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BAYESIAN PREDICTION OF WEIBULL DISTRIBUTION BASED ON FIXED AND RANDOM SAMPLE SIZE

A. H. Abd Ellah

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Abstract. We consider the problem of predictive interval for future observations from Weibull distribution. We consider two cases they are: (i) fixed sample size (FSS), (ii) random sample size (RSS). Further, we derive the predictive function for both FSS and RSS in closed forms. Next, the upper and lower 1%, 2.5%, 5% and 10% critical points for the predictive functions are calculated. To show the usefulness of our results, we present some simulation examples. Finally, we apply our results to some real data set in life testing given in Lawless [16].

1. Introduction. In many applications, the sample size could be random rather than fixed, Lingappaiah [19] has discussed the prediction problem for future samples based on FSS and RSS. Also, he has obtained the Bayesian predictive distribution of the range when the parent distribution is a one parameter Weibull. Upadhyay and Pandey [26] have provided predicted intervals for future

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observations from one parameter Weibull distribution in the case of FSS. Problems for constructing prediction limits have also been considered by Giesser [11], Soliman and Abd-Ellah [25], Abd Ellah [1], Abd Ellah and Sultan [2], Soliman [23, 24], Balakrishnan and Basu [5], Dunsmore [10], Aitchison and Dunsmore [4], Hahan and Meeker [15], Lawless [17], Nelson [21], Patel [22], Arnold, Balakrishnan and Nagaraja [3], Geisser [12] and Nagaraja [20]. Balakrishnan and Lin [7] have developed exact prediction intervals for failure times from one-parameter and two-parameter exponential distributions based on doubly Type-II censored samples. Balakrishnan and Lin [6] have developed exact prediction intervals for failure times of the items censored at the last observation from one-parameter and two-parameter exponential distributions based general progressively Type-II censored samples.

In this paper, we consider the Bayesian prediction for future observations from Weibull distribution using both FSS and RSS. In section 2, we derive the exact Bayesian predictive function for the FSS. The Bayesian predictive function for the RSS is considered in section 3. Some applications of the findings of the papers are given in section 4. Finally, some concluding remarks are added in section 5.

Let $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{r:n}$, be the first r ordered lifetimes fail in a sample of n $(r \leq n)$ components from Weibull *pdf*

$$f(x|\lambda) = c\lambda x^{c-1} e^{-\lambda x^c}, \qquad \lambda, x > 0, \tag{1.1}$$

Let $x_{r+1:n} \leq x_{r+2:n} \leq \cdots \leq x_{n:n}$, be the remaining $(n-r)$ lifetimes from the same distribution.

The joint density function of $X_{r:n}$ and $X_{s:n}$, $r < s$ is

$$f_{r,s:n}(x,y) = C_{r,s:n} F(x)^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y), \tag{1.2}$$

where $f(\cdot)$ and $F(\cdot)$ are respectively, the *pdf* and *cdf* of the Weibull population given in (1.1) and

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}. \tag{1.3}$$

For more details, see David [8], Arnold, Balakrishnan and Nagaraja[3] and David and Nagaraja [9].

Lawless [16] and Lingappaiah [18] have used two different statistics to predict the future observation $x_{s:n}$ based on $x_{1:n}, x_{2:n}, \ldots x_{r:n}$ when the sample size n is fixed. Their classical approach was based on two statistics given below:

$$\text{Statistic-1:} \quad W = \frac{X_{s:n}^c - X_{r:n}^c}{S_r}, \text{(see [16])}, \tag{1.4}$$

$$\text{Statistic-2:} \quad U = \frac{X_{s:n}^c - X_{r:n}^c}{X_{r:n}^c}, \text{ (see [18])}, \tag{1.5}$$

where

$$S_r = \sum_{i=1}^{r} x_{i:n}^c + (n-r)x_{r:n}^c. \tag{1.6}$$

The distributions of Statistic-1 and Statistic-2 given in (1.4) and (1.5), require the *pdf* of $Z^c = X_{s:n}^c - X_{r:n}^c$, that can be obtained as

$$f(z|\lambda) = \frac{c\lambda z^{c-1}(e^{-\lambda z^c})^{n-s+1}(1-e^{-\lambda z^c})^{s-r-1}}{\beta(s-r, n-s+1)}. \tag{1.7}$$

