin{aligned}
\lim_{t\to\infty} \inf |G(t) - F(t)| \quad &\leq \quad k(T_0)\omega(T_0) \\
&\qquad - \lim_{t\to\infty} \sup \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\rho(s)k(s)q(s)ds \\
&\leq k(T_0)\omega(T_0) - \Omega(T_0) < \infty. \qquad (12)
\end{aligned}$$

Now, we claim that

$$\int_{T_0}^{\infty} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}} < \infty. \tag{13}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}} = \infty. \tag{14}$$

By (6), there is a positive constant η satisfying

$$\inf_{s\,\geq\, t_0} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)}\right] > \eta. \tag{15}$$

On the other hand, by (14) for any positive number μ there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}}ds \geq \frac{\mu}{(\alpha-1)\,\eta} \quad \text{for all} \quad t \geq T_1.$$

So for all $t \geq T_1$

$$G(t) = \frac{(\alpha-1)}{H(t,T_0)} \int_{T_0}^{t} H(t,s) d \left[\int_{T_0}^{s} k(u) \frac{|\omega(u)|^{\alpha/(\alpha-1)}}{[\rho(u)r(u)]^{1/(\alpha-1)}} du \right]$$

$$= \frac{(\alpha-1)}{H(t,T_0)} \int_{T_0}^{t} \left[\frac{-\partial H(t,s)}{\partial s} \right] d \left[\int_{T_0}^{s} k(u) \frac{|\omega(u)|^{\alpha/(\alpha-1)}}{[\rho(u)r(u)]^{1/(\alpha-1)}} du \right] ds$$

$$\geq \frac{(\alpha-1)}{H(t,T_0)} \int_{T_1}^{t} \left[\frac{-\partial H(t,s)}{\partial s} \right] d \left[\int_{T_0}^{s} k(u) \frac{|\omega(u)|^{\alpha/(\alpha-1)}}{[\rho(u)r(u)]^{1/(\alpha-1)}} du \right] ds$$

$$\geq \frac{\mu}{(\alpha-1)\,\eta} \, \frac{(\alpha-1)}{H(t,T_0)} \int_{T_1}^{t} \left[\frac{-\partial H(t,s)}{\partial s} \right] ds = \frac{\mu H(t,T_1)}{\eta H(t,T_0)}. \tag{16}$$

From (15) we have

$$\lim_{t\to\infty} \inf \frac{H(t,T_1)}{H(t,t_0)} > \eta > 0.$$

So there exists $T_2 \geq T_1$ such that $\dfrac{H(t,T_1)}{H(t,t_0)} \geq \eta$ for all $t \geq T_2$. Therefore by (16) $G(t) \geq \mu$ for all $t \geq T_2$, and since μ is arbitrary constant, we conclude that

$$\lim_{t\to\infty} G(t) = \infty. \tag{17}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} [G(t_n) - F(t_n)] = \lim_{t\to\infty} \sup\, [G(t) - F(t)]\,.$$

In view of (12), there exists a constant M such that

$$G(t_n) - F(t_n) \leq M \quad \text{for all sufficient large } n. \tag{18}$$

It follows from (17) that

$$\lim_{n\to\infty} G(t_n) = \infty. \tag{19}$$

This and (18) give

$$\lim_{n\to\infty} F(t_n) = \infty. \tag{20}$$

Then, by (18) and (19),

$$\frac{F(t_n)}{G(t_n)} - 1 \geq \frac{-M}{G(t_n)} > \frac{-1}{2} \quad \text{for } n \text{ large enough.}$$

Thus,

$$\frac{F(t_n)}{G(t_n)} > \frac{1}{2} \quad \text{for } n \text{ large enough.}$$

This and (20) imply that

$$\lim_{n\to\infty} \frac{F^{\alpha}(t_n)}{G^{\alpha-1}(t_n)} = \infty. \tag{21}$$

