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EVEN AND OLD OVERDETERMINED STRATA FOR DEGREE 6 HYPERBOLIC POLYNOMIALS*

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Abstract. In the present paper we consider degree 6 hyperbolic polynomials (HPs) in one variable (i.e. real and with all roots real). We are interested in such HPs whose number of equalities between roots of the polynomial and/or its derivatives is higher than expected. We give the complete study of the four families of such degree 6 even HPs and also of HPs which are primitives of degree 5 HPs.

1. Introduction.

1.1. An example – the Gegenbauer polynomial of degree 4. In this paper we consider real polynomials in one variable. We are particularly interested in *hyperbolic polynomials (HPs)* (resp. *strictly hyperbolic polynomials (SHPs)*), i.e. with all roots real (resp. real and distinct). One of the things we are looking

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for are HPs in which the number of equalities between roots of the polynomial and its derivatives is higher than expected. Notice that the derivatives of order 1, …, $n-1$ of a degree n (S)HP are (S)HPs. To fix the ideas we present an example first.

Definition 1.1. *In the present paper* the Gegenbauer polynomial G_n *is defined as the unique polynomial of the kind*

$$x^n - x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_0 \tag{1}$$

which is divisible by its second derivative. One can prove that it is strictly hyperbolic, and that it is odd or even together with n. The general definition of Gegenbauer polynomials $C_n^{(\lambda)}(x)$ depends on a parameter λ. In this paper we consider only $C_n^{(-1/2)}(x)$.

The Gegenbauer polynomial $G_4 := x^4 - x^2 + 5/36$ has (by definition) two roots in common with G_4'' (they equal $\pm 1/\sqrt{6}$), and G_4' has 0 as a common root with G_4'''. This makes three equalities between roots from the set of 10 roots, the ones of G_4, G_4', G_4'', G_4''' altogether. Of these equalities none is a formal algebraic corollary of the other two, but when one imposes two of these equalities as conditions, they define a unique polynomial for which it turns out that the third equality also holds.

On the other hand, for a monic degree 4 HP not reduced to x^4 one should expect the total number of such equalities to be not more than 2. Indeed, an affine change of the coordinate of the x-axis and a multiplication by a non-zero constant bring such a polynomial to the form $P = x^4 - x^2 + ax + b$. (If the coefficient of x^3 is zero, and if P is hyperbolic, then the one of x^2 is non-zero (apply the Descartes rule); the latter cannot be positive because P'' must be hyperbolic.) One expects that it is possible to vary the two parameters a, b so that to obtain exactly two *formally independent* equalities among the 10 roots of P, P', P'', P'''. "Formally independent" means "independent as algebraic equalities". E.g. the equalities $a = b$, $b = c$ and $c = d$ are formally independent while $a = b$, $b = c$ and $a = c$ are not. Another type of examples of formally dependent equalities is obtained when P has a multiple, say, double root – the equality between the two roots of P and the fact that the root of P' is between them implies that all three roots are equal.

In the same way one shows that for $n \geq 4$ there are at least $n - 2 + [(n-2)/2]$ equalities between roots of the Gegenbauer polynomial and its derivatives ($n-2$ equalities between roots of G_n and G_n''; 0 is a common root for all $[(n+1)/2]$ derivatives which are odd polynomials). On the other hand there are $n-$

2 coefficients a_j in formula (1) (see Definition 1.1) which play the role of free parameters, therefore one can control $n-2$ equalities between roots of G_n, ..., $G_n^{(n-1)}$. In this sense in the case of G_n the expected number of such equalities is exceeded by $[(n-2)/2]$.

In what follows we say that *an HP has k equalities* if k is the exact number of formally independent equalities between the roots of the HP and of all its nonconstant derivatives.

2.1. Root arrangements and overdetermined strata. We give now the general definitions of root arrangement, multiplicity vector and overdetermined stratum. The latter was proposed by B. Z. Shapiro, see paper [11].

Notation 1.2. *Denote by $x_1 \leq \cdots \leq x_n$ the roots of a degree n HP and in the same way by f_j, s_j, t_j, F_j and l_j the roots of its first, second, third, fourth and fifth derivatives. Our examples concern polynomials of degree ≤ 6 and the notation l_j is chosen to match the first letter of "last" (i.e. "fifth"). When necessary we use also the notation x_i^j for the i-th root of the j-th derivative of an HP where $x_1^j \leq \cdots \leq x_{n-j}^j$.*

Recall that by the Rolle theorem one has

$$x_j \leq f_j \leq x_{j+1} \ , \ j = 1, \ldots n-1 \tag{2}$$

and accordingly for the higher order derivatives. If one has equality from left or right, then one has both equalities and the HP has a multiple root. Thus a multiple root of the derivative of an HP is a multiple root of the HP as well.

Definition 1.3. *The* multiplicity vector (MV) *of a given HP is a vector whose components are the multiplicities of the roots of the HP given in the increasing order. The* arrangement *(or* configuration*) of the roots of a degree n HP and of its derivatives of order 1, ..., $n-1$ is defined when all these roots are written in a string with the sign $<$ or $=$ between any two consecutive roots. Another way to define an arrangement is to give the corresponding* configuration vector (CV)*, i.e. a vector on which the positions of the roots of the HP and its first, ..., fifth derivatives are denoted by* 0, f, s, t, F *and* l *(compare with Notation* 1.2*) and coinciding roots are put in square brackets. When necessary we indicate under the CV which these coinciding roots are in accordance with Notation* 1.2.

Example 1.4. Consider the polynomial $U := x^3(x+1)^2(x-2) = x^6 - 3x^4 - 2x^3$. Its MV equals $(2, 3, 1)$. An easy computation shows that

$t_1 < -2/3 < f_2 < -1/\sqrt{5} = F_1 < s_2 < t_2 < 0 = l_1$. Hence its roots and the roots of its derivatives define the following arrangement:

$$x_1 = f_1 = x_2 < s_1 < t_1 < f_2 < F_1 < s_2 < t_2 < x_3 = f_3 = x_4 = s_3 = f_4 = x_5 =$$

$$= l_1 < F_2 < t_3 < s_4 < f_5 < x_6 \ .$$

The CV looks like this:

$$\begin{array}{cccccccccccccc} ([0f0], & s, & t, & f, & F, & s, & t, & [0f0sf0l], & F, & t, & s, & f, & 0) \\ x_1\, f_1\, x_2 & s_1 & t_1 & f_2 & F_1 & s_2 & t_2 & x_3\, f_3\, x_4\, s_3\, f_4\, x_5\, l_1 & F_2 & t_3 & s_4 & f_5 & x_6 \end{array}$$

Remark 1.5. All arrangements compatible with the Rolle theorem are not realizable by HPs when the degree n is ≥ 4. (The number of all such nondegenerate arrangements, i.e. without equalities, is

$$N(n) = \left(\begin{array}{c} n+1 \\ 2 \end{array} \right)! \frac{1!2! \cdots (n-1)!}{1!3! \cdots (2n-1)!} ,$$

see [13].) For $n = 4$ the negative answer is given in [1]; two of the twelve non-degenerate arrangements are not realizable by HPs (see also [4]). For $n = 5$ it is shown in [3] that only 116 out of 286 nondegenerate arrangements compatible by the Rolle theorem are realizable by HPs. The presence of overdetermined strata (see the definition below) is closely related to this non-realizability, see [4]. For $n = 4$ one can realize all arrangements compatible with the Rolle theorem by degree n *hyperbolic polynomial-like functions (HPLFs)* (i.e. smooth functions having n real zeros and whose n-th derivatives vanish nowhere), see [6], but for $n \geq 5$ this is not true, see [7], [8], [9] and [10]. (Degree n HPLFs are termed also as functions which are *convex of order* n, see [2], p. 23.) In [12] the more delicate question is considered – for which positions of the roots $x_1 \leq f_1 \leq x_2 \leq f_2 \leq x_3$ and s_1 ($f_1 \leq s_1 \leq f_2$) of a degree 2 HPLF and its first two derivatives does there exist such an HPLF.

Denote by $\mathrm{Pol}_n^{\mathbf{C}}$ (resp. $\mathrm{Pol}_n^{\mathbf{R}}$) the space of all monic degree n polynomials in one variable with complex (resp. real) coefficients. When we do not specify whether the coefficients are real or complex we write Pol_n. Denote by $\mathcal{PP}_n^{\mathbf{C}}$ (resp. $\mathcal{PP}_n^{\mathbf{R}}$) the cartesian product $\mathrm{Pol}_n^{\mathbf{C}} \times \cdots \times \mathrm{Pol}_1^{\mathbf{C}}$ (resp. $\mathrm{Pol}_n^{\mathbf{R}} \times \cdots \times \mathrm{Pol}_1^{\mathbf{R}}$). Its points are n-tuples of polynomials $(P_n, P_{n-1}, \ldots, P_1)$ of respective degrees. One can decompose (i.e. stratify) this space according to the presence of multiple roots and their multiplicities and the existence of common zeros of the P_i's. The strata are defined by *coloured partitions* of the $n(n+1)/2$ not necessarily distinct points of **C** (resp. **R**) divided into groups of points with different colours, of cardinalities respectively n, $n-1$, ..., 1.

There is a natural embedding map $\pi : \mathrm{Pol}_n \hookrightarrow \mathcal{PP}_n$ sending each monic polynomial P of degree n to the n-tuple of monic polynomials $(P, P'/n, P''/n(n-1), \ldots, P^{(n-1)}/n!)$.

Suppose that λ is a coloured partition of $n(n+1)/2$ coloured points, $St_\lambda \subset \mathcal{PP}_n$ is the corresponding stratum and $\pi(St_\lambda) = St_\lambda \cap \pi(\mathrm{Pol}_n^{\mathbf{C}})$ is its (probably empty) intersection with the embedded space of polynomials $\pi(\mathrm{Pol}_n^{\mathbf{C}})$. E.g. if on a stratum one has $x_1 = f_1 \neq x_2$ (see (2) and the lines after it), then this intersection is empty. Notice that $\dim St_\lambda$ equals the number of parts in λ.