Our goal is to predict the future order statistics based on the observed sample, $x_{1:n}, x_{2:n}, \ldots, x_{r:n}$ using both FSS and RSS cases. In the case of FSS, the Bayes predictive density function of $y = {x_{s:n}}^c$ for given $x = (x_{1:n}, x_{2:n}, \ldots, x_{r:n})$ can be written as

$$h(y|x) = \int_0^\infty f(y|\lambda)\pi(\lambda|x)\, d\lambda, \tag{1.8}$$

where $f(y|x)$ is the conditional *pdf* of the future observation and $\pi(\lambda|x)$ is the posterior *pdf*.

In the case of RSS, the predictive distribution function of y when the sample size n is random variable is given by (see [14])

$$q(y|x,n) = \frac{1}{Pr(n \geq s)} \sum_{n=s}^{\infty} r(n)h(y|x), \tag{1.9}$$

where $r(n)$ is the pmf of n and $h(y|x)$ is given in (1.8).

2. Prediction for fixed sample size. In this section, we derive the predictive distribution function (1.8) based on the two different statistics given in (1.4) and (1.5). The posterior *pdf* $\pi(\lambda|x)$ in (1.8) requires different priors. In the case of Statistic-1 gamma *pdf* will be the suitable while the *pdf* of the r-th order statistic will be suitable prior for Statistic-2 (see [13] and [26]). Also, obtaining the posterior requires the likelihood function which is

$$L(x|\lambda) = \frac{n!}{(n-r)!} c^r \lambda^r \prod_{i=1}^{r} x_i^{c-1} e^{-\lambda S_r} \tag{2.1}$$

where S_r is given in (1.6).

2.1. Prediction based on Statistic-1. Under Statistic-1, we assume the 2-parameter gamma *pdf* as a prior for the Weibull parameter λ as

$$g(\lambda|a,b) = \frac{1}{\Gamma(a) b^a} \lambda^{a-1} e^{-\lambda/b}, \quad a, b > 0, \tag{2.2}$$

where a and b are known. Combining (2.2) with the likelihood function in (2.1), then the posterior *pdf* of λ is obtained as

$$\pi(\lambda|x) = \frac{1}{\Gamma(R)} \lambda^{R-1} A^R e^{-\lambda A}, \tag{2.3}$$

and

$$R = r + a \text{ and } A = S_r + 1/b, \tag{2.4}$$

and S_r is given in (1.6).

By using (1.7) and (2.3) in (1.8), the Bayesian predictive *pdf* of W given in (1.4) is obtained as (see Appendix A)

$$h_1(w|x) = \frac{R c w^{c-1} A^R}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} \frac{\binom{s-r-1}{i}(-1)^i}{(A + (n-s+i+1)w^c)^{R+1}}, \tag{2.5}$$

while the Bayesian predictive *cdf* is given by

$$\begin{aligned} H_1(t) &= Pr(W \le t|x) \\ &= 1 - \frac{A^R}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} \frac{\binom{s-r-1}{i}(-1)^i (n-s+i+1)^{-1}}{(A + (n-s+i+1)t^c)^R}, \end{aligned} \tag{2.6}$$

where A and R are given in (2.4). The percentage points of the predictive *cdf* given in (2.6) can be easily obtained by solving the following nonlinear equation

$$H_1(t) = 1 - \alpha. \tag{2.7}$$

Then the exact two sided $(1-\alpha)100\%$ Bayesian interval for the future observation $x_{s:n}$ can be constructed as

$$\left(t_{\alpha/2} + x_{r:n}, t_{1-\alpha/2} + x_{r:n}\right), \tag{2.8}$$

where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ are the lower and upper percentage points of $H_1(t)$.

