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WEAK AND SEMI COMPATIBLE MAPS IN PROBABILISTIC METRIC SPACE USING IMPLICIT RELATION*

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Abstract. The concept of semi compatibility is given in probabilistic metric space and it has been applied to prove the existence of unique common fixed point of four self-maps with weak compatibility satisfying an implicit relation. At the end we provide examples in support of the result.

1. Introduction. Menger [2] in 1942 introduced the notation of the probabilistic metric space. The probabilistic generalization of metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds.

Cho et al. [1] introduced the notation of semi compatible maps in a topological space. According to them a pair of self-maps (S, T) to be semi compatible if condition (i) $Sy = Ty \Rightarrow STy = TSy$; (ii) the sequence $\{x_n\}$ in X and $x \in X$,

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$\{Sx_n\} \to x$, $\{Tx_n\} \to x$ then $STx_n = Tx$ as $n \to \infty$, hold. We define semi compatible self-maps in probabilistic metric space by (ii) only. Popa in [3] used the family Φ of implicit function to find the fixed points of two pairs of semi compatible maps in a d complete topological space, where Φ be the family of real continuous function $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}$ satisfying the properties

($\mathrm{F_h}$) for every $u \geq 0$, $v \geq 0$ with $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ we have $u \geq v$.

($\mathrm{F_u}$) $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

The main object of this paper is to obtain fixed point theorem in the setting of probabilistic metric space using weak compatibility, semi compatibility and an implicit relation. Also in support of the result we furnish examples.

2. Preliminaries.

Let us recall some definitions:

Definition 2.1. *A probabilistic metric space (PM space) is an ordered pair $(X, \mathcal{F})$ consisting of a nonempty set X and a mapping $\mathcal{F}$ from $X \times X$ into the collections of all distribution functions $F \in \mathbb{R}$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in $\mathbb{R}$.*

The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(a) $F_{x,y}(u) = 1 \quad \forall\ u > 0$ iff $x = y$,

(b) $F_{x,y}(0) = 0 \quad \forall\ x, y$ in X,

(c) $F_{x,y} = F_{y,x} \quad \forall\ x, y$ in X,

(d) If $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$ then $F_{x,z}(u+v) = 1\ \forall\ x, y, z$ in X and $u, v > 0$.

Definition 2.2. *A commutative, associative and non-decreasing mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if and only if $t(a, 1) = a$ for all $a \in [0, 1]$, $t(0, 0) = 0$ and $t(c, d) \geq t(a, b)$ for $c \geq a, d \geq b$.*

Definition 2.3. *A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space, t is a t-norm and the generalized triangle inequality*

$$F_{x,z}(u + v) \geq t(F_{x,z}(u), F_{y,z}(v)) \quad \text{holds for all} \quad x, y, z \text{ in } X, \quad u, v > 0.$$

The concept of neighborhoods in Menger space was introduced by Schweizer and Sklar [4].

Definition 2.4. *Let $(X, \mathcal{F}, t)$ be a Menger space. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then (ε, λ)-neighborhood of x, called $U_x(\varepsilon, \lambda)$, is defined by*

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > (1 - \lambda)\}.$$

An (ε, λ)-topology in X is the topology induced by the family

$$\{U_x(\varepsilon, \lambda) : x \in X, \quad \varepsilon > 0 \quad and \quad \lambda \in (0,1)\}$$

of neighborhood.

Remark. If t is continuous, then Menger space $(X, \mathcal{F}, t)$ is a Hausdorff space in (ε, λ)-topology.

Let $(X, \mathcal{F}, t)$ be a complete Menger space and $A \subset X$. Then A is called a bounded set if $\lim\limits_{u \to \infty} \inf\limits_{x,y \in A} F_{x,y}(u) = 1$.

