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GENERALIZED D-SYMMETRIC OPERATORS I

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Abstract. Let H be an infinite-dimensional complex Hilbert space and let $A,\ B \in \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the algebra of operators on H into itself. Let $\delta_{AB}\colon \mathcal{L}(H) \to \mathcal{L}(H)$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$. This note will initiate a study on the class of pairs (A, B) such that $\overline{\mathcal{R}(\delta_{AB})} = \overline{\mathcal{R}(\delta_{B^*A^*})}$; i.e. $\overline{\mathcal{R}(\delta_{AB})}$ is self-adjoint.

Introduction. Let $\mathcal{L}(H)$ the algebra of all bounded operators on an infinite dimensional complex Hilbert space H. The generalized derivation operator δ_{AB} associated with (A, B), defined on $\mathcal{L}(H)$ by $\delta_{AB}(X) = AX - XB$ was systematically studied for the first time in [6]. The properties of such operators have been studied extensively (see for example [2, 5, 8, 9, 10]).

The D-symmetric operators (A is D-symmetric if $\overline{\mathcal{R}(\delta_A)}$ is self-adjoint, where $\overline{\mathcal{R}(\delta_A)}$ is the closure of the range $\mathcal{R}(\delta_A)$ of δ_A in the norm topology) were studied by J. H. Anderson, J. W. Bunce, J. A. Deddens and J. P. Williams [1], S. Bouali and J. Charles [3, 4] and J. G. Stampfli [8].

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We consider the class of pairs (A, B) such that $\overline{\mathcal{R}(\delta_{AB})}$ is self-adjoint, we call such pairs D-symmetric. In this work we extend the results of the D-symmetric operators to D-symmetric pairs.

In the first part we give some properties and characterizations which concern the D-symmetric pairs. The second part contains a description of the sets:

$$\mathcal{C}(A, B) = \{C \in \mathcal{L}(H), \;\; C\mathcal{L}(H) + \mathcal{L}(H)C \subset \overline{\mathcal{R}(\delta_{AB})}\}$$

and

$$\mathcal{I}(A, B) = \{Z \in \mathcal{L}(H), \;\; Z\mathcal{R}(\delta_{AB}) + \mathcal{R}(\delta_{AB})Z \subset \overline{\mathcal{R}(\delta_{AB})}\}$$

which generalize those introduced by J. P. Williams in [10].

Notations.

1. Let $\mathcal{K}(H)$ be the ideal of all compact operators. For $A \in \mathcal{L}(H)$, let $[A]$ denote the coset of A in the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$.

2. $\mathcal{C}_1(H)$ is the ideal of trace class operators.

3. For $A, B \in \mathcal{L}(H)$, $\overline{\mathcal{R}(\delta_{AB})}^U$ denotes the ultraweak closure of $\mathcal{R}(\delta_{AB})$, and $\mathcal{L}(H)'^U$ denotes the bounded linear forms in ultraweak topology.

4. Let M be a subspace of $\mathcal{L}(H)$. We denote the orthogonal of M in the duality $\mathcal{L}(H)$, $\mathcal{L}(H)'$ by M^o.

5. For g and ω two vectors in H, we define $g \otimes \omega \in \mathcal{L}(H)$ as follows:

$$g \otimes \omega(x) = < x, \omega > g \;\; \text{for all } x \in H.$$

1. Properties of D-symmetric Pairs.

Definition 1.1. *Let $A, B \in \mathcal{L}(H)$.*

(1) *If $\overline{\mathcal{R}(\delta_{AB})}$ is self-adjoint i.e. $\overline{\mathcal{R}(\delta_{AB})} = \overline{\mathcal{R}(\delta_{B^*A^*})}$, we say that (A, B) is D-symmetric pair of operators. We denote the set of such pairs by $\mathcal{GD}(H)$.*

(2) *Let $\delta_{[A][B]}$ the generalized derivation operator defined on $\mathcal{C}(H)$ by $\delta_{[A][B]}([X]) = [\delta_{AB}(X)]$. If $\overline{\mathcal{R}(\delta_{[A][B]})}$ is self-adjoint i.e. $\overline{\mathcal{R}(\delta_{[A][B]})} = \overline{\mathcal{R}(\delta_{[B^*][A^*]})}$, we say that $([A], [B])$ is D-symmetric in $\mathcal{C}(H)$.*

Lemma 1.1. *If $A, B \in \mathcal{L}(H)$, then*

$$\mathcal{R}(\delta_{AB})^0 \simeq \mathcal{R}(\delta_{AB})^0 \cap \mathcal{K}(H)^0 \oplus \ker(\delta_{BA}) \cap \mathcal{C}_1(H).$$

The proof of Lemma 1.1 is the same as the proof of Theorem 3 in [11].