On the other hand, by the Holder's inequality, we have

$$\begin{aligned}
F(t_n) &= \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} |h(t_n,s)|\, |\omega(s)|\, ds \\
&\leq \left\{ \frac{\alpha-1}{H(t_n,T_0)} \int_{T_0}^{t_n} H(t_n,s)k(s) \frac{|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}} ds \right\}^{(\alpha-1)/\alpha} \\
&\quad \times \left\{ \frac{1}{(\alpha-1)^{\alpha-1} H(t_n,T_0)} \int_{T_0}^{t_n} \frac{\rho(s)r(s)\, |h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds \right\}^{1/\alpha} \\
&\leq \frac{G^{(\alpha-1)/\alpha}(t_n)}{(\alpha-1)^{(\alpha-1)/\alpha}} \left\{ \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} \frac{\rho(s)r(s)\, |h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds \right\}^{1/\alpha},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\frac{F^{\alpha}(t_n)}{G^{(\alpha-1)}(t_n)} &\leq \frac{1}{(\alpha-1)^{(\alpha-1)} H(t_n,T_0)} \int_{T_0}^{t_n} \frac{\rho(s)r(s)\, |h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds \\
&\leq \frac{1}{(\alpha-1)^{(\alpha-1)} \eta H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)\, |h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds,
\end{aligned}$$

for all large n. It follows from (21) that

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)\, |h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds = \infty, \tag{22}$$

that is,

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{T_0}^{t} \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} ds = \infty,$$

which contradicts (7). Hence, (13) holds. Then, it follows from (10) that

$$\int_{T_0}^{t} \frac{\Omega_{+}^{\alpha/(\alpha-1)}(s)}{[k(s)\rho(s)r(s)]^{1/(\alpha-1)}} ds \le \int_{T_0}^{t} k(s) \frac{|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}} ds < \infty,$$

which contradicts (8). This completes the proof of Theorem 2. □

Theorem 3. *Suppose that* (1) *and* (2) *hold, and let the functions* H, h, ρ *and* k *be the same as in Theorem* 1. *such that* (6) *and*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds < \infty, \tag{23}$$

hold. If there exists a function $\Omega \in C([t_0,\infty),\mathbf{R})$ *such that* (8) *hold for every* $T \ge t_0$ *and*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\gamma\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\,(H(t,s)k(s))^{\alpha-1}}\right] ds \ge \Omega(T), \tag{24}$$

then equation (E_1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E_1) such that $x(t) \neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0 \ge t_0$. Define $\omega(t)$ as of Theorem 1. As in the proofs of Theorem 1 and 2, we can obtain (4), (5) and (10). From (23) it follow that

$$\begin{aligned}
\limsup_{t\to\infty} |G(t) - F(t)| &\le k(t_0)\omega(t_0) \\
&\quad - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds \\
&< \infty, \qquad (25)
\end{aligned}$$

where $F(t)$ and $G(t)$ are defined as in the proof of Theorem 2. By (24) we have

$$\begin{aligned}
\Omega(t_0) \le{} & \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds \\
& - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\gamma\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\,(H(t,s)k(s))^{\alpha-1}} ds.
\end{aligned}$$

This and (23) imply that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} < \infty,$$

considering a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0,∞) with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)\,|h(t_n,s)|^{\alpha}}{(H(t_n,s)k(s))^{\alpha-1}} ds$$

$$= \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} ds < \infty. \tag{26}$$

Now, suppose that (14) holds. With the same argument as in Theorem 2, we conclude that (17) is satisfied. By (25), there exists a constant M such that (18) is fulfilled. Then, following the procedure of the proof of Theorem 2, we see that (22) holds, which contradicts (26). This contradiction proves that (26) fails. The remainder of the proof is similar to that of Theorem 2, so we omit the details. This completes the proof of Theorem 3. □

Theorem 4. *Suppose that* (1) *and* (2) *hold, and let the functions H, h, ρ and k be the same as in Theorem* 1 *such that* (6), *and*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} ds < \infty, \tag{27}$$

hold. If there exists a function $\Omega \in C([t_0,\infty),\mathbf{R})$ such that (8) *and* (24) *hold for every $T \geq t_0$, then equation (E_1) is oscillatory.*

Remark 1. If $p(t) \equiv 0$, and $\psi(x) \equiv 1$, then the above Theorems 1, 2 and 4 extend and improve Theorems 1, 2 and 3 of Manojlovic [12], and Theorems $1-3$ reduce to Theorems $1-3$ of Wang [14], respectively.