Definition 2.5. *A sequence $\{x_n\}$ in $(X, \mathcal{F}, t)$ is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x(\varepsilon, \lambda)$ for all $n \geq N$ or equivalently $F(x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$.*

Definition 2.6. *A sequence $\{x_n\}$ in $(X, \mathcal{F}, t)$ is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $F(x_n, x_m; \varepsilon) > 1 - \lambda$ for all $n, m \geq N$.*

Definition 2.7. *A Menger space $(X, \mathcal{F}, t)$ with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X.*

Definition 2.8. *Let $(X, \mathcal{F}, t)$ be a Menger space. Two mappings $f, g : X \to X$ are said to be weakly comptable if they commute at coincidence point.*

Lemma 1. *Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is continuous and $t(p, p) \geq p$ for all $p \in [0, 1]$. If there exists a constant $k(0, 1)$ such that for all $p > 0$ and $n \in N$*

$$F(x_n, x_{n+1}; kp) \geq F(x_{n-1}, x_n; p),$$

then $\{x_n\}$ is Cauchy sequence.

Lemma 2. *If (X, d) is a metric space, then the metric d induces a mapping $\mathcal{F} : X \times X \to L$ defined by $F(p, q) = H(x - d(p, q))$, $p, q \in \mathbb{R}$. Further if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.*

3. Main results.

Theorem 3.1. *Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and $t(p,p) \geq p$ for all $pin[0,1]$. Let A, B, S and T be self mappings from X into itself such that*

(I) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(II) *the pair (A, S) is semi-compatible and (B, T) is weak-compatible;*

(III) *one of A or S is continuous.*

For some $\phi \in \Phi$, there exist $k \in (0,1)$ such that for all $x, y \in X$ and $p > 0$

(IV) $\phi(t(F(Ax, By, kp)), t(F(Sx, Ty, p)), t(F(Ax, Sx, p)), t(F(By, Ty, kp))) \geq 0$;

(V) $\phi(t(F(Ax, By, kp)), t(F(Sx, Ty, p)), t(F(Ax, Sx, kp)), t(F(By, Ty, p))) \geq 0$.

Then A, B, S and T have unique common fixed point in X.

Proof. Let x_0 be any arbitrary point of X, as $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ there exists x_1, x_2 in X such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, we construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$.

Now by (IV)

$$\phi(t(F(Ax_{2n}, Bx_{2n+1}, kp)), t(F(Sx_{2n}, Tx_{2n+1}, p)), t(F(Ax_{2n}, Sx_{2n}, p)), t(F(Bx_{2n+1}, Tx_{2n+1}, kp))) \geq 0$$

$$\Rightarrow \phi(t(F(y_{2n+1}, y_{2n+2}, kp)), t(F(y_{2n}, y_{2n+1}, p)), t(F(y_{2n+1}, y_{2n}, p)), t(F(y_{2n+2}, y_{2n+1}, kp))) \geq 0.$$

By $(\mathrm{F_h})$

$$t(F(y_{2n+2}, y_{2n+1}, kp)) \geq t(F(y_{2n+1}, y_{2n}, p))$$

$$\Rightarrow F(y_{2n+2}, y_{2n+1}, kp) \geq F(y_{2n+1}, y_{2n}, p).$$

Substituting again $x = x_{2n+2}$ and $y = x_{2n+1}$ in (V), we have

$$\phi(t(F(y_{2n+3}, y_{2n+2}, kp)), t(F(y_{2n+1}, y_{2n+2}, p)), t(F(y_{2n+3}, y_{2n+2}, kp)), t(F(y_{2n+1}, y_{2n+2}, p))) \geq 0.$$

By ($\mathrm{F_h}$)

$$F(y_{2n+3}, y_{2n+2}, kp) \geq F(y_{2n+2}, y_{2n+1}, p)$$

Hence by Lemma 1, $\{y_n\}$ is Cauchy sequence in X. Therefore $\{y_n\}$ converge to u in X, and its subsequences $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+2}\}$ also converge to u.

Case 1. If S is continuous, we have

$$SAx_{2n} \to Su, \qquad SSx_{2n} \to Su.$$

So, weak compatibility of the pair (A, S) gives $ASx_{2n} \to Su$ as $n \to \infty$.

