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NEW UPPER BOUND FOR THE EDGE FOLKMAN NUMBER $F_e(3, 5; 13)$

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Abstract. For a given graph G let $V(G)$ and $E(G)$ denote the vertex and the edge set of G respevtively. The symbol $G \xrightarrow{e} (a_1, \dots, a_r)$ means that in every r-coloring of $E(G)$ there exists a monochromatic a_i-clique of color i for some $i \in \{1, ..., r\}$. The edge Folkman numbers are defined by the equality

$$F_e(a_1, \dots, a_r; q) = \min\{|V(G)| : G \xrightarrow{e} (a_1, \dots, a_r; q) \text{ and } \mathrm{cl}(G) < q\}.$$

In this paper we prove a new upper bound on the edge Folkman number $F_e(3, 5; 13)$, namely

$$F_e(3, 5; 13) \leq 21.$$

This improves the bound

$$F_e(3, 5; 13) \leq 24,$$

proved in [5].

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1. Introduction. Only finite non-oriented graphs without multiple edges and loops are considered. We call a p-clique of the graph G a set of p vertices each two of which are adjacent. The largest positive integer p such that G contains a p-clique is denoted by $\mathrm{cl}(G)$. A set of vertices of the graph G none two of which are adjacent is called an independent set. The largest positive integer p such that G contains an independent set on p vertices is called the independence number of the graph G and is denoted by $\alpha(G)$. In this paper we shall also use the following notations:

- $V(G)$ is the vertex set of the graph G;
- $E(G)$ is the edge set of the graph G;
- $N(v)$, $v \in V(G)$ is the set of all vertices of G adjacent to v;
- $G[V]$, $V \subseteq V(G)$ is the subgraph of G induced by V;
- K_n is the complete graph on n vertices;
- $\overline{G}$ is the complementary graph of G.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$ where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$. It is clear that

$$\mathrm{cl}(G_1 + G_2) = \mathrm{cl}(G_1) + \mathrm{cl}(G_2). \tag{1}$$

Definition 1. *Let $a_1, \dots, a_r$ be positive integers. The symbol $G \overset{v}{\rightarrow} (a_1, \dots, a_r)$ means that in every r-coloring of $V(G)$ there is a monochromatic a_i-clique in the i-th color for some $i \in \{1, \dots, r\}$.*

Definition 2. *Let $a_1, \dots, a_r$ be positive integers. We say that an r-coloring of $E(G)$ is $(a_1, \dots, a_r)$-free if for each $i = 1, \dots, r$ there is no monochromatic a_i-clique in the i-th color. The symbol $G \overset{e}{\rightarrow} (a_1, \dots, a_r)$ means that there is no $(a_1, \dots, a_r)$-free coloring of $E(G)$.*

The smallest positive integer n for which $K_n \overset{e}{\rightarrow} (a_1, \dots, a_r)$ is called a Ramsey number and is denoted by $R(a_1, \dots, a_r)$. Note that the Ramsey number $R(a_1, a_2)$ can be interpreted as the smallest positive integer n such that for every n-vertex graph G either $\mathrm{cl}(G) \geq a_1$ or $\alpha(G) \geq a_2$. The existence of such numbers was proved by Ramsey in [16]. We shall use only the values $R(3, 3) = 6$ and $R(3, 4) = 9$, [3].

The edge Folkman numbers are defined by the equality

$$F_e(a_1,\dots,a_r;q)=\min\{|V(G)|:G\overset{e}{\rightarrow}(a_1,\dots,a_r;q)\text{ and }\operatorname{cl}(G)<q\}.$$

It is clear that $G\overset{e}{\rightarrow}(a_1,\dots,a_r)$ implies $\operatorname{cl}(G)\geq\max\{a_1,\dots,a_r\}$. There exists a graph G such that $G\overset{e}{\rightarrow}(a_1,\dots,a_r)$ and $\operatorname{cl}(G)=\max\{a_1,\dots,a_r\}$. In the case $r=2$ this was proved in [1] and in the general case in [14]. Therefore

$$F_e(a_1,\dots,a_r;q)\text{ exists if and only if }q>\max\{a_1,\dots,a_r\}.$$

It follows from the definition of $R(a_1,\dots,a_r)$ that

$$F_e(a_1,\dots,a_r;q)=R(a_1,\dots,a_r)\text{ if }q>R\ (a_1,\dots,a_r).$$

The smaller the value of q in comparison to $R(a_1,\dots,a_r)$ the more difficult the problem of computing the number $F_e(a_1,\dots,a_r;q)$.