Definition 1.6. *The stratum St_λ is called* overdetermined *if the codimension of St_λ in $\mathcal{PP}_n$ is greater than the codimension of $\pi(St_\lambda)$ in $\pi(\mathrm{Pol}_n)$. (Here we assume that $\pi(St_\lambda) \neq \emptyset$.) We denote by $\mathcal{Q}$ the difference between these two codimensions.*

Example 1.7. *If P has a double root $x_1 = x_2$, then to define the corresponding stratum in $\mathcal{PP}_n$ this requires two equalities – $x_1 = x_2$ and $x_1 = f_1$ – whereas in $\pi(\mathrm{Pol}_n)$ the first of them is sufficient (it implies the second). Therefore the codimensions mentioned in the above definition equal respectively* 2 *and* 1*. In the same way one shows that more generally whenever the polynomial P has a multiple root, this defines an overdetermined stratum St_λ.*

Example 1.8. *For $n \geq 4$ the Gegenbauer polynomial G_n defines an overdetermined stratum.*

In the present paper we do not discuss in detail overdetermined strata in which the quantity $\mathcal{Q}$ (see Definition 1.6) results *only* from the presence of multiple roots in P. Such overdetermined strata are called *trivial*. There are two more classes of overdetermined strata whose existence is evident.

Definition 1.9. *An overdetermined stratum is* old *if the embedding $\tilde{\pi} : \mathrm{Pol}_{n-1} \hookrightarrow \mathcal{PP}_{n-1}$, $\tilde{\pi}(P) = (P'/n, P''/n(n-1), \ldots, P^{(n-1)}/n!)$ defines an overdetermined stratum in $\mathcal{PP}_{n-1}$. ("Old" is used in the sense of "previously known", i.e. known already for $n-1$.) When n is odd (resp. even), an overdetermined stratum is called* odd *(resp.* even*) if it is defined by an odd (resp. even) polynomial. Such are the strata defined by Gegenbauer polynomials.*

Example 1.10. *The stratum U (see the first CV in the formulation of Theorem* 2.11*) is at the same time even and old. It is obtained by integrating the degree* 5 *hyperbolic odd polynomial $x^3(x^2-1)$ followed by a rescaling and multiplication by a nonzero constant. The quantity $\mathcal{Q}$ for this stratum equals* 6 *– to define the corresponding point in $\pi(St_\lambda)$ one needs only the three equalities $x_2 = x_3 = x_4 = x_5$ and the equality $l_1 = x_2$ whereas to define the corresponding* *point of $\mathcal{PP}_n^{\mathbf{R}}$ one needs ten equalities: $x_2 = f_2 = s_2 = t_2 = x_3 = f_3 = s_3 = l_1 = x_4 = f_4 = x_5$. This stratum is trivial – all equalities between roots result from the four-fold root at 0 and from the equality $l_1 = x_2$. The latter is independent of the former, therefore the surplus $\mathcal{Q}$ of equalities is due only to the multiple root at 0.*

1.3. Aim, scope and plan of the present paper. In what follows we often use family (1) for $n = 6$ (see also Notation 1.2) which we present in the form

$$P = x^6 - x^4 + ax^3 + bx^2 + cx + d \tag{3}$$

Proposition 1.11. 1) *Any triple of equalities $f_3 = t_2$, $t_2 = l_1(= 0)$, $x_{i_1}^{j_1} = x_{i_2}^{j_2}$, $j_1 < j_2$ where the roots $x_{i_1}^{j_1}$, $x_{i_2}^{j_2}$ are not among f_3, t_2, l_1, implies that such a polynomial belongs to an overdetermined even stratum in family* (3).

2) *More generally, a degree n polynomial of family* (1) *for which there hold the equalities $(0 =)x_1^{n-1} = x_2^{n-3} = \cdots = x_{[(n-1)/2]+1}^{n-1-2[(n-1)/2]}$, $x_{i_1}^{j_1} = x_{i_2}^{j_2}$, where the roots $x_{i_1}^{j_1}$, $x_{i_2}^{j_2}$ are not among x_1^{n-1}, x_3^{n-3}, $\cdots$, $x_{[(n-1)/2]+1}^{n-1-2[(n-1)/2]}$, belongs to an overdetermined even or odd stratum.*

3) *A degree n HP with $n - 1$ or more formally independent equalities $x_{i_1}^{j_1} = x_{i_2}^{j_2}$ belongs to an overdetermined stratum.*

Proof. Part 3) results directly from Definition 1.6. To prove part 1) observe that the first two equalities imply $a = c = 0$. The third equality implies the system $P^{(j_1)}(x_{i_1}^{j_1}) = 0 = P^{(j_2)}(x_{i_2}^{j_2})$. As P is even, this system is equivalent to $P^{(j_1)}(-x_{i_1}^{j_1}) = 0 = P^{(j_2)}(-x_{i_2}^{j_2})$ where $-x_{i_k}^{j_k} = x_{n+1-j_k-i_k}^{j_k}$, $k = 1, 2$. This means that the three equalities imply the fourth equality $x_{n+1-j_1-i_1}^{j_1} = x_{n+1-j_2-i_2}^{j_2}$ which is formally independent of them. By Definition 1.6 the polynomial belongs to an overdetermined stratum. Part 2) is proved by complete analogy with part 1). □

The aim of the present paper is to study the even and the old overdetermined strata in family (3) defined by HPs. For degree 4 and 5 polynomials such a study has been done in papers [3] and [4]. Up to now the authors have not found other kinds of overdetermined strata defined by degree 6 HPs. It would be nice (and seemingly very hard) to prove or disprove for any n that in the case of HPs all nontrivial and not old overdetermined strata are even or odd together with n. In the case of complex polynomials this is not true, see [5].

There are two reasons to be interested in the study of even and old overdetermined strata in the case $n = 6$. The first is that there are nontrivial strata defined by polynomials having 4 (i.e. less than $n - 1 = 5$) equalities. Such polynomials are related to part 1) of the proposition. There are no such examples for

$n = 4$ or 5. The second reason is that one can find families of old nontrivial strata containing polynomials with different numbers of equalities for different values of the integration constant. This does not occur for $n = 4$ or 5 which is not explained by the lack of parameters, but by arithmetic reasons, see Remark 3.1.

In Section 2 we list the four one-parameter families of even strata, in Section 3 we discuss old strata.

2. Families of even overdetermined strata.

2.1. The six subfamilies. In the present section we consider four one parameter families of even overdetermined strata. Inspired by part 1) of Proposition 1.11 we define the even strata by the integers j_1 and j_2. We assume that $0 \leq j_1 \leq j_2 - 2$, $j_2 \leq 4$. We do not study systematically old strata in this section (they will be dealt with in Section 3). Therefore we assume that $j_1 = 0$. The four possible values of j_2 are 4, 3, 2 and 1. For all of them $P^{(5)}$ divides $P^{(3)}$. Adding to the three equalities $f_3 = t_2$, $t_2 = l_1 (= 0)$, $x_{i_1}^{j_1} = x_{i_2}^{j_2}$ (which define the family) the equality $x_{n+1-j_1-i_1}^{j_1} = x_{n+1-j_2-i_2}^{j_2}$ resulting from them, see the proof of Proposition 1.11, we characterize the families as follows:

Family A): $P^{(4)}$ divides P and $P^{(5)}$ divides $P^{(3)}$;

Family B): P'''/x divides P and $P^{(5)}$ divides $P^{(3)}$;

Family C): P'' and P have two or four roots in common and $P^{(5)}$ divides $P^{(3)}$.

Family D): P has a multiple root.

The parametrization of each family is described in one of the following four subsections. Two of the families (C) and D)) are divided in two subfamilies, so from now on we speak about the six (sub)families A), B), C1), C2), D1) and D2). There are polynomials belonging to two or three subfamilies at the same time. To understand better the intersections between the subfamilies we present on Fig. 1 a scheme of their intersections. On the figure the subfamilies are indicated by lines of different styles (like the scheme of the lines of an underground). The "stations" are the arrangements with more equalities between roots of the polynomial and/or its derivatives than for neighbouring values of the parameters.

The "stations" are named by letters; in the description of each subfamily we explain how the root arrangement depends on the parameter and we indicate by the same letters the arrangements corresponding to the "stations". Thus for example the first arrangement in Family A) (indicated by the letter Θ) corresponds to the "station" indicated by Θ on the scheme. Family A) is presented there by a dash-dot-dot-dotted line, the third arrangement of that family is presented by the "station" named H, the second arrangement of the family is "line A) of the underground between stations Θ and H" etc.

Of the 20 "stations" three belong to just one line (these are ζ, ξ and η indicated by circles), 16 belong to two and one (namely, U) belongs to three lines. Notice that an arrangement (i.e. a "station") belonging to two (or three) subfamilies is, in general, obtained for different values of the parameters when considered as belonging to the first or to the second (third) subfamily.

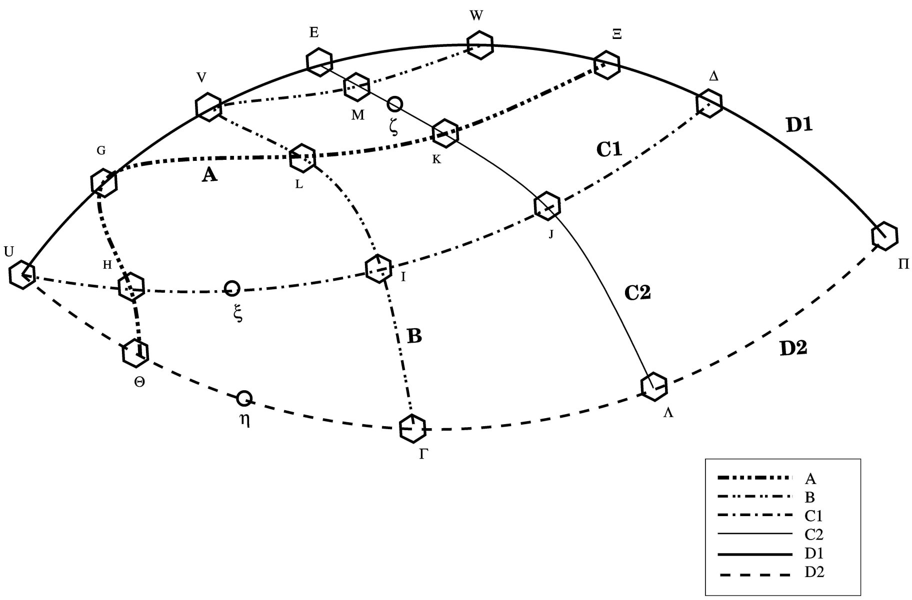

Fig. 1. The six subfamilies

Remark 2.1. 1) In the description of the (sub)families we give beside the arrangements of the polynomials of the strata of the family the approximative values of the parameters corresponding to the "stations" and the letters by which the "stations" are denoted on Fig. 1.