Remark 2. If $p(t) \equiv 0$, then Theorems $1-4$, extend and improve Theorems $1-4$ of Wu et al., [16].

Example 1. Consider the differential equation

$$\frac{d}{dt}\left(t^{-\frac{3}{2}}(1+e^{-|x(t)|})\frac{dx}{dt}\right) + t^{-5/2}x(t) = 0 \quad \text{for} \quad t \geq t_0 > 0.$$

We note that

$$\alpha = 2,\; r(t) = t^{-\frac{3}{2}},\; q(t) = t^{-5/2} \quad \text{and } \psi(x) = 1 + e^{-|x(t)|}.$$

let

$$\rho(s) = s,\ k(s) = s \ \text{ and } H(t,s) = (t-s)^2.$$

Then

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\gamma\rho(s)r(s)\,|h(t,s)|^\alpha}{\alpha^\alpha\,(H(t,s)k(s))^{\alpha-1}} \right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} \left(t^2 s^{\frac{-1}{2}} - 2ts^{\frac{1}{2}} + s^{\frac{3}{2}} - 8s^{\frac{-1}{2}} + 8s^{\frac{-1}{2}} t - s^{\frac{-3}{2}} t^2 \right) ds = \infty.$$

Hence, this equation is oscillatory by Theorem 1. However, none of the results of [12], [14] and [16] are applicable for this equation.

Example 2. Consider the differential equation

$$\frac{d}{dt}\left(\frac{2+\cos^2(\ln t)}{1+3\cos^2(\ln t)} \frac{1+3x^2(t)}{2+x^2(t)} \frac{d}{dt}x(t) \right) + \frac{1}{t}\frac{d}{dt}x(t) + \frac{1}{t^2}x(t) = 0,$$

for $t \geq t_0 = 1$. We note that

$$\alpha = 2,\ r(t) = \frac{2+\cos^2(\ln t)}{1+3\cos^2(\ln t)},\ q(t) = \frac{1}{t^2},\ p(t) = \frac{1}{t},$$

and

$$0 < \psi(x) = \frac{1+3x^2}{2+x^2} \leq 3.$$

If we take

$$\rho(s) = 1,\ k(s) = s \ \text{ and } \ H(t,s) = (t-s)^2,$$

then $h(t,s) = 2s(t-s)$ and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\gamma\rho(s)r(s)\,|h(t,s)|^\alpha}{\alpha^\alpha\,(H(t,s)k(s))^{\alpha-1}} \right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[(t-s)^2 \frac{1}{s} - 3s \left(\frac{2+\sin^2(\ln s)}{1+3\sin^2(\ln s)} \right) \right] ds$$

$$\geq \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[(t-s)^2 \frac{1}{s} - 6s \right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2}\left[t^2\ln t - \frac{9}{2}t^2 + 2t + \frac{5}{2}\right] = \infty.$$

Hence by Theorem 1 this equation is oscillatory. One such solution of this equation is $x(t) = \cos(\ln t)$.

3. Oscillation creteria for $(\boldsymbol{E_2})$.

Theorem 5. *Suppose that* (1) *and*

$$\frac{\frac{d}{dx}g(x)}{\left(\psi(x)\left|g(x)^{\alpha-2}\right|\right)^{1/(\alpha-1)}} \geq \delta > 0 \quad \text{for} \quad x \neq 0, \tag{28}$$

hold, and let the functions H, h, ρ and k be the same as in Theorem 1*. Then equation (E_2) is oscillatory if*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)k(s)q(s) - \frac{\beta^{1-\alpha}\rho(s)r(s)\left|h(t,s)\right|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds = \infty,$$

where $\beta = \frac{\delta}{\alpha-1}$.