Theorem 1.1. *For $A,\ B \in \mathcal{L}(H)$ the following are equivalent:*
(1). (A, B) *is D-symmetric;*
(2). *a.* $([A], [B])$ *is D-symmetric in* $\mathcal{C}(H)$*, and*
b. $BT = TA$ *implies* $BT^* = T^*A$ *for all* $T \in \mathcal{C}_1(H)$*;*
(3). *c.* $([A], [B])$ *is D-symmetric in* $\mathcal{C}(H)$*, and*
d. $\overline{\mathcal{R}(\delta_{AB})}^U = \overline{\mathcal{R}(\delta_{B^*A^*})}^U$.

P r o o f. Note that $\overline{\mathcal{R}(\delta_{AB})}^U$ is self-adjoint if and only if $\mathcal{R}(\delta_{AB})^0 \cap \mathcal{L}(H)'^U$ is self-adjoint. Using Lemma 1.1 we have

$$\mathcal{R}(\delta_{AB})^0 \cap \mathcal{L}(H)'^U \simeq \ker(\delta_{BA}) \cap \mathcal{C}_1(H).$$

Consequently we obtain: $\overline{\mathcal{R}(\delta_{AB})}^U$ is self-adjoint if and only if $\ker(\delta_{BA}) \cap \mathcal{C}_1(H)$ is self-adjoint. Thus (2) $\Leftrightarrow$ (3).

The equivalence of (1) and (2) is a consequence of Lemma 1.1. □

Theorem 1.2. *Let $A, B \in \mathcal{L}(H)$. If there exists $\lambda \in \mathbb{C}$ such that $(B - \lambda)(A - \lambda) = (A - \lambda)^2 = 0$, $A - \lambda \neq 0$ and $B - \lambda \neq 0$, then (A, B) is not D-symmetric.*

P r o o f. Since for all $\lambda \in \mathbb{C}$, $\mathcal{R}(\delta_{AB}) = \mathcal{R}(\delta_{(A-\lambda)(B-\lambda)})$, we may assume without loss of generality that $\lambda = 0$. The condition $A^*A \neq 0$ $(A \neq 0)$ implies that there exists an vector $f = Ah \neq 0$, such that $A^*f \neq 0$. Then $Bf = 0$. Since $A^*B^* = 0$, we choose $g \neq 0$ such that $A^*g = 0$. We put $A^*f = \omega$;

$$\langle \omega, f\rangle = \langle A^*f, f\rangle = \langle f, Af\rangle = \langle f, A^2h\rangle = 0$$

i.e. ω and f are orthogonal. If $X = \|\omega\|^{-2}(g \otimes \omega)$ and $Y \in \mathcal{L}(H)$, then it follows that:

$$\begin{aligned}
\langle (B^*X - XA^*)f, g\rangle &= \langle B^*Xf, g\rangle - \langle XA^*f, g\rangle \\
&= \langle 0, g\rangle - \langle X\omega, g\rangle \\
&= -\langle g, g\rangle \\
&= -\|g\|^2
\end{aligned}$$

and

$$\langle (AY - YB)f, g\rangle = \langle Yf, A^*g > - < 0, g\rangle = 0.$$

Suppose that $B^*X - XA^* \in \overline{\mathcal{R}(\delta_{AB})}^U$. Then there exists a net $(Y_\alpha)_\alpha \subset \mathcal{L}(H)$ such that, for all x and y in H, we have:

$$\langle (AY_\alpha - Y_\alpha B)x, y\rangle \longrightarrow \langle (B^*X - XA^*)x, y\rangle.$$

So that,

$$0 = \langle (AY_\alpha - Y_\alpha B)f, g\rangle \longrightarrow \langle (B^*X - XA^*)f, g\rangle = -\|g\|^2.$$

It follows that $g = 0$; this proves that $B^*X - XA^* \notin \overline{\mathcal{R}(\delta_{AB})}^U$. Consequently we obtain that (A, B) is not D-symmetric by Theorem 1.1. □