2) For $j_1 = 1$, $j_2 = 3$ and for $j_1 = 1$, $j_2 = 4$ one can obtain in the same way two more lines of the underground described in parts D) and E) of Subsection 3. These lines are not drawn on Fig. 1, but it is easy to imagine how they pass (the first through the "stations" V, ζ and Ξ, the second through G, ξ and η). For $j_1 = 2$, $j_3 = 4$ one obtains only one "station" (namely, E), not a true line, see parts B) and C) of Subsection 3.

Remark 2.2. Consider for the polynomial $Y := (x^2-1)^{2k}$, $k \in \mathbf{N}$, $k \geq 2$. Then $Y^{(k+s)}$ divides $Y^{(k-s)}$ for $s = 1, 2, \ldots, k-1$ (see for the proof of this fact Proposition 8 in [3]). The stratum Ξ (see in Theorem 2.5 the description of Family A)) is defined by a polynomial playing the role of Y'' for $k = 4$. The stratum E (see Theorem 2.9) is defined by one playing the role of Y for $k = 3$. These strata have more equalities than the other "stations".

2.2. Family A). In the present subsection we consider Family A).

Proposition 2.3. *When $P^{(4)}$ divides P and $P^{(5)}$ divides $P^{(3)}$, then $P^{(5)}$ divides P' as well. Hence, the polynomials of the family define overdetermined strata.*

Indeed, present the polynomials in form (3). The roots of $P^{(4)}$ equal $\pm 1/\sqrt{15}$ and the condition $P^{(4)}$ to divide P implies the equalities $1/15^3 - 1/15^2 \pm a/15\sqrt{15} + b/15 \pm c/\sqrt{15} + d = 0$. They are equivalent to

$$\frac{1}{15^3} - \frac{1}{15^2} + \frac{b}{15} + d = 0, \quad a + 15c = 0.$$

The condition $P^{(5)}$ to divide $P^{(3)}$ implies $a = 0$, hence, $c = 0$. The last equality means that 0 is a root of P' as well. The conditions $P^{(4)}$ to divide P and $P^{(5)}$ to divide $P^{(3)}$ are equivalent to the following three equalities $x_j = F_1$, $x_{6-j} = F_2$, $l_1 = t_2$. The equality $l_1 = f_3$ (resulting from $c = 0$) is formally independent of them. This equality is fulfilled for every polynomial of the family, therefore all polynomials of the family belong to overdetermined strata.

One can consider b as a real parameter, hence, the family can be presented in the form

$$P_b \; : \; x^6 - x^4 + bx^2 - b/15 + 14/3375 = (x^2 - 1/15)(x^4 - 14x^2/15 + b - 14/225) \quad \square$$

It would not be correct to say that the polynomials of the family define only one overdetermined stratum because the root arrangement depends on the value of the parameter b. In the present section we describe this arrangement as a function of the value of b.

Lemma 2.4. *The polynomial P_b is hyperbolic if and only if*
$$b \in [14/225, 7/25].$$

Indeed, two of the roots of P_b are roots of $P_b^{(4)}$ as well. These are $\pm 1/\sqrt{15}$. The remaining four roots are $\pm\sqrt{105 \pm 45\sqrt{7-25b}}/15$. All of them are real if and only if one has $7 - 25b \geq 0$ and $105 \geq 45\sqrt{7-25b}$. These inequalities are equivalent to $b \in [14/225, 7/25]$. $\square$

Theorem 2.5. *The CV of the polynomial P_b depends in the following way on the parameter b:*

for $b = \dfrac{14}{225} = 0.062222\ldots,\qquad \Theta$

$$(0,\ f,\ s,\ t,\ [0F],\ f,\ s,\ [0ftl0],\ s,\ f,\ [0F],\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ F_1\ \ f_2\ \ s_2\ \ x_3\ f_3\ t_2\ l_1\ x_4\ \ s_3\ \ f_4\ \ x_5\ F_2\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b \in \left(\dfrac{14}{225}, \dfrac{25}{147} - \dfrac{8\sqrt{78}}{735}\right)$

$$(0,\ f,\ s,\ t,\ [0F],\ f,\ s,\ 0,\ [ftl],\ 0,\ s,\ f,\ [0F],\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ F_1\ \ f_2\ \ s_2\ \ x_3\ \ f_3\ t_2\ l_1\ \ x_4\ \ s_3\ \ f_4\ \ x_5\ F_2\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b = \dfrac{25}{147} - \dfrac{8\sqrt{78}}{735} = 0.073945\ldots,\qquad H$

$$(0,\ f,\ s,\ t,\ [0F],\ f,\ [s0],\ [ftl],\ [0s],\ f,\ [0F],\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ F_1\ \ f_2\ \ s_2\ x_3\ \ f_3\ t_2\ l_1\ \ x_4\ s_3\ \ f_4\ \ x_5\ F_2\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b \in \left(\dfrac{25}{147} - \dfrac{8\sqrt{78}}{735}, \dfrac{3}{25}\right)$

$$(0,\ f,\ s,\ t,\ [0F],\ f,\ 0,\ s,\ [ftl],\ s,\ 0,\ f,\ [0F],\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ F_1\ \ f_2\ \ x_3\ \ s_2\ \ f_3\ t_2\ l_1\ \ s_3\ \ x_4\ \ f_4\ \ x_5\ F_2\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b = \dfrac{3}{25} = 0.12000\ldots,\qquad G$

$$(0,\ f,\ s,\ t,\ [0Ff0],\ s,\ [ftl],\ s,\ [0fF0],\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ F_1\ f_2\ x_3\ \ s_2\ \ f_3\ t_2\ l_1\ \ s_3\ \ x_4\ f_4\ x_5\ F_2\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b \in \left(\dfrac{3}{25}, \dfrac{47}{225}\right)$

$$(0,\ f,\ s,\ t,\ 0,\ f,\ [0F],\ s,\ [ftl],\ s,\ [0F],\ f,\ 0,\ t,\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ \ x_2\ \ f_2\ \ x_3\ F_1\ \ s_2\ \ f_3\ t_2\ l_1\ \ s_3\ \ x_4\ F_2\ \ f_4\ \ x_5\ \ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b = \dfrac{47}{225} = 0.20889\ldots,\qquad L$

$$(0,\ f,\ s,\ [t0],\ f,\ [0F],\ s,\ [ftl],\ s,\ [0F],\ f,\ [0t],\ s,\ f,\ 0)$$
$$x_1\ \ f_1\ \ s_1\ \ t_1\ x_2\ \ f_2\ \ x_3\ F_1\ \ s_2\ \ f_3\ t_2\ l_1\ \ s_3\ \ x_4\ F_2\ \ f_4\ \ x_5\ t_3\ \ s_4\ \ f_5\ \ x_6$$

for $b \in \left(\dfrac{47}{225}, \dfrac{25}{147} + \dfrac{8\sqrt{78}}{735}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & s, & 0, & t, & f, & [0F], & s, & [ftl\ , & s, & [0F], & f, & t, & 0, & s, & f, & 0) \\
x_1 & f_1 & s_1 & x_2 & t_1 & f_2 & x_3\,F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4\,F_2 & f_4 & t_3 & x_5 & s_4 & f_5 & x_6
\end{array}$$

for $b = \dfrac{25}{147} + \dfrac{8\sqrt{78}}{735} = 0.26620\ldots, \qquad K$

$$\begin{array}{cccccccccccccccc}
(0, & f, & [s0], & t, & f, & [0F], & s, & [ftl], & s, & [0F], & f, & t, & [0s], & f, & 0) \\
x_1 & f_1 & s_1\,x_2 & t_1 & f_2 & x_3\,F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4\,F_2 & f_4 & t_3 & x_5\,s_4 & f_5 & x_6
\end{array}$$

for $b \in \left(\dfrac{25}{147} + \dfrac{8\sqrt{78}}{735}, \dfrac{7}{25}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & 0, & s, & t, & f, & [0F], & s, & [ftl], & s, & [0F], & f, & t, & s, & 0, & f, & 0) \\
x_1 & f_1 & x_2 & s_1 & t_1 & f_2 & x_3\,F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4\,F_2 & f_4 & t_3 & s_4 & x_5 & f_5 & x_6
\end{array}$$

for $b = \dfrac{7}{25} = 0.28000\ldots, \qquad \Xi$

$$\begin{array}{ccccccccccc}
([0f0], & s, & [tf], & [0F], & s, & [ftl], & s, & [0F], & [ft], & s, & [0f0]) \\
x_1\,f_1\,x_2 & s_1 & t_1\,f_2 & x_3\,F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4\,F_2 & f_4\,t_3 & s_4 & x_5\,f_5\,x_6
\end{array}$$

P r o o f. One can check directly that when $b \in [14/225, 7/25]$, then the following happens:

A) for and only for $b = 14/225$ does the polynomial P_b have a double root at 0;

B) for and only for $b = (25/147) \pm (8\sqrt{78}/735)$ does one have $\mathrm{Res}(P_b, P_b'') = 0$;

C) for and only for $b = 3/25$ and $b = 7/25$ does the polynomial P_b have a double non-zero root;

D) for and only for $b = 47/225$ does one have $\mathrm{Res}(P_b, P_b''') = 0$.