Among the edge Folkman numbers of the kind $F_e(a_1,\dots,a_r;R(a_1,\dots,a_r))$ only the following ones are known:

$F_e(3,3;6)=8$, [2];
$F_e(3,4;9)=14$, [11];
$F_e(3,5;14)=16$, [4];
$F_e(4,4;18)=20$, [4];
$F_e(3,3,3;17)=19$ [4].

Only three edge Folkman numbers of the kind $F_e(a_1,\dots,a_r;R(a_1,\dots,a_r)-1)$ are known, namely $F_e(3,4;8)=16$, $F_e(3,3;5)=15$ and $F_e(3,3,3;16)=21$. The number $F_e(3,4;8)=16$, was computed in the papers [6], [5]. The inequality $F_e(3,3;5)\leq 15$ was proved in [12] and the inequality $F_e(3,3;5)\geq 15$ was obtained by the means of computer in [15]. The inequality $F_e(3,3,3;16)\geq 21$ was proved in [4] and the opposite inequality $F_e(3,3,3;16)\leq 21$ in [8]. At the end of this exposition we shall mention that we know only one edge Folkman number of the kind $F_e(a_1,\dots,a_r;R(a_1,\dots,a_r)-2)$, namely $F_e(3,3,3;15)=23$, [9] and only one edge Folkman number of the kind $F_e(a_1,\dots,a_r;R(a_1,\dots,a_r)-3)$, namely $F_e(3,3,3;14)=25$, [10]. No other edge Folkman numbers are known.

This paper is dedicated to the Folkman number $F_e(3,5;13)$.

The best known lower bound on this number is $F_e(3,5;13)\geq 18$, which was proved by Lin in [4]. Later Nenov proved in [13] that equality $F_e(3,5;13)=18$ can be achieved only for the graph $K_8+C_5+C_5$. Thus if $K_8+C_5+C_5\overset{e}{\rightarrow}(3,5)$ then $F_e(3,5;13)=18$ and otherwise $F_e(3,5;13)>18$. So far nobody was able to check whether $K_8+C_5+C_5 \xrightarrow{e} (3,5)$. The best known upper bound is $F_e(3,5;13) \leq 24$, [5].

We consider the graph Q, which was first introduced in [3] and whose complementary graph is given in the Figure 1.

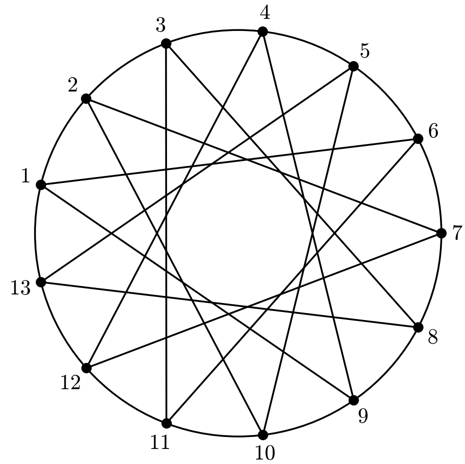

Fig. 1. Graph $\overline{Q}$

We shall use the following properties of the graph Q:

(2) $$\mathrm{cl}(Q) = 4, \ [3];$$

(3) $$\alpha(Q) = 2, \ [3];$$

(4) $$Q \xrightarrow{v} (3,4), \ [7].$$

The goal of this paper is to prove the following

Theorem. *Let* $G = K_8 + Q$. *Then* $G \xrightarrow{e} (3,5)$.

It follows from (1) and (2) that $\mathrm{cl}(G) = 12$. Since $|V(G)| = 21$ we obtain from the Theorem the following

Corollary. $F_e(3,5;13) \leq 21$.

2. Proof of the theorem. Assume the opposite: that there exists a $(3,5)$-free 2-coloring of $E(G)$

(5) $$E(G) = E_1 \cup E_2, \qquad E_1 \cap E_2 = \emptyset.$$

We shall call the edges in E_1 blue and the edges in E_2 red.

We define for an arbitrary vertex $v \in V(G)$ and index $i = 1, 2$:

$$N_i(v) = \{x \in N(v) \mid [v, x] \in E_i\},$$
$$G_i(v) = G[N_i(v)]$$
$$A_i(v) = N_i(v) \cap V(Q)$$

Let H be a subgraph of G. We say that H is a monochromatic subgraph in the blue-red coloring (5) if $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$. If $E(H) \subseteq E_1$ we say that H is a blue subgraph and if $E(H) \subseteq E_2$ we say that H is a red subgraph.