Step (i) Substituting $x = Sx_{2n}$, $y = x_{2n+1}$ in (IV), we obtain that

$$\phi(t(F(ASx_{2n}, Bx_{2n+1}, kp)), t(F(SSx_{2n}, Tx_{2n+1}, p)), t(F(ASx_{2n}, SSx_{2n}, p)),\\ t(F(Bx_{2n+1}, Tx_{2n+1}, kp))) \geq 0.$$

Now letting $n \to \infty$ and by the continuity of the t-norm, we have

$$\phi(t(F(Su, u, kp)), t(F(Su, u, p)), t(F(Su, Su, p)), t(F(u, u, kp))) \geq 0$$

$$\Rightarrow \phi(F(Su, u, kp), F(Su, u, p), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\Rightarrow \phi(F(Su, u, p), F(Su, u, p), 1, 1) \geq 0$$

Using ($\mathrm{F_u}$), we get $F(Su, u, p) \geq 1$, for all $p > 0$, which gives $F(Su, u, p) = 1$, that is, $Su = u$.

Step (ii) Substituting $x = u$ and $y = x_{2n+1}$ in (IV), we obtain that

$$\phi(t(F(Au, Bx_{2n+1}, kp)), t(F(Su, Tx_{2n+1}, p)), t(F(Au, Su, p)),\\ t(F(Bx_{2n+1}, Tx_{2n+1}, kp))) \geq 0.$$

Taking the limit $n \to \infty$ and as $Su = u$ and Bx_{2n+1}, $Tx_{2n+1} \to u$, we get

$$\phi(F(Au, u, kp), 1, F(Au, u, p), 1) \geq 0.$$

Now as ϕ is non-decreasing in the first argument, we have

$$\phi(F(Au, u, p), 1, F(Au, u, p), 1) \geq 0.$$

Using ($\mathrm{F_h}$), we get $F(Au, u, p) \geq 1$, for all $p > 0$, which gives $F(Au, u, p) = 1$, that is, $Au = u = Su$.

Step (iii) By (I) $A(X) \subseteq T(X)$, then there exists w in X such that $Au = u = Su = Tw$.

Substituting $x = x_{2n}$ and $y = w$ in (IV), we obtain that

$$\phi(t(F(Ax_{2n}, Bw, kp)), t(F(Sx_{2n}, Tw, p)), t(F(Ax_{2n}, Sx_{2n}, p)), t(F(Bw, Tw, kp))) \geq 0.$$

Taking the limit $n \to \infty$ and as Ax_{2n}, $Sx_{2n} \to u$, we get

$$\phi(F(u, Bw, kp), 1, 1, F(Bw, u, kp)) \geq 0.$$

Using ($\mathrm{F_h}$), we get $F(u, Bw, kp) \geq 1$, for all $p > 0$, which gives $F(u, Bw, p) = 1$, that is, $Bw = u$. Therefore $Bw = Tw = u$. Since (B, T) is weak compatible, we get $TBw = BTw$, which implies $Bu = Tu$.

Step (iv) Now substituting $x = u$ and $y = u$ in (IV) and as $Au = u = Su$ and $Bu = Tu$, we get that

$$\phi(t(F(Au, Bu, kp)), t(F(Su, Tu, p)), t(F(Au, Su, p)), t(F(Bu, Tu, kp))) \geq 0,$$

$$\phi(t(F(Au, Bu, kp)), t(F(Su, Tu, p)), 1, 1)) \geq 0.$$

Now as ϕ is non-decreasing in the first argument, we have

$$\phi(F(Au, Bu, p), F(Au, Bu, p), 1, 1) \geq 0.$$

Using ($\mathrm{F_u}$), we get $F(Au, Bu, p) \geq 1$, for all $p > 0$, which gives $F(Au, Bu, p) = 1$, that is, $Au = Bu$. Thus $u = Au = Su = Bu = Tu$.

Case 2. If A is continuous, we have $ASx_{2n} \to Au$. Also the pair (A, S) is semi-compatible, therefore $ASx_{2n} \to Su$. By the uniqueness of the limit $Au = Su$.

Step (v) Substituting $x = u$ and $y = x_{2n+1}$ in (IV), we get

$$\phi(t(F(Au, Bx_{2n+1}, kp)), t(F(Su, Tx_{2n+1}, p)), t(F(Au, Su, p)), t(F(Bx_{2n+1}, Tx_{2n+1}, kp))) \geq 0.$$

Taking the limit $n \to \infty$ and as Bx_{2n+1}, $Tx_{2n+1} \to u$, we get

$$\phi(F(Au, u, kp), 1, F(Au, u, p), 1) \geq 0.$$

Now as ϕ is non-decreasing in the first argument, we have

$$\phi(F(Au, u, p), 1, F(Au, u, p), 1) \geq 0.$$

Using (F_h), we have $F(Au,u,p) \geq 1$ for all $p > 0$, which gives $u = Au$.