The conditions $P^{(4)}$ to divide P and $P^{(5)}$ to divide $P^{(3)}$ and Proposition 2.3 hold for all values of $b \in [14/225, 7/25]$. Hence, for values of b close to and greater than $14/225$ a splitting of the root of P_b at 0 occurs and the second of the listed arrangements is valid. For $b \in (14/225, (25/147) - (8\sqrt{78}/735))$ no confluence of any roots is possible; this follows from A) – D).

For $b = (25/147) - (8\sqrt{78}/735)$ the only possible confluence is the one between s_2 and x_3 and by parity the one between s_3 and x_4.

For $b = 3/25$ one must have $x_2 = x_3$ and $x_4 = x_5$. Indeed, the absence of a root of P_b at 0 implies that one has either $x_1 = x_2$, $x_5 = x_6$ or $x_2 = x_3$, $x_4 = x_5$ or both. The first possibility is not realizable because it would imply $x_1 = t_1$, i.e. the even polynomial P_b must have a non-zero root of multiplicity ≥ 4. As there are no confluences for $b \in ((25/147) - (8\sqrt{78}/735), 3/25)$, on this interval one must have $x_3 < s_2$ and $s_3 < x_4$. This justifies the fourth arrangement of the list.

For b close to and greater than $3/25$ the double roots $x_2 = x_3$ and $x_4 = x_5$ have split and for $b = 47/225$ one must have the equalities $t_1 = x_2$ and $t_3 = x_5$. Indeed, it is impossible to have $t_1 = x_1$ and $t_3 = x_6$, see above, and as P_b has no root at 0, the root t_2 is not equal to any of the roots of P_b.

For $b = (25/147) + (8\sqrt{78}/735)$ one has $x_2 = s_1$ and $x_5 = s_4$. Indeed, it is to be checked directly that for this value of b one does not have $x_3 = s_2 = F_1$, $x_4 = s_3 = F_2$.

The last arrangement on the list is checked straightforwardly. The last but one can be deduced from the surrounding two by continuity. $\square$

2.3. Family B). The conditions that the polynomials of Family B) are even and have $t_{1,3} = \pm 1/\sqrt{5}$ as roots imply that the family can be presented in the form

$$R_b : x^6 - x^4 + bx^2 - \frac{b}{5} + \frac{4}{125} = \left(x^2 - \frac{1}{5}\right)\left(x^4 - \frac{4x^2}{5} + b - \frac{4}{25}\right).$$

Lemma 2.6. *The polynomial R_b is hyperbolic if and only if*

$$b \in [4/25, 8/25].$$

Indeed, the roots of R_b which are different from $\pm 1/\sqrt{5}$ equal $\pm\sqrt{10 \pm 5\sqrt{8 - 25b}}/5$. They are all real exactly when the two inequalities hold: $8 - 25b \geq 0$, $10 - 5\sqrt{8 - 25b} \geq 0$. $\square$

Theorem 2.7. *The CV of the polynomial R_b depends in the following way on the parameter b:*

for $b = \dfrac{4}{25} = 0.16000\ldots,\quad \Gamma$

$$\begin{array}{ccccccccccccccc}
(0, & f, & s, & [t0], & f, & F, & s, & [0ftl0], & s, & F, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1 & s_2 & x_3\,f_3\,t_2\,l_1\,x_4 & s_3 & F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

for $b \in \left(\dfrac{4}{25}, \dfrac{123 - 3\sqrt{113}}{490}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & s, & [t0], & f, & F, & s, & 0, & [ftl], & 0, & s, & F, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1 & s_2 & x_3 & f_3\,t_2\,l_1 & x_4 & s_3 & F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b = \frac{123 - 3\sqrt{113}}{490} = 0.18594\ldots, \qquad I$$

$$\begin{array}{cccccccccccccc}
(0, & f, & s, & [t0], & f, & F, & [s0], & [ftl], & [0s], & F, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1 & s_2\,x_3 & f_3\,t_2\,l_1 & x_4\,s_3 & F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b \in \left(\frac{123 - 3\sqrt{113}}{490}, \frac{47}{225}\right)$$

$$\begin{array}{cccccccccccccccc}
(0, & f, & s, & [t0], & f, & F, & 0, & s, & [ftl], & s, & 0, & F, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1 & x_3 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4 & F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b = \frac{47}{225} = 0.20889\ldots, \qquad L$$

$$\begin{array}{cccccccccccccc}
(0, & f, & s, & [t0], & f, & [F0], & s, & [ftl], & s, & [0F], & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1\,x_3 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4\,F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b \in \left(\frac{47}{225}, \frac{7}{25}\right)$$

$$\begin{array}{cccccccccccccccc}
(0, & f, & s, & [t0], & f, & 0, & F, & s, & [ftl], & s, & F, & 0, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b = \frac{7}{25} = 0.28000\ldots, \qquad V$$

$$\begin{array}{cccccccccccc}
(0, & f, & s, & [0ft0], & F, & s, & [ftl], & s, & F, & [0ft0], & s, & f, & 0) \\
x_1 & f_1 & s_1 & x_2\,f_2\,t_1\,x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4\,f_4\,t_3\,x_5 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b \in \left(\frac{7}{25}, \frac{123 + 3\sqrt{113}}{490}\right)$$

$$\begin{array}{cccccccccccccccc}
(0, & f, & s, & 0, & f, & [t0], & F, & s, & [ftl], & s, & F, & [0t], & f, & 0, & s, & f, & 0) \\
x_1 & f_1 & s_1 & x_2 & f_2 & t_1\,x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4\,t_3 & f_4 & x_5 & s_4 & f_5 & x_6
\end{array}$$

$$\textit{for } b = \frac{123 + 3\sqrt{113}}{490} = 0.31610\ldots, \qquad M$$

$$\begin{array}{cccccccccccccc}
(0, & f, & [s0], & f, & [t0], & F, & s, & [ftl], & s, & F, & [0t], & f, & [0s], & f, & 0) \\
x_1 & f_1 & s_1\,x_2 & f_2 & t_1\,x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4\,t_3 & f_4 & x_5\,s_4 & f_5 & x_6
\end{array}$$

for $b \in \left(\frac{123+3\sqrt{113}}{490}, \frac{8}{25}\right)$

$$\begin{array}{cccccccccccccccc} (0, & f, & 0, & s, & f, & [t0], & F, & s, & [ftl], & s, & F, & [0t], & f, & s, & 0, & f, & 0) \\ x_1 & f_1 & x_2 & s_1 & f_2 & t_1 x_3 & F_1 & s_2 & f_3 t_2 l_1 & s_3 & F_2 & x_4 t_3 & f_4 & s_4 & x_5 & f_5 & x_6 \end{array}$$

for $b = \frac{8}{25} = 0.32000\ldots,$ $\quad W$

$$\begin{array}{cccccccccccc} ([0f0], & s, & f, & [t0], & F, & s, & [ftl], & s, & F, & [0t], & f, & s, & [0f0]) \\ x_1 f_1 x_2 & s_1 & f_2 & t_1 x_3 & F_1 & s_2 & f_3 t_2 l_1 & s_3 & F_2 & x_4 t_3 & f_4 & s_4 & x_5 f_5 x_6 \end{array}$$

Proof. 1^0. One checks directly that the roots $t_{1,3} = \pm 1/\sqrt{5}$ of R_b''' are roots of R_b' exactly if $b = 7/25$. For $b < 7/25$ one has $R_b'(1/\sqrt{5}){=}(1/\sqrt{5})(-14/25+2b) < 0$, hence one has $t_1 < f_2$ and $t_3 > f_4$; in the same way $R_b'(1/\sqrt{5}) > 0$ for $b > 7/25$ and one has $t_1 > f_2$ and $t_3 < f_4$.

2^0. The roots $F_{1,2} = \pm 1/\sqrt{15}$ of $R_b^{(4)}$ are roots of R_b as well exactly if $b = 47/225$. This is the only polynomial belonging at the same time to Family A) and Family B). For $b < 47/225$ (resp. for $b > 47/225$) one has $R_b(1/\sqrt{15}) > 0$ (resp. $R_b(1/\sqrt{15}) < 0$).

3^0. One can have $R_b'(\pm 1/\sqrt{15}) = 0$ only for $b = 3/25 \notin [4/25, 8/25]$. Hence, in Family B) the roots of $R_b^{(4)}$ are never roots of R_b'.

4^0. One can have $R_b''(\pm 1/\sqrt{15}) = 0$ exactly if $b = 1/3 > 8/25$. Hence, in Family B) one has $R_b''(\pm 1/\sqrt{15}) < 0$ for all values of b (it suffices to check the inequality for $b = 4/15$ because $R_b''(\pm 1/\sqrt{15})$ is an affine function of b).

5^0. Using 3^0 and 4^0, one concludes that in Family B) one has always $f_2 < F_1 < s_2$ and $s_3 < F_2 < f_4$. Hence, one has $x_2 < f_2 < F_1$ and $F_2 < f_4 < x_5$, therefore for $b = 47/225$ (see 2^0) one has $F_1 = x_3$ and $F_2 = x_4$. For $b < 47/225$ (resp. for $b > 47/225$) one has $F_1 < x_3$, $F_2 > x_4$ (resp. $F_1 > x_3$, $F_2 < x_4$).

6^0. To understand for which values of b there are common roots of R_b and R_b'' one has to solve the system

$$x^6 - x^4 + bx^2 - \frac{b}{5} + \frac{4}{125} = 0, \quad 30x^4 - 12x^2 + 2b = 0$$

whose solutions are $x^2 = 1/\sqrt{5}$, $b = 3/5 \notin [4/25, 8/25]$ and $x^2 = (13 \pm \sqrt{113})/70$, $b = b_\pm = (123 \pm 3\sqrt{113})/490$. For $b = b_-$ one has $R_b'(\sqrt{(13-\sqrt{113})/70}) > 0$, hence one has $x_3 = s_2$ and $x_4 = s_3$. For $b = b_+$ one has $R_b'(\sqrt{(13+\sqrt{113})/70}) < 0$, hence $x_2 = s_1$ and $x_5 = s_4$.