Proof. Let $x(t)$ a nonoscillatory solution of equation (E_2). Without loss of generality, we may assume that $x(t) \neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0 \geq t_0$. Define $\omega(t)$ as

$$\omega(t) = \rho(t)\frac{r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}}{g(x)} \qquad \text{for} \qquad t \geq t_0.$$

Thus,

$$\frac{d}{dt}\omega(t) = \frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)}{g(x)}$$

$$- \frac{\frac{d}{dx}g(x)}{\left(\psi(x)\left|g(x)^{\alpha-2}\right|\right)^{1/(\alpha-1)}}\frac{\left|\omega(t)\right|^{\frac{\alpha}{\alpha-1}}}{\left[\rho(t)r(t)\right]^{1/(\alpha-1)}}.$$

This and equation (E_2) imply

$$\frac{d}{dt}\omega(t) \leq \frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t) - \rho(t)[q(t) + p(t)\varphi(1, \frac{\omega(t)}{\rho(t)})]$$

$$- \frac{\frac{d}{dx}g(x)}{(\psi(x)\,|g(x)^{\alpha-2}|)^{1/(\alpha-1)}} \frac{|\omega(t)|^{\frac{\alpha}{\alpha-1}}}{[\rho(t)r(t)]^{1/(\alpha-1)}}.$$

From (1) and (28) we have

$$\frac{d}{dt}\omega(t) \leq \frac{\frac{d}{dt}\rho(t)}{\rho(t)}\omega(t) - \rho(t)q(t) - p(t)\omega(t) - \frac{\delta}{[\rho(t)r(t)]^{1/(\alpha-1)}}\,|\omega(t)|^{\frac{\alpha}{\alpha-1}}.$$

Multiply the above inequality by $H(t,s)k(s)$ and integrate from T to t we obtain

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \leq \int_T^t H(t,s)k(s)\left(\frac{\frac{d}{ds}\rho(s)}{\rho(s)} - p(s)\right)\omega(s)ds$$

$$-\int_T^t H(t,s)k(s)\frac{d}{ds}\omega(s)ds - \delta\int_T^t H(t,s)k(s)[\gamma\rho(s)r(s)]^{\frac{-1}{\alpha-1}}\,|\omega(s)|^{\frac{\alpha}{\alpha-1}}\,ds.$$

Since

$$-\int_T^t H(t,s)k(s)\frac{d}{ds}\omega(s)ds = H(t,T)k(T)\omega(T) + \int_T^t \frac{\partial}{\partial s}(H(t,s)k(s))\omega(s)ds.$$

The previous inequality becomes

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \leq H(t,T)k(T)\omega(T) + \int_T^t |h(t,s)\omega(s)|\,ds$$

$$- (\alpha-1)\int_T^t \frac{\beta H(t,s)k(s)\,|\omega(s)|^{(\alpha-1)/\alpha}}{[\rho(s)r(s)]^{1/(\alpha-1)}}ds. \tag{29}$$

Define

$$X = \frac{1}{\alpha}\left[\beta^{(1-\alpha)/\alpha}\,[H(t,s)k(s)]^{(1-\alpha)/\alpha}\,[\rho(s)r(s)]^{1/\alpha}\,|h(t,s)|\right],$$

$$Y = \left(\beta^{(\alpha-1)/\alpha}\left[H(t,s)k(s)\right]^{(\alpha-1)/\alpha}\left[\rho(s)r(s)\right]^{-1/\alpha}|\omega(s)|\right)^{1/(\alpha-1)}.$$

Then use the lemma 1, we have

$$|h(t,s)\omega(s)| - (\alpha-1)\frac{\beta H(t,s)k(s)\,|\omega(s)|^{\frac{\alpha}{\alpha-1}}}{[\rho(s)r(s)]^{1/(\alpha-1)}} \le \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}.$$

From (29) we have

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T) + \int_T^t \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}ds,$$

or

$$\int_T^t \left[H(t,s)k(s)\rho(s)q(s) - \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}\right] ds \le H(t,T)k(T)\omega(T).$$

The remainder of the proof proceeds as in the proof of Theorem 1. The proof is complete. □

Following the procedure of the proof of Theorem 2 and 3, we can also prove the following theorems.