The rest of the proof follows from step (iii) onwards of the **Case 1**. □

Uniqueness of common fixed point. Let v be another common fixed point of A, S, B and T, then $v = Av = Sv = Bv = Tv$. Now putting $x = u$ and $y = v$ in (IV), we get

$$\phi(t(F(Au,Bv,kp)), t(F(Su,Tv,p)), t(F(Au,Su,p)), t(F(Bv,Tv,kp))) \geq 0$$

$$\Rightarrow \phi(t(F(u,v,kp)), t(F(u,v,p)), t(F(u,u,p)), t(F(v,v,kp))) \geq 0$$

$$\Rightarrow \phi(t(F(u,v,kp)), t(F(u,v,p)), 1, 1)) \geq 0.$$

Now as ϕ is non-decreasing in the first argument, we have

$$\phi((F(u,v,p)), (F(u,v,p)), 1, 1)) \geq 0.$$

By Using (F_h), we have $\mathcal{F}(u,v,p) \geq 1$ for all $p > 0$, which gives $u = v$.

Corollary 3.2. *Let $(X,\mathcal{F},t)$ be a complete Menger space, where t is continuous and $t(p,p) \geq p$ for all a in $[0,1]$. Let A, B, S and T be self mappings from X into itself such that*

(I) *$A(X) \subseteq T(X) \cap S(X)$;*

(II) *the pair (A,S) is semi compatible and (A,T) is weak compatible;*

(III) *one of A or S is continuous.*

For some $\phi \in \Phi$, there exist $k \in (0,1)$ such that for all $x,y \in X$ and $p > 0$

(IV) $\phi(t(F(Ax,Ay,kp)), t(F(Sx,Ty,p)), t(F(Ax,Sx,p)), t(F(Ay,Ty,kp))) \geq 0$;

(V) $\phi(t(F(Ax,Ay,kp)), t(F(Sx,Ty,p)), t(F(Ax,Sx,kp)), t(F(Ay,Ty,p))) \geq 0$.

Then A, S and T have unique common fixed point in X.

Proof. Put $B = A$ in Theorem 3.1. □

Corollary 3.3. *Let $(X,\mathcal{F},t)$ be a complete Menger space, where t is continuous and $t(p,p) \geq p$ for all a in $[0,1]$. Let A, B, S and T be self mappings from X into itself such that*

(I) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(II) *the pairs (A,S) and (A,T) are semi-compatible;*

(III) *one of A, B, T or S is continuous.*

For some $\phi \in \Phi$, there exist $k \in (0,1)$ such that for all $x, y \in X$ and $p > 0$

(IV) $\phi(t(F(Ax, By, kp)), t(F(Sx, Ty, p)), t(F(Ax, Sx, p)), t(F(By, Ty, kp))) \geq 0;$

(V) $\phi(t(F(Ax, By, kp)), t(F(Sx, Ty, p)), t(F(Ax, Sx, kp)), t(F(By, Ty, p))) \geq 0.$

Then A, B, S and T have unique common fixed point in X.

Proof. As semi-compatible mappings are weak compatible, the proof follows from Theorem 3.1. $\square$

4. Examples.

Example 4.1. Let $X = [0,1]$ and the metric d be defined by $d(x,y) = |x-y|$. For each p define

$$F(x,y,p) = \begin{cases} 1 & \text{for} \quad x = y \\ H(p) & \text{for} \quad x \neq y \end{cases}, \quad \text{where} \quad H(p) = \begin{cases} 0 & \text{if} \quad p \leq 0 \\ p & \text{if} \quad 0 < p < 1 \\ 1 & \text{if} \quad p \geq 1 \end{cases}.$$

Clearly, $(X, \mathcal{F}, t)$ is a complete probabilistic space where t is defined by $t(p,p) \geq p$. Consider the sequence $x_n = 1/n$. Let A, B, S and T be defined as $Ax = x/6$, $Tx = x$, $Bx = x/5$ and $Sx = x/2$. Fix $k = 1$ and $p = 1$. So, we see the all conditions of **Theorem 3.1** are satisfied and hence 0 is the common fixed point in X.