7^0. To sum up the proof of the theorem write in a string the values of the parameter b for which the root arrangement changes and under them the parts of the proof of the theorem or Proposition 2.6 where the change of the arrangement is described:

4/25	b_-	47/225	7/25	b_+	8/25	
Proposition 2.6	6^0	2^0 , 5^0	1^0	6^0	Proposition 2.6	□

2.4. Family C). To define Family C) by a formula we use the conditions that P and P'' have a root τ in common:

$$\tau^6 - \tau^4 + b\tau^2 + d = 0 \ \ , \ \ 30\tau^4 - 12\tau^2 + 2b = 0.$$

From these equalities and setting $v := \tau^2$ one finds that $b = 6v - 15v^2$, $d = -5v^2 + 14v^3$, so one can define Family C) by the formula

$$S_v := x^6 - x^4 + (6v - 15v^2)x^2 - 5v^2 + 14v^3 = (x^2 - v)(x^4 + (v-1)x^2 + 5v - 14v^2),\ v \geq 0.$$

Lemma 2.8. *The polynomial S_v is hyperbolic if and only if $v \in J$ where $J = [0, 1/19] \cup [1/3, 5/14]$.*

Indeed, the polynomial $x^4 + (v-1)x^2 + 5v - 14v^2$ has roots $\pm\sqrt{(1 - v \pm \sqrt{57v^2 - 22v + 1})/2}$. All four roots are real when for $v \geq 0$ the two inequalities hold:

$$57v^2 - 22v + 1 \geq 0, \quad 1 - v - \sqrt{57v^2 - 22v + 1} \geq 0.$$

The last two inequalities and $v \geq 0$ are equivalent to $v \in [0, 1/19] \cup [1/3, 5/14]$.

In what follows we speak about Subfamily C1) and Subfamily C2) which means Family C) defined respectively for $v \in [0, 1/19]$ and $v \in [1/3, 5/14]$.

Theorem 2.9. *The CV of the polynomial S_v depends in the following way on the parameter v:*

for $v \in \left[0, \frac{1}{19}\right]$ (*Subfamily C1)*)

for $v = 0$, U

$(0,\ f,\ s,\ t,\ F,\ [0fst0fsl0f0],\ F,\ t,\ s,\ f,\ 0)$

$x_1\ f_1\ s_1\ t_1\ F_1\ x_2\, f_2\, s_2\, t_2\, x_3\, f_3\, s_3\, l_1\, x_4\, f_4\, x_5\ F_2\ t_3\ s_4\ f_5\ x_6$

for $v \in \left(0, \frac{19}{105} - \frac{2\sqrt{78}}{105}\right)$

$(0,\ f,\ s,\ t,\ F,\ 0,\ f,\ [0s],\ [ftl],\ [s0],\ f,\ 0,\ F,\ t,\ s,\ f,\ 0)$

$x_1\ f_1\ s_1\ t_1\ F_1\ x_2\ f_2\ x_3\, s_2\ f_3\, t_2\, l_1\ s_3\, x_4\ f_4\ x_5\ F_2\ t_3\ s_4\ f_5\ x_6$

for $v = \frac{19}{105} - \frac{2\sqrt{78}}{105} = 0.01272\ldots, \qquad H$

$$\begin{array}{ccccccccccccccc}
(0, & f, & s, & t, & [F0], & f, & [0s], & [ftl], & [s0], & f, & [0F], & t, & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1 & F_1\,x_2 & f_2 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & f_4 & x_5\,F_2 & t_3 & s_4 & f_5 & x_6
\end{array}$$

for $v \in \left(\frac{19}{105} - \frac{2\sqrt{78}}{105}, \frac{1}{5} - \frac{2\sqrt{5}}{25} \right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & s, & t, & 0, & F, & f, & [0s], & [ftl], & [s0], & f, & F, & 0, & t, & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1 & x_2 & F_1 & f_2 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & f_4 & F_2 & x_5 & t_3 & s_4 & f_5 & x_6
\end{array}$$

for $v = \frac{1}{5} - \frac{2\sqrt{5}}{25} = 0.02111\ldots, \qquad \xi$

$$\begin{array}{ccccccccccccccc}
(0, & f, & s, & t, & 0, & [Ff], & [0s], & [ftl], & [s0], & [fF], & 0, & t, & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1 & x_2 & F_1\,f_2 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & f_4\,F_2 & x_5 & t_3 & s_4 & f_5 & x_6
\end{array}$$

for $v \in \left(\frac{1}{5} - \frac{2\sqrt{5}}{25}, \frac{13}{70} - \frac{\sqrt{113}}{70} \right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & s, & t, & 0, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & 0, & t, & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1 & x_2 & f_2 & F_1 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & F_2 & f_4 & x_5 & t_3 & s_4 & f_5 & x_6
\end{array}$$

for $v = \frac{13}{70} - \frac{\sqrt{113}}{70} = 0.03385\ldots, \qquad I$

$$\begin{array}{ccccccccccccccc}
(0, & f, & s, & [t0], & f, & F, & [0s], & [ftl], & [s0], & F, & f, & [0t], & s, & f, & 0) \\
x_1 & f_1 & s_1 & t_1\,x_2 & f_2 & F_1 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & F_2 & f_4 & x_5\,t_3 & s_4 & f_5 & x_6
\end{array}$$

for $v \in \left(\frac{13}{70} - \frac{\sqrt{113}}{70}, \frac{1}{5} - \frac{2\sqrt{7}}{35} \right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & s, & 0, & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & 0, & s, & f, & 0) \\
x_1 & f_1 & s_1 & x_2 & t_1 & f_2 & F_1 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & F_2 & f_4 & t_3 & x_5 & s_4 & f_5 & x_6
\end{array}$$

for $v = \frac{1}{5} - \frac{2\sqrt{7}}{35} = 0.04881\ldots, \qquad J$

$$\begin{array}{ccccccccccccccc}
(0, & f, & [s0], & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & [0s], & f, & 0) \\
x_1 & f_1 & s_1\,x_2 & t_1 & f_2 & F_1 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & F_2 & f_4 & t_3 & x_5\,s_4 & f_5 & x_6
\end{array}$$

for $v \in \left(\frac{1}{5} - \frac{2\sqrt{7}}{35}, \frac{1}{19}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & 0, & s, & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & s, & 0, & f, & 0) \\
x_1 & f_1 & x_2 & s_1 & t_1 & f_2 & F_1 & x_3\, s_2 & f_3\, t_2\, l_1 & s_3\, x_4 & F_2 & f_4 & t_3 & s_4 & x_5 & f_5 & x_6
\end{array}$$

for $v = \frac{1}{19} = 0.052632\ldots, \quad \Delta$

$$\begin{array}{ccccccccccccc}
([0f0], & s, & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & s, & [0f0]) \\
x_1\, f_1\, x_2 & s_1 & t_1 & f_2 & F_1 & x_3\, s_2 & f_3\, t_2\, l_1 & s_3\, x_4 & F_2 & f_4 & t_3 & s_4 & x_5\, f_5\, x_6
\end{array}$$

$v \in \left[\frac{1}{3}, \frac{5}{14}\right]$ (*Subfamily C*2))

for $v = \frac{1}{3} = 0.3333\ldots, \quad E$

$$\begin{array}{ccccccc}
([0fs0f0], & t, & [Fs], & [ftl], & [sF], & t, & [0fs0f0]) \\
x_1\, f_1\, s_1\, x_2\, f_2\, x_3 & t_1 & F_1\, s_2 & f_3\, t_2\, l_1 & s_3\, F_2 & t_3 & x_4\, f_4\, s_4\, x_5\, f_5\, x_6
\end{array}$$

for $v \in \left(\frac{1}{3}, \frac{13}{70} + \frac{\sqrt{113}}{70}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & [0s], & f, & 0, & t, & F, & s, & [ftl], & s, & F, & t, & 0, & f, & [s0], & f, & 0) \\
x_1 & f_1 & x_2\, s_1 & f_2 & x_3 & t_1 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & t_3 & x_4 & f_4 & s_4\, x_5 & f_5 & x_6
\end{array}$$

for $v = \frac{13}{70} + \frac{\sqrt{113}}{70} = 0.33757\ldots, \quad M$

$$\begin{array}{ccccccccccccccc}
(0, & f, & [0s], & f, & [0t], & F, & s, & [ftl], & s, & F, & [t0], & f, & [s0], & f, & 0) \\
x_1 & f_1 & x_2\, s_1 & f_2 & x_3\, t_1 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & t_3\, x_4 & f_4 & s_4\, x_5 & f_5 & x_6
\end{array}$$

for $v \in \left(\frac{13}{70} + \frac{\sqrt{113}}{70}, \frac{1}{5} + \frac{2\sqrt{30}}{75}\right)$

$$\begin{array}{ccccccccccccccccc}
(0, & f, & [0s], & f, & t, & 0, & F, & s, & [ftl], & s, & F, & 0, & t, & f, & [s0], & f, & 0) \\
x_1 & f_1 & x_2\, s_1 & f_2 & t_1 & x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4 & t_3 & f_4 & s_4\, x_5 & f_5 & x_6
\end{array}$$

for $v = \dfrac{1}{5} + \dfrac{2\sqrt{30}}{75} = 0.34606\ldots, \quad \zeta$

$$\begin{array}{ccccccccccccccc} (0, & f, & [0s], & [ft], & 0, & F, & s, & [ftl], & s, & F, & 0, & [tf], & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & f_2\,t_1 & x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4 & t_3\,f_4 & s_4\,x_5 & f_5 & x_6 \end{array}$$