Theorem 6. *Suppose that* (1) *and* (28) *hold, and let the functions* H, h, ρ *and* k *be the same as in Theorem* 1. *If there exist a functions* $\Omega \in C([t_0,\infty),\mathbf{R})$ *such that*

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\left(H(t,s)k(s)\right)^{\alpha-1}}ds < \infty, \tag{30}$$

and that for every $T \ge t_0$,

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\rho(s)k(s)q(s) - \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}\right] ds \ge \Omega(T), \tag{31}$$

and (8) *hold, then every solution of* (E_2) *is oscillatory.*

Theorem 7. *Suppose that* (1) *and* (28) *hold, and let the functions* H, h, ρ *and* k *be the same as in Theorem* 1. *If there exist a function* $\Omega \in C([t_0,\infty),\mathbf{R})$ *such that*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)k(s)q(s)ds < \infty, \tag{32}$$

and that (8) *and* (31) *hold, then every solution of* (E_2) *is oscillatory.*

Theorem 8. *Suppose that* (1) *and* (28) *hold, and let the functions* H, h, ρ *and* k *be the same as in Theorem* 1. *If there exist a function* $\Omega \in C([t_0,\infty),\mathbf{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)\,|h(t,s)|^{\alpha}}{(H(t,s)k(s))^{\alpha-1}} ds < \infty, \tag{33}$$

and that for every $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\,(H(t,s)k(s))^{\alpha-1}} \right] ds \geq \Omega(T), \tag{34}$$

and (8) *hold, then every solution of* (E_2) *is oscillatory.*

Remark 3. If $p(t) \equiv 0$, then Theorem 5, 6 and 8 extend and improve Theorem 1, 3 and 2 of Manojlovic [13].

Example 3. Consider the differential equation

$$\frac{d}{dt}\left(t^{-1}x^2(t)\frac{dx(t)}{dt} \right) + t^{-2}x^3(t) = 0 \quad \text{for} \quad t \geq t_0 > 0.$$

We note that

$$\alpha = 2 \quad \text{and} \quad \frac{\frac{d}{dx}g(x)}{\psi(x)} = 3.$$

Let

$$\rho(s) = 1,\ k(s) = s \quad \text{and} \quad H(t,s) = (t-s)^2.$$

Then

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)k(s)q(s) - \frac{\beta^{1-\alpha}\rho(s)r(s)\,|h(t,s)|^{\alpha}}{\alpha^{\alpha}\,(H(t,s)k(s))^{\alpha-1}} \right] ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} \{ \frac{t^2}{s} - 2t + s - \frac{t^2}{12s} + \frac{t}{2s} - \frac{3}{4} \} ds = \infty.$$

Hence, this equation is oscillatory by Theorem 5. However the result of Manojlovic [13] do not apply to this equation.

Example 4. Consider the differential equation

$$\frac{d}{dt}\left(\frac{2+\sin^2(\ln t)}{1+3\sin^2(\ln t)}\frac{1+3x^2(t)}{2+x^2(t)}\frac{dx}{dt}(t)\right)+\frac{1}{t}\frac{dx}{dt}+\frac{(x(t)+x^3(t))}{t^2(1+3\sin^2(\ln t))}=0,$$

for $t \geq t_0 = 1$. We note that

$$\frac{\frac{d}{dx}g(x)}{\psi(x)} = 2+x^2 \geq 2 = \delta.$$

If we take

$$\rho(s) = 1,\ k(s) = s \text{ and } H(t,s) = (t-s)^2,$$

then $h(t,s) = 2s(t-s)$ and

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\rho(s)k(s)q(s)-\frac{\beta^{1-\alpha}\rho(s)r(s)\left|h(t,s)\right|^{\alpha}}{\alpha^{\alpha}\left(H(t,s)k(s)\right)^{\alpha-1}}\right]ds$$

$$=\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_{1}^{t}\left[\left(\frac{(t-s)^2}{s(1+3\sin^2(\ln s))}\right)-\frac{s}{2}\left(\frac{2+\sin^2(\ln s)}{1+3\sin^2(\ln s)}\right)\right]ds$$

$$\geq\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_{1}^{t}\left[\left(\frac{(t-s)^2}{4s}\right)-s\right]ds$$

$$=\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\left[\frac{1}{4}t^2\ln t-\frac{7}{8}t^2+\frac{1}{2}t+\frac{3}{8}\right]=\infty.$$

Hence by Theorem 1 this equation is oscillatory. One such solution of this equation is $x(t) = \sin(\ln t)$.

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