Theorem 1.3. *If H is separable, then $\mathcal{GD}(H)$ is not norm-closed in $(\mathcal{L}(H))^2$.*

Proof. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis for H. Define a sequence of operators $(S_n)_{n\geq 1}$ as follows:

$$S_n(e_k) = \begin{cases} \frac{1}{n} e_2, & \text{if} \quad k = 1; \\ e_{k+1}, & \text{if} \quad k \geq 2. \end{cases}$$

Corollary 3 in [7] asserts that for every $n \geq 1$ $\mathcal{K}(H) \subset \overline{\mathcal{R}(\delta_{S_n})}$. It follows from [11, Corollary 1, p. 277] that $\{S_n\}' \cap \mathcal{C}_1(H) = \{0\}$, then Theorem 1.1 implies that $(S_n, S_n) \in \mathcal{GD}(H)$ for all $n \geq 1$. Let

$$S(e_k) = \begin{cases} 0, & \text{if} \quad k = 1; \\ e_{k+1}, & \text{if} \quad k \geq 2. \end{cases}$$

It is clear that $\|(S_n, S_n) - (S, S)\| \longrightarrow 0$. Let $f = e_1 + e_2$, $\omega = e_3$ and $g = e_1$. Since $S^*f = 0$, $Sf = \omega$ and $Sg = 0$, It follows from the proof of Theorem 1.2 that (S^*, S^*) is not D-symmetric. Thus $(S, S) \notin \mathcal{GD}(H)$, which completes the proof. □

2. Properties and Descriptions of $\mathcal{C}(A, B)$ and $\mathcal{I}(A, B)$. Consider the natural closed subalgebras of $\mathcal{L}(H)$ associated with (A, B):

$$\mathcal{C}(A, B) = \{C \in \mathcal{L}(H), \ \ C\mathcal{L}(H) + \mathcal{L}(H)C \subset \overline{\mathcal{R}(\delta_{AB})}\}$$

and

$$\mathcal{I}(A, B) = \{Z \in \mathcal{L}(H), \ \ Z\mathcal{R}(\delta_{AB}) + \mathcal{R}(\delta_{AB})Z \subset \overline{\mathcal{R}(\delta_{AB})}\}$$

It is clear that; if $\mathcal{R}(\delta_{AB})$ is norm-dense in $\mathcal{L}(H)$, $\mathcal{I}(A, B) = \mathcal{C}(A, B) = \mathcal{L}(H)$ (for example $A = 2B = 2I$). Thus $\mathcal{C}(A, B) \neq \{0\}$ and $\mathcal{I}(A, B)$ contains non-scalar operators in general.

Theoren 2.1. *If (A, B) is D-symmetric, then:*

ı. $\mathcal{C}(A,B)$ *and* $\mathcal{I}(A,B)$ *are norm closed* C^*-*algebras in* $\mathcal{L}(H)$*;*
ıı. $\mathcal{C}(A,B)$ *is a two-sided ideal of* $\mathcal{I}(A,B)$*.*

P r o o f. *ı.* It is clear that $\mathcal{C}(A,B)$ and $\mathcal{I}(A,B)$ are norm closed algebras in $\mathcal{L}(H)$. Since $\overline{\mathcal{R}(\delta_{AB})}$ is self-adjoint, $\mathcal{C}(A,B)$ and $\mathcal{I}(A,B)$ are C^*-algebras.

ıı. If $Z\in\mathcal{I}(A,B)$ and $C\in\mathcal{C}(A,B)$, then for all $X\in\mathcal{L}(H)$ we have:

$$X(CZ)=(XC)Z\in\overline{\mathcal{R}(\delta_{AB})}Z\subset\overline{\mathcal{R}(\delta_{AB})},$$

and $(CZ)X=C(ZX)\in\overline{\mathcal{R}(\delta_{AB})}$. Thus $\mathcal{C}(A,B)$ is a right ideal of $\mathcal{I}(A,B)$. Since $\mathcal{C}(A,B)$ and $\mathcal{I}(A,B)$ are C^*-algebras, $\mathcal{C}(A,B)$ is a two-sided ideal of $\mathcal{I}(A,B)$. □