for $v \in \left(\dfrac{1}{5} + \dfrac{2\sqrt{30}}{75}, \dfrac{19}{105} + \dfrac{2\sqrt{78}}{105}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & [0s], & t, & f, & 0, & F, & s, & [ftl], & s, & F, & 0, & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & t_1 & f_2 & x_3 & F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2 & x_4 & f_4 & t_3 & s_4\,x_5 & f_5 & x_6 \end{array}$$

for $v = \dfrac{19}{105} + \dfrac{2\sqrt{78}}{105} = 0.34918\ldots, \quad K$

$$\begin{array}{ccccccccccccccc} (0, & f, & [0s], & t, & f, & [0F], & s, & [ftl], & s, & [F0], & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & t_1 & f_2 & x_3\,F_1 & s_2 & f_3\,t_2\,l_1 & s_3 & F_2\,x_4 & f_4 & t_3 & s_4\;x_5 & f_5 & x_6 \end{array}$$

for $v \in \left(\dfrac{19}{105} + \dfrac{2\sqrt{78}}{105}, \dfrac{1}{5} + \dfrac{2\sqrt{7}}{35}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & [0s], & t, & f, & F, & 0, & s, & [ftl], & s, & 0, & F, & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & t_1 & f_2 & F_1 & x_3 & s_2 & f_3\,t_2\,l_1 & s_3 & x_4 & F_2 & f_4 & t_3 & s_4\,x_5 & f_5 & x_6 \end{array}$$

for $v = \dfrac{1}{5} + \dfrac{2\sqrt{7}}{35} = 0.35119\ldots, \quad J$

$$\begin{array}{ccccccccccccccc} (0, & f, & [0s], & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & t_1 & f_2 & F_1 & x_3\,s_2 & f_3\,t_2\,l_1 & s_3\,x_4 & F_2 & f_4 & t_3 & s_4\,x_5 & f_5 & x_6 \end{array}$$

for $v \in \left(\dfrac{1}{5} + \dfrac{2\sqrt{7}}{35}, \dfrac{5}{14}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & [0s], & t, & f, & F, & s, & 0, & [ftl], & 0, & s, & F, & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\;s_1 & t_1 & f_2 & F_1 & s_2 & x_3 & f_3\;t_2\;l_1 & x_4 & s_3 & F_2 & f_4 & t_3 & s_4\;x_5 & f_5 & x_6 \end{array}$$

for $v = \dfrac{5}{14} = 0.35714\ldots, \quad \Lambda$

$$\begin{array}{ccccccccccccccc} (0, & f, & [0s], & t, & f, & F, & s, & [0ftl0], & s, & F, & f, & t, & [s0], & f, & 0) \\ x_1 & f_1 & x_2\,s_1 & t_1 & f_2 & F_1 & s_2 & x_3\,f_3\,t_2\,l_1\,x_4 & s_3 & F_2 & f_4 & t_3 & s_4\,x_5 & f_5 & x_6 \end{array}$$

P r o o f. 1^0. The theorem states that the root arrangement is constant on certain open intervals and changes at their extremities (the first open interval being $(0, 19/105 - 2\sqrt{78}/105)$).

One checks directly the following facts:

a) the roots $t_{1,3} = \pm 1/\sqrt{5}$ of S_v'''

a1) are roots of S_v exactly when $v = 1/5 \notin J$ or when $v = (13 + \sqrt{113})/70 \in (1/3, 5/14)$ or when $v = (13 - \sqrt{113})/70 \in (0, 1/19)$;

a2) are roots of S_v' exactly when $v = 1/5 - 2\sqrt{30}/75 \notin J$ or $v = 1/5 + 2\sqrt{30}/75 \in (1/3, 5/14)$;

b) the roots $F_{1,2} = \pm 1/\sqrt{15}$ of $S_v^{(4)}$

b1) are roots of S_v exactly when $v = 1/15 \notin J$ or when $v = 19/105 + 2\sqrt{78}/105 \in (1/3, 5/14)$ or when $v = 19/105 - 2\sqrt{78}/105 \in (0, 1/19)$;

b2) are roots of S_v' exactly when $v = 1/5 - 2\sqrt{5}/25 \in (0, 1/19)$ or when $v = 1/5 + 2\sqrt{5}/25 > 5/14$;

b3) are roots of S_v'' exactly when $v = 1/3$ or $v = 1/15 \notin J$;

c) the root arrangements for $v = 0$, $v = 1/19$, $v = 1/3$ and $v = 5/14$ are the ones indicated in the theorem. (E.g. to check that $F_1 = -1/\sqrt{15}$ is between f_2 and f_3 for $v = 5/14$ one has to show that $S_{5/14}'(F_1) < 0$.)

2^0. For $v > 0$ and close to 0 the roots s_1 and s_4 of S_v'' cannot be equal to roots of S_v. Therefore one has $s_2 = x_3$ and $s_3 = x_4$. (It is impossible to have $s_2 = x_2$ because this would imply $s_2 = f_2$, hence $x_2 = f_2 = x_3 = s_2 = f_3$ and by symmetry $f_3 = s_3 = x_4 = f_4 = x_5$ which happens only for $v = 0$.)

3^0. For $v = 0$ (resp. $v = 1/19$) one has $F_1 < x_2$, $F_2 > x_5$ (resp. $F_1 > x_2$, $F_2 < x_5$). There is only one value of $v \in (0, 1/19)$ for which $F_1 = x_2$ and $F_2 = x_5$ (namely, $v = 19/105 - 2\sqrt{78}/105$, see b1)). This is the only value of v from the interval $[0, 1/19]$ for which the roots $F_{1,2}$ are roots of S_v. This justifies the relative position of $F_{1,2}$ and the roots x_j of S_v for $v \in [0, 1/19]$. (We use the fact that the roots of the polynomials $S_v^{(k)}$, $0 \leq k \leq 4$, depend continuously on v.)

4^0. In the same way one justifies the relative position of $F_{1,2}$ and the roots f_i of S_v' (see b2)) and the one of $t_{1,3}$ and x_ν (see a1)) for $v \in [0, 1/19]$. The relative positions of $t_{1,3}$ and x_ν and the one of $t_{1,3}$ and f_i do not change for $v \in [0, 1/19]$. This justifies the first seven arrangements from the list given in the theorem. It justifies also the eighth arrangement for v bigger than and close to $(13 - \sqrt{113})/70$, and the tenth one for v smaller than and close to $1/19$.

5^0. For $v \in [0, 1/19]$ there exists a single value of v (namely, $1/5 - 2\sqrt{7}/35$) for which one has $x_2 = s_1$, $x_3 = s_2$, $x_4 = s_3$ and $x_5 = s_4$. This follows from the uniqueness of the Gegenbauer polynomial (see Definition 1.1) – the coefficient $6v - 15v^2$ of x^2 in S_v is a monotonous function of $v \in [0, 1/19]$, therefore the family S_v contains the Gegenbauer polynomial for only one value of v. This justifies the remaining arrangements for $v \in [0, 1/19]$.

6^0. For $v \in [1/3, 5/14]$ a similar reasoning (left for the reader) justifies all arrangements. $\square$

2.5. Family D). When $P = x^6 - x^4 + bx^2 + d$ has a multiple root w, then the conditions $P(w) = P'(w) = 0$ imply that one has either $b = -3w^4 + 2w^2$ and $d = 2w^6 - w^4$ or $w = 0$ and $d = 0$. In the first case we speak about Subfamily D1), in the second – about Subfamily D2). Set $r := w^2$, $r \geq 0$. Subfamily D1) can be parametrized as follows:

$$T_r = x^6 - x^4 + (-3r^2 + 2r)x^2 + 2r^3 - r^2.$$

Lemma 2.10. *For $r \geq 0$ the polynomial T_r is hyperbolic if and only if $r \in [0, 1/2]$.*

Indeed, one has $T_r = (x^2 - r)^2(x^2 + 2r - 1)$ and the factor $x^2 + 2r - 1$ is hyperbolic if and only if $r \in [0, 1/2]$.

Theorem 2.11. *The CV of the polynomial T_r of Subfamily D1) depends in the following way on the parameter r:*

for $r = 0$, $\quad U$

$$\begin{array}{ccccccccccc} (0, & f, & s, & t, & F, & [0fst0fsl0f0], & F, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & F_1 & x_2\, f_2\, s_2\, t_2\, x_3\, f_3\, s_3\, l_1\, x_4\, f_4\, x_5 & F_2 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $r \in \left(0, \dfrac{1}{15}\right)$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & F, & [0f0], & s, & [ftl], & s, & [0f0], & F, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & F_1 & x_2\, f_2\, x_3 & s_2 & f_3\, t_2\, l_1 & s_3 & x_4\, f_4\, x_5 & F_2 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $r = \dfrac{1}{15} = 0,06666\ldots,$ $\quad G$

$$\begin{array}{ccccccccccccc} (0, & f, & s, & t, & [0Ff0], & s, & [ftl], & s, & [0fF0], & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2\, F_1\, f_2\, x_3 & s_2 & f_3\, t_2\, l_1 & s_3 & x_4\, f_4\, F_2\, x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $r \in \left(\dfrac{1}{15}, \dfrac{1}{5}\right)$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & [0f0], & F, & s, & [ftl], & s, & F, & [0f0], & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2\, f_2\, x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4\, f_4\, x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $r = \dfrac{1}{5} = 0,20000\ldots,$ $\quad V$

$$\begin{array}{ccccccccccccc} (0, & f, & s, & [0tf0], & F, & s, & [ftl], & s, & F, & [0ft0], & s, & f, & 0) \\ x_1 & f_1 & s_1 & x_2\, t_1\, f_2\, x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4\, f_4\, t_3\, x_5 & s_4 & f_5 & x_6 \end{array}$$

for $r \in \left(\frac{1}{5}, \frac{1}{3}\right)$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & [0f0], & t, & F, & s, & [ftl], & s, & F, & t, & [0f0], & s, & f, & 0) \\ x_1 & f_1 & s_1 & x_2\, f_2\, x_3 & t_1 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & t_3 & x_4\, f_4\, x_5 & s_4 & f_5 & x_6 \end{array}$$