Example 4.2. Let $X = [0,2]$ and the metric d be defined by $d(x,y) = |x-y|$. For each $p > 0$ we define

$$F(x,y,p) = \begin{cases} \dfrac{p}{p + d(x,y)} & \text{if} \quad p > 0 \\ 0 & \text{if} \quad p = 0 \end{cases}$$

Define self maps A, S, B and T as follows $Sx = \begin{cases} \dfrac{1}{2} & 0 \leq x < 1 \\ x & 1 \leq x \leq 2 \end{cases}$, $Ax = \dfrac{x+4}{5}$, $Bx = \dfrac{1+x}{2}$ and $Tx = \begin{cases} 1 & 0 \leq x < 1 \\ \dfrac{3-x}{2} & 1 \leq x \leq 2 \end{cases}$. The sequence $\{x_n\}$ is defined as $x_n = 1 - \dfrac{1}{2n}$. $B1 = 1$ and $T1 = 1 \Rightarrow TB1 = BT1$, clearly $\{B, T\}$ is weak compatible.

$Sx_n = 1 - \dfrac{1}{2n}$ and $Ax_n = 1 - \dfrac{1}{10n}$, clearly $Ax_n \to 1$ and $Sx_n \to 1$, i.e. $u = 1$.

$ASx_n = 1 - \frac{1}{20n}$, $SAx_n = \frac{1}{2}$. Now $\lim F(ASx_n, Su, p) = F(1,1,p) = 1$. Hence $\{A, S\}$ is semi compatible but not compatible as $\lim F(ASx_n, SAx_n, p) = \lim F\left(1 - \frac{1}{20n}, \frac{1}{2}, p\right) = \frac{p}{p+\frac{1}{2}} < 1, \forall\, p$.

So, for all $k \in (0,1)$ we see the all conditions of **Theorem 3.1** are satisfied and hence 1 is the common fixed point in X.

Example 4.3. Let $X = [0,2]$ and the metric d be defined by $d(x,y) = \frac{|x-y|}{1+|x-y|}$. For each p define $\mathcal{F}(x,y,p) = \begin{cases} 1 & \text{for} \quad x = y \\ H(p) & \text{for} \quad x \neq y \end{cases}$, where $H(p) = \begin{cases} 0 & \text{if} \quad p \leq 0 \\ p.d(x,y) & \text{if} \quad 0 < p < 1 \\ 1 & \text{if} \quad p \geq 1 \end{cases}$. Clearly, $(X, \mathcal{F}, t)$ is a complete probabilistic space where t is defined by $t(p,p) \geq p$. $Ax = \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{4-x}{2} & 1 < x \leq 2 \end{cases}$, $Sx = \begin{cases} 1 & x = 1 \\ \frac{x+3}{5} & \text{otherwise} \end{cases}$, $Bx = \begin{cases} \frac{x}{2} & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$ and $Tx = \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{x}{2} & 1 < x \leq 2 \end{cases}$.

The sequence $\{x_n\}$ is defined as $x_n = 2 - \frac{1}{2n}$.

$B1 = 1$ and $T1 = 1 \Rightarrow TB1 = BT1$ and $B2 = T2 = 1 \Rightarrow TB2 = BT2$. Clearly $\{B, T\}$ is weak compatible. $Sx_n = 1 - \frac{1}{10n}$ and $Ax_n = 1 + \frac{1}{4n}$, clearly $Ax_n \to 1$ and $Sx_n \to 1$. That is $u = 1.ASx_n = 1$, $SAx_n = \frac{4}{5} + \frac{1}{20n}$. Now $\lim F(ASx_n, Su, p) = F(1,1,p) = 1$. Hence $\{A, S\}$ is semi compatible but not compatible as

$$\lim F(ASx_n, SAx_n, p) = \lim F\left(1, \frac{4}{5} + \frac{1}{20n}, p\right) = p \cdot \frac{1}{5} < 1.$$

So, for all $k \in (0,1)$ we see the all conditions of **Theorem 3.1** are satisfied and hence 1 is the common fixed point in X.

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