Lemma 2.1. *Let* $A,\ B\in\mathcal{L}(H)$*, then;*

$$\mathcal{I}(A,B)=\{Z\in\mathcal{L}(H),\ \ \delta_Z(A)\mathcal{L}(H)+\mathcal{L}(H)\delta_Z(B)\subset\overline{\mathcal{R}(\delta_{AB})}\}.$$

P r o o f. If $Z\in\mathcal{I}(A,B)$ and $X\in\mathcal{L}(H)$, then

$$\delta_Z(A)X=Z\delta_{AB}(X)-\delta_{AB}(ZX),\quad\text{and}\quad X\delta_Z(B)=\delta_{AB}(X)Z-\delta_{AB}(XZ).$$

This implies that $\delta_Z(A)X\in\overline{\mathcal{R}(\delta_{AB})}$ and $X\delta_Z(B)\in\overline{\mathcal{R}(\delta_{AB})}$. Thus

$$\delta_Z(A)\mathcal{L}(H)+\mathcal{L}(H)\delta_Z(B)\subset\overline{\mathcal{R}(\delta_{AB})}.$$

The reverse inclusion follows from the identities:

$$Z\delta_{AB}(X)=\delta_Z(A)X+\delta_{AB}(ZX),\quad\text{and}\quad\delta_{AB}(X)Z=X\delta_Z(B)+\delta_{AB}(XZ).$$

□

Theorem 2.2. *Let* $A,\ B\in\mathcal{L}(H)$*. If* $\overline{\mathcal{R}(\delta_{AB})}$ *does not contain any nonzero positive operator, then* $\mathcal{C}(A,B)=\{0\}$ *and* $\mathcal{I}(A,B)=\{A\}'\cap\{B\}'$*.*

P r o o f. If $C\in\mathcal{C}(A,B)$ then $CC^*\in\overline{\mathcal{R}(\delta_{AB})}$; consequently we have $C=0$. Thus $\mathcal{C}(A,B)=\{0\}$.

Let $Z\in\mathcal{I}(A,B)$, $\delta_Z(A)\mathcal{L}(H)\subset\overline{\mathcal{R}(\delta_{AB})}$ and $\mathcal{L}(H)\delta_Z(B)\subset\overline{\mathcal{R}(\delta_{AB})}$ by Lemma 2.1.

Consequently we obtain $\delta_Z(A)(\delta_Z(A))^*=(\delta_Z(B))^*\delta_Z(B)=0$. Thus $Z\in\{A\}'\cap\{B\}'$.

Conversely; if $Z\in\{A\}'\cap\{B\}'$, then $\delta_Z(A)=\delta_Z(B)=0$. It follows from Lemma 2.1 that $Z\in\mathcal{I}(A,B)$. □

REFERENCES

[1] J. H. Anderson, J. W. Bunce, J. A. Deddens, J. P. Williams. C*-algebras and derivation ranges. *Acta Sci. Math. (Szeged)*, **40** (1978), 211–227.

[2] J. H. Anderson, C. Foias. Properties which normal operators share with normal derivation and related operators. *Pacific J. Math.*, **61** (1976), 313–325.

[3] S. Bouali, J. Charles. Extension de la notion d'opérateurs d-symétriques I. *Acta Sci. Math. (Szeged)* **58** (1993), 517–525.

[4] S. Bouali, J. Charles , Extension de la notion d'opérateurs d-symétriques II. *Linear Algebra And Its Applications* **225** (1995), 175–185.

[5] D. A. Herrero. Approximation of Hilbert space operators, I. Pitman, Advanced publishing program, Boston – Melbourne, 1982.

[6] M. A. Rosenblum. On the operator equation $BX - XA = Q$. *Duke Math. J.* **23** (1956), 263–269.

[7] C. Rosentrater. Compact operators and derivations induced by weighted shifts. *Pacific J. Math.* **104** (1983), 465–470.

[8] J. G. Stampfli. On self-adjoint derivation ranges. *Pacific J. Math.* **82** (1979), 257–277.

[9] J. G. Stampfli. The norm of a derivation. *Pacific J. Math.* **33** (1970), 737–747.

[10] J. P. Williams. Derivation ranges: Open problems. Topics in Modern Operator Theory, Birkhauser-Verlag, 1981, 319–328.

[11] J. P. Williams. On the range of a derivation. *Pacific J. Math.* **38** (1971), 273–279.

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