for $r = \frac{1}{3} = 0,33333\ldots, \quad E$

$$\begin{array}{ccccccc} ([0f0sf0], & t, & [Fs], & [ftl], & [sF], & t, & [0fs0f0]) \\ x_1\, f_1\, x_2\, s_1\, f_2\, x_3 & t_1 &]F_1\, s_2 & f_3\, t_2\, l_1 & s_3\, F_2 & t_3 & x_4\, f_4\, s_4\, x_5\, f_5\, x_6 \end{array}$$

for $r = \left(\frac{1}{3}, \frac{2}{5}\right)$

$$\begin{array}{ccccccccccccccc} ([0f0], & s, & f, & 0, & t, & F, & s, & [ftl], & s, & F, & t, & 0, & f, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & f_2 & x_3 & t_1 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & t_3 & x_4 & f_4 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r = \frac{2}{5} = 0,40000\ldots, \quad W$

$$\begin{array}{ccccccccccccc} ([0f0], & s, & f, & [0t], & F, & s, & [ftl], & s, & F, & [t0], & f, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & f_2 & x_3\, t_1 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & t_3\, x_4 & f_4 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r \in \left(\frac{2}{5}, \frac{7}{15}\right)$

$$\begin{array}{ccccccccccccccc} ([0f0], & s, & f, & t, & 0, & F, & s, & [ftl], & s, & F, & 0, & t, & f, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & f_2 & t_1 & x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4 & t_3 & f_4 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r = \frac{7}{15} = 0,466666\ldots, \quad \Xi$

$$\begin{array}{ccccccccccc} ([0f0], & s, & [ft], & [0F], & s, & [ftl], & s, & [F0], & [tf], & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & f_2\, t_1 & x_3\, F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2\, x_4 & t_3\, f_4 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r \in \left(\frac{7}{15}, \frac{9}{19}\right)$

$$\begin{array}{ccccccccccccccc} ([0f0], & s, & t, & f, & F, & 0, & s, & [ftl], & s, & 0, & F, & f, & t, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & t_1 & f_2 & F_1 & x_3 & s_2 & f_3\, t_2\, l_1 & s_3 & x_4 & F_2 & f_4 & t_3 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r = \frac{9}{19} = 0,47368\ldots, \quad \Delta$

$$\begin{array}{ccccccccccccc} ([0f0], & s, & t, & f, & F, & [0s], & [ftl], & [s0], & F, & f, & t, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & t_1 & f_2 & F_1 & x_3\, s_2 & f_3\, t_2\, l_1 & s_3\, x_4 & F_2 & f_4 & t_3 & s_4 & x_5\, f_5\, x_6 \end{array}$$

for $r \in \left(\dfrac{9}{19}, \dfrac{1}{2}\right)$

$([0f0],$	$s,$	$t,$	$f,$	$F,$	$s,$	$0,$	$[ftl],$	$0,$	$s,$	$F,$	$f,$	$t,$	$s,$	$[0f0])$
$x_1\, f_1\, x_2$	s_1	t_1	f_2	F_1	s_2	x_3	$f_3\, t_2\, l_1$	x_4	s_3	F_2	f_4	t_3	s_4	$x_5\, f_5\, x_6$

for $r = \dfrac{1}{2} = 0,50000\ldots,$ $\quad \Pi$

$([0f0],$	$s,$	$t,$	$f,$	$F,$	$s,$	$[0ftl0],$	$s,$	$F,$	$f,$	$t,$	$s,$	$[0f0])$
$x_1\, f_1\, x_2$	s_1	t_1	f_2	F_1	s_2	$x_3\, f_3\, t_2\, l_1\, x_4$	s_3	F_2	f_4	t_3	s_4	$x_5\, f_5\, x_6$

P r o o f. 1^0. The arrangement for $r = 0$ is evident and the ones for $r = 1/3$ and $r = 1/2$ are to be checked directly. (E.g. for $r = 1/2$ it suffices to check that $T'_{1/2}(F_1) < 0$ and $T''_{1/2}(F_1) < 0$, $F_1 = -1/\sqrt{15}$.)

2^0. The roots $\pm\sqrt{r}$ of T_r are roots of $T_r^{(4)}$, T_r''', T_r'' respectively for (and only for) $r = 1/15$, $1/5$ and $1/3$. For $r \in [0, 1/3) \cup (1/3, 1/2]$ (resp. for $r = 1/3$) one has $T_r''(F_1) < 0$ (resp. $T_r''(F_1) = 0$). This justifies all arrangements for $r \in [0, 1/3]$.

3^0. If $r > 0$, then the roots of $x^2 + 2r - 1$ are roots of T_r'', T_r''' and $T_r^{(4)}$ respectively for (and only for) $r = 1/3$, $2/5$ and $7/15$. The roots $t_{1,3} = \pm 1/\sqrt{5}$ of T_r''' are roots of T_r' for and only for $r = 1/5$ and $r = 7/15$. It is only for $r = 1/3$ and $r = 9/19$ that $x^2 + 2r - 1$ and T_r'' have a root in common. This is sufficient to justify all arrangements for $r \in [1/3, 1/4]$. $\square$

Subfamily D2) can be parametrized by the polynomial $U_b := x^6 - x^4 + bx^2$ having a double root at 0.

Proposition 2.12. *The polynomial* U_b *is hyperbolic if and only if* $b \in [0, 1/4]$.

Indeed, the roots of U_b/x^2 equal $\pm\sqrt{1 \pm \sqrt{1 - 4b}}$ and the condition all roots to be real is equivalent to the two inequalities $1 - 4b \geq 0$ and $1 - \sqrt{1 - 4b} \geq 0$, i.e. to $b \in [0, 1/4]$.

Theorem 2.13. *The CV of the polynomial* U_b *of Subfamily D2) depends in the following way on the parameter* b*:*

$b = 0,$ $\quad U$

$(0,$	$f,$	$s,$	$t,$	$F,$	$[0fst0fsl0f0],$	$F,$	$t,$	$s,$	$f,$	$0)$
x_1	f_1	s_1	t_1	F_1	$x_2\, f_2\, s_2\, t_2\, x_3\, f_3\, s_3\, l_1\, x_4\, f_4\, x_5$	F_2	t_3	s_4	f_5	x_6

for $b \in \left(0, \frac{14}{225}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & t, & F, & 0, & f, & s, & [0ftl0], & s, & f, & 0, & F, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & F_1 & x_2 & f_2 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & f_4 & x_5 & F_2 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $b = \frac{14}{225} = 0,062222\ldots, \qquad \Theta$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & [F0], & f, & s, & [0ftl0], & s, & f, & [0F], & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & F_1\, x_2 & f_2 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & f_4 & x_5\, F_2 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $b \in \left(\frac{14}{225}, \frac{3}{25}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & t, & 0, & F, & f, & s, & [0ftl0], & s, & f, & F, & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & F_1 & f_2 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & f_4 & F_2 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $b = \frac{3}{25} = 0,12000\ldots, \qquad \eta$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & 0, & [Ff], & s, & [0ftl0], & s, & [fF], & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & F_1\, f_2 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & f_4\, F_2 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

$b \in \left(\frac{3}{25}, \frac{4}{25}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & t, & 0, & f, & F, & s, & [0ftl0], & s, & F, & f, & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $b = \frac{4}{25} = 0,16000\ldots, \qquad \Gamma$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & [t0], & f, & F, & s, & [0ftl0], & s, & F, & f, & [0t], & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1\, x_2 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & x_5\, t_3 & s_4 & f_5 & x_6 \end{array}$$

for $b \in \left(\frac{4}{25}, \frac{45}{196}\right)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & 0, & t, & f, & F, & s, & [0ftl0], & s, & F, & f, & t, & 0, & s, & f, & 0) \\ x_1 & f_1 & s_1 & x_2 & t_1 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & t_3 & x_5 & s_4 & f_5 & x_6 \end{array}$$

for $b = \frac{45}{196} = 0,22958\ldots, \qquad \Lambda$

$$\begin{array}{ccccccccccccccc} (0, & f, & [s0], & t, & f, & F, & s, & [0ftl0], & s, & F, & f, & t, & [0s], & f, & 0) \\ x_1 & f_1 & s_1\, x_2 & t_1 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & t_3 & x_5\, s_4 & f_5 & x_6 \end{array}$$

for $b \in \left(\frac{45}{196}, \frac{1}{4}\right)$

$$\begin{array}{cccccccccccccccccc} (0, & f, & 0, & s, & t, & f, & F, & s, & [0ftl0], & s, & F, & f, & t, & s, & 0, & f, & 0) \\ x_1 & f_1 & x_2 & s_1 & t_1 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & t_3 & s_4 & x_5 & f_5 & x_6 \end{array}$$

for $b = \frac{1}{4} = 0,25000\ldots,$ $\quad \Pi$

$$\begin{array}{ccccccccccccc} ([0f0], & s, & t, & f, & F, & s, & [0ftl0], & s, & F, & f, & t, & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & t_1 & f_2 & F_1 & s_2 & x_3\, f_3\, t_2\, l_1\, x_4 & s_3 & F_2 & f_4 & t_3 & s_4 & x_5\, f_5\, x_6 \end{array}$$

Proof. 1^0. The arrangements for $b = 0$ and $b = 1/4$ are the ones for $r = 0$ and $r = 1/2$ of Subfamily D1).

2^0. The roots $t_{1,3} = \pm 1/\sqrt{5}$ of U_b''' are never roots of U_b'. Indeed, this would imply $b = 7/25 \notin [0, 1/4]$. The roots $F_{1,2} = \pm 1/\sqrt{15}$ of $U_b^{(4)}$ are roots of U_b' only for $b = 3/25$. For $b = 0$ (hence, for $b \in [0, 3/25)$) one has $F_1 < f_2$ and $F_2 > f_4$, for $b = 1/4$ (hence, for $b \in (3/25, 1/4]$) one has $F_1 > f_2$ and $F_2 < f_4$.