It follows from the assumption that the coloring (5) is (3,5)-free that

(6) $$\mathrm{cl}(G_1(v)) \leq 4 \text{ and } \mathrm{cl}(G_2(v)) \leq 8 \text{ for each } v \in V(G)$$

Indeed, assume that $\mathrm{cl}(G_1(v)) \geq 5$. Then there must be no blue edge connecting any two of the vertices in $\mathrm{cl}(G_1(v))$ because otherwise this blue edge together with the vertex v would give a blue triangle. As we assumed $\mathrm{cl}(G_1(v)) \geq 5$ then we have a red 5-clique. Analogously assume $\mathrm{cl}(G_2(v)) \geq 9$. Since $R(3, 4) = 9$, then we have either a blue 3-clique or a red 4-clique in $G_2(v)$. If we have a blue 3-clique in $G_2(v)$ then we are through. If we have a red 4-clique then this 4-clique together with the vertex v gives a red 5-clique. Thus (6) is proved.

We shall prove that

(7) $$\mathrm{cl}(G[A_1(v)]) + \mathrm{cl}(G[A_2(v)]) \leq 5 \text{ for each } v \in V(K_8)$$

Assume that (7) is not true, i.e. that there exists a vertex $v \in V(K_8)$ such that

$$\mathrm{cl}(G[A_1(v)]) + \mathrm{cl}(G[A_2(v)]) \geq 6.$$

Then as there are seven more vertices in $V(K_8)$ with the exception of v, it follows that

$$\mathrm{cl}(G_1(v)) + \mathrm{cl}(G_2(v)) \geq 13.$$

It follows from the pigeonhole principle that either $\mathrm{cl}(G_1(v)) \geq 5$ or $\mathrm{cl}(G_2(v)) \geq 9$, which contradicts (6). Thus (7) is proved.

Now we shall prove that

(8) $$\mathrm{cl}(G[A_1(v)]) = 4 \text{ or } \mathrm{cl}(G[A_2(v)]) = 4 \text{ for each } v \in V(K_8)$$

By (2) we have

(9) $$\mathrm{cl}(G[A_i(v)]) \leq 4,\ \ i = 1, 2.$$

Assume that (8) is not true. Then we obtain from (9) that

(10) $\mathrm{cl}(G[A_1(v)]) \leq 3$ and $\mathrm{cl}(G[A_2(v)]) \leq 3$ for some $v \in V(K_8)$.

It follows from (4) that in every 2-coloring of $V(Q)$, in which there are no 4-cliques in none of the two colors then there are 3-cliques in the both colors. Therefore the inequalities in (10) are in fact equalities, which contradicts (7). Thus (8) is proved.

We shall prove that there are at least 7 vertices $v \in V(K_8)$ such that

$$\mathrm{cl}(G[A_2(v)]) = 4.$$

Assume the opposite. Then it follows from (8) that there are at least 2 vertices v_1, v_2 in $V(K_8)$ such that $\mathrm{cl}(G[A_1(v_1)]) = \mathrm{cl}(G[A_1(v_2)]) = 4$. Now we conclude from (6) that all edges from v_1, v_2 to all vertices in $V(K_8)$ (including the edge $[v_1, v_2]$) are red. Since $R(3,3) = 6$ there is a monochromatic 3-clique in the other 6 vertices in $V(K_8)$ excluding v_1, v_2. If this monochromatic 3-clique is blue then we are through. If it is red then this monochromatic 3-clique together with the edge $[v_1, v_2]$ forms a red 5-clique which is a contradiction. Thus we proved that there are at least 7 vertices $v \in V(K_8)$ such that $\mathrm{cl}(G[A_2(v)]) = 4$.

We obtain from $R(3,3) = 6$ that there is a monochromatic 3-clique among these 7 vertices sufficing $\mathrm{cl}(G[A_2(v)]) = 4$. This 3-clique is red (otherwise we are through). Let us denote its vertices by a_1, a_2, a_3. It follows from (7) that

$$\mathrm{cl}(G[A_1(a_i)]) \leq 1, \ i = 1, 2, 3.$$

Now we have from (3) that $|A_1(a_i)| \leq 2$. Then there are at least 7 vertices in $V(Q)$ from which there are only red edges to a_1, a_2, a_3. As $R(3,3) = 6$ and $\alpha(Q) = 2$ it follows that there is a 3-clique among these 7 vertices. If this 3-clique is monochromatic blue then we are through. Therefore it is not monochromatic blue and hence there is a red edge in it. This red edge together with a_1, a_2, a_3 gives a monochromatic red 5-clique.

The Theorem is proved. □

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