3^0. For $b \in [0, 1/4]$ there is a single value of b for which the roots x_2, x_5 of U_b are roots also of U_b'' (resp. U_b''', resp. of $U_b^{(4)}$), this value is 45/196 (resp. 4/25, resp. 14/225). This is to be checked directly. The theorem follows now from the arrangements for $b = 0$ and $b = 1/4$ and from the order of the numbers $0 < 14/225 < 3/25 < 4/25 < 45/196 < 1/4$. $\square$

3. Comments on old strata. One can obtain overdetermined strata in the family of degree 6 HPs by integrating once or several times HPs of lower degree. By abuse of language we say "to integrate a stratum" meaning "to integrate an HP defining this stratum".

There are two nontrivial overdetermined strata of degree 4. We list the corresponding arrangements and we give examples of HPs realizing them:

$$A \ : \quad (\ [0f0],\ s,\ [ft],\ s,\ [0f0]\ ) \qquad x^4 - x^2 + \frac{1}{4} = \left(x^2 - \frac{1}{2}\right)^2$$

$$B \ : \ (\ 0,\ f,\ [0s],\ [ft],\ [s0],\ f,\ 0\ ) \qquad x^4 - x^2 + \frac{5}{36}$$

(the stratum B is defined by the Gegenbauer polynomial of degree 4).

There are three nontrivial overdetermined strata of degree 5:

$$\Sigma \ : \ (\ 0,\ f,\ s,\ [0t],\ f,\ [0sF],\ f,\ [t0],\ s,\ f,\ 0\ ) \qquad x^5 - x^3 + \frac{9}{100}x$$

$$\Phi \ : \ (\ 0,\ f,\ [0s],\ t,\ f,\ [0sF],\ f,\ t,\ [s0],\ f,\ 0\ ) \qquad x^5 - x^3 + \frac{21}{100}x$$

$$F \ : \ (\ [0f0],\ s,\ [ft],\ [0sF],\ [tf],\ s,\ [0f0]\ ) \qquad x^5 - x^3 + \frac{1}{4}x$$

(the stratum Φ is defined by the Gegenbauer polynomial of degree 5).

A) When integrating (once, not twice) the stratum A one cannot obtain an HP because one has $(x^2 - 1/2)^2 \geq 0$ with equality only for $x = \pm 1/\sqrt{2}$.

B) When integrating (once) the stratum B, one obtains the stratum F. This follows from the arrangements defined by the two strata. The polynomial $F := x^5 - x^3 + x/4$ (realizing the stratum F) is only up to rescaling and multiplication by a non-zero constant a primitive of the polynomial $(x^2 - 1/2)^2$ (realizing the stratum B). All such "rescaled and normalized" primitives are of the form $F + c$, but only for $c = 0$ does one obtain an HP. Indeed, for $c = 0$ one has double zeros at $-1/\sqrt{2}$ (a local maximum) and at $1/\sqrt{2}$ (a local minimum); hence, choosing other values of c leads to the loss of real roots.

C) When integrating the stratum F one obtains the stratum E, see the description of Subfamily D1) in Theorem 2.11. The latter is defined by the polynomial $(x^2 - 1/3)^3$ having triple roots for $x = \pm 1/\sqrt{3}$. All other "rescaled and normalized" primitives (of the polynomial F) are of the form $S(M_2) + c$ and only for $c = 0$ does one obtain an HP. Hence, E is the only old stratum of degree 6 obtained by integrating twice the stratum B. It is also the only old stratum of degree 6 obtained by integrating once the stratum F.

D) When integrating the stratum Φ one obtains (depending on the constant of integration) one of the following strata realizable by the family of polynomials $x^6 - x^4 + \frac{7}{25}x^2 + \nu$. We set

$$\nu_0 = -\frac{3}{125},\ \nu_1 = -\frac{(45 + 6\sqrt{30})^3}{11390625} + \frac{(45 + 6\sqrt{30})^2}{50625} - \frac{7}{125} - \frac{14\sqrt{30}}{1875},\ \nu_2 = -\frac{49}{3375}.$$

for $\nu = \nu_0 = -0.02400$, $\quad V$

$$\begin{array}{ccccccccccccc} (0, & f, & s, & [0ft0], & F, & s, & [ftl], & s, & F, & [0tf0], & s, & f, & 0) \\ x_1 & f_1 & s_1 & x_2\, f_2\, t_1\, x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4\, t_3\, f_4\, x_5 & s_4 & f_5 & x_6 \end{array}$$

for $\nu \in (\nu_0, \nu_1)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & 0, & [ft], & 0, & F, & s, & [ftl], & s, & F, & 0, & [tf], & 0, & s, & f, & 0) \\ x_1 & f_1 & s_1 & x_2 & f_2\, t_1 & x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4 & f_4\, t_3 & x_5 & s_4 & f_5 & x_6 \end{array}$$

for $\nu = \nu_1 = -0.018582$, $\quad \zeta$

$$\begin{array}{ccccccccccccccc} (0, & f, & [0s], & [ft], & 0, & F, & s, & [ftl], & s, & F, & 0, & [tf], & [s0], & f, & 0) \\ x_1 & f_1 & x_2\, s_1 & f_2\, t_1 & x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4 & f_4\, t_3 & s_4\, x_5 & f_5 & x_6 \end{array}$$

for $\nu \in (\nu_1, \nu_2)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & 0, & s, & [ft], & 0, & F, & s, & [ftl], & s, & F, & 0, & [tf], & s, & 0, & f, & 0) \\ x_1 & f_1 & x_2 & s_1 & f_2\, t_1 & x_3 & F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2 & x_4 & f_4\, t_3 & s_4 & x_5 & f_5 & x_6 \end{array}$$

for $\nu = \nu_2 = -0.014518$, $\quad \Xi$

$$\begin{array}{ccccccccccc} ([0f0], & s, & [ft], & [0F], & s, & [ftl], & s, & [F0], & [tf], & s, & [0f0]) \\ x_1\, f_1\, x_2 & s_1 & f_2\, t_1 & x_3\, F_1 & s_2 & f_3\, t_2\, l_1 & s_3 & F_2\, x_4 & t_3\, f_4 & s_4 & x_5\, f_5\, x_6 \end{array}$$

E) When integrating the stratum Σ one obtains (depending on the constant of integration) one of the strata realizable by the family of polynomials $x^6 - x^4 + \frac{3}{25}x^2 + \lambda$. We set

$$\lambda_0 = -\frac{13}{3375},\ \lambda_1 = -\frac{(5-2\sqrt{5})^3}{15625} + \frac{(5-2\sqrt{5})^2}{625} - \frac{3}{125} + \frac{6\sqrt{5}}{625},\ \lambda_2 = 0.$$

for $\lambda = \lambda_0 = -0.003851$, $\quad G$

$$\begin{array}{ccccccccccccc} (0, & f, & s, & t, & [0fF0], & s, & [ftl], & s, & [0Ff0], & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2\, f_2\, F_1\, x_3 & s_2 & f_3\, t_2\, l_1 & s_3 & x_4\, F_2\, f_4\, x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $\lambda \in (\lambda_0, \lambda_1)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & t, & 0, & [fF], & 0, & s, & [ftl], & s, & 0, & [Ff], & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & f_2\, F_1 & x_3 & s_2 & f_3\, t_2\, l_1 & s_3 & x_4 & F_2\, f_4 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $\lambda = \lambda_1 = -0.002097$, $\quad \xi$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & 0, & [fF], & [0s], & [ftl], & [s0], & [Ff], & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & f_2\, F_1 & x_3\, s_2 & f_3\, t_2\, l_1 & s_3\, x_4 & F_2\, f_4 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $\lambda \in (\lambda_1, \lambda_2)$

$$\begin{array}{ccccccccccccccccc} (0, & f, & s, & t, & 0, & [fF], & s, & 0, & [ftl], & 0, & s, & [Ff], & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & f_2\,F_1 & s_2 & x_3 & f_3\,t_2\,l_1 & x_4 & s_3 & F_2\,f_4 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

for $\lambda = \lambda_2 = 0, \quad \eta$

$$\begin{array}{ccccccccccccccc} (0, & f, & s, & t, & 0, & [fF], & s, & [0ftl0], & s, & [Ff], & 0, & t, & s, & f, & 0) \\ x_1 & f_1 & s_1 & t_1 & x_2 & f_2\,F_1 & s_2 & x_3\,f_3\,t_2\,l_1\,x_4 & s_3 & F_2\,f_4 & x_5 & t_3 & s_4 & f_5 & x_6 \end{array}$$

F) One cannot obtain a degree 6 old stratum by integrating a degree 5 stratum with an MV $(1, 2, 2)$. Indeed, a multiple root of the derivative of an HP is a multiple root of the polynomial itself, therefore the MV of the corresponding degree 6 polynomial should be $(3, 3)$ which after derivation gives the MV $(2, 1, 2)$, not $(1, 2, 2)$. A similar reasoning allows to exclude the MVs $(2, 2, 1)$, $(2, 3)$ and $(3, 2)$. The MV $(2, 1, 2)$ gives by integration the MV $(3, 3)$ which (up to an affine change of the variable x) is defined by an even HP. Hence, its derivative must be an odd HP, i.e. only such degree 5 HPs give by integration old degree 6 overdetermined strata.

Remark 3.1. The situation with the strata B and F, see B), might in a sense seem contrary to intuition – one would expect the two critical values of the primitive of G_4 to be different and there to be an interval of values of the integration constant for which this primitive is an SHP. (This is the case of the strata Φ, see D), or of Σ, see E)). Or not to be any values for which this primitive is hyperbolic. However, overdetermined strata arise in situations when the roots of an HP and its derivatives satisfy certain algebraic equations, therefore intuitive reasoning applicable to a generic situation about real polynomials is not always applicable here. This is the reason why families of old strata like the ones described in D) and E) occur for the first time in degree 6 and not in degree 5.

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