Example 1. In this example, we generate 5 order statistics from the Weibull *pdf* given in (1.1) as: 0.017, 0.363, 0.365, 0.438 and 0.456. By using these data, we construct 90% and 95% confidence interval for the observation $x_{s:n}$, $s = 6, 7, 8, 9, 10$ through (2.8) as given below

r	s	90% *P.I*	95% *P.I*
5	6	(0.4591, 0.7246)	(0.4574, 0.8131)
5	7	(0.4803, 0.9697)	(0.4723, 1.1166)
5	8	(0.5180, 1.2877)	(0.5019, 1.5106)
5	9	(0.5765, 1.7695)	(0.5494, 2.1117)
5	10	(0.6815, 2.7888)	(0.6347, 3.4048)

In the calculation, the values of a and b in the gamma distribution are chosen randomly positive.

2.2. Prediction based on Statistic-2. Under Statistic-2, we suggest the following prior *pdf* for the Weibull parameter λ as

$$g(\lambda) = \frac{n!}{(r-1)!(n-r)!} c x^{c-1} \lambda (1 - e^{-\lambda x^c})^{r-1} e^{-(n-r+1)\lambda x^c}. \tag{2.9}$$

From (2.1) and (2.9), the posterior *pdf* of λ is obtained as

$$\pi(\lambda|x) = \frac{1}{K\Gamma(r+2)} \lambda^{r+1} (1 - e^{-\lambda x^c})^{r-1} e^{-\lambda(S_r + (n-r+1)x^c)}, \tag{2.10}$$

where

$$K = \sum_{\ell=0}^{r-1} \frac{\binom{r-1}{\ell}(-1)^{\ell}}{(S_r + (n-r+\ell+1)x^c)^{r+2}}. \tag{2.11}$$

The Bayesian predictive density function of Statistic-2 can be derived upon using (1.7) and (2.10) in (1.8) as

$$\begin{aligned} h_2(u|x) &= \frac{(r+2)cu^{c-1}}{K\beta(s-r,n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \binom{r-1}{i}\binom{s-r-1}{j} \\ &\quad \times \frac{(-1)^{j+i}}{[S_r + (n-r+j+1)x^c + (n-s+i+1)u^c]^{r+3}}, \end{aligned} \tag{2.12}$$

with *cdf* as

$$\begin{aligned} H_2(t) &= Pr(U \le t|x) \\ &= \frac{1}{K\beta(s-r,n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \frac{\binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i}}{(n-s+i+1)} \\ &\quad \times \left[\frac{1}{(S_r + (n-r+j+1)x^c + (n-s+i+1)t^c)^{r+2}} \right. \\ &\quad \left. - \frac{1}{(S_r + (n-r+j+1)x^c)^{r+2}}\right], \end{aligned} \tag{2.13}$$

where S_r and K are given in (1.6) and (2.11), respectively.

The percentage points of the predictive *cdf* given in (2.6) can be easily obtained by solving the following nonlinear equation

$$H_2(t) = 1 - \alpha. \tag{2.14}$$

Then the exact two sided $(1-\alpha)100\%$ Bayesian predictive interval for the future observation $x_{s:n}$ is given in (2.8), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $H_2(t)$.

Example 2. In this example, we use the same sample as in Example 1, for constructing 90% and 95% confidence interval for the observation $x_{s:n}$,

$s = 6, 7, 8, 9, 10$ through (2.8), (2.13) and (2.14) as given below

r	s	90% $P.I$	95% $P.I$
5	6	(0.4608, 0.7781)	(0.4584, 0.8659)
5	7	(0.4920, 1.0507)	(0.4804, 1.1839)
5	8	(0.5490, 1.3997)	(0.5256, 1.5914)
5	9	(0.6388, 1.9312)	(0.5996, 2.2189)
5	10	(0.8002, 3.0781)	(0.7328, 3.6004)

3. Prediction for random sample size.

3.1. Sample size has Poisson distribution. In this section, we assume that the sample size n has a Poisson distribution with pmf as

$$p(n;\theta) = \frac{e^{-\theta}\theta^n}{n!}, \qquad n = 0, 1, 2, \ldots, \theta > 0. \tag{3.1}$$

Replacing $r(n)$ in (1.9) by (3.1), we get

$$q(y|x,n) = \frac{1}{1-P(s-1)} \sum_{n=s}^{\infty} \frac{e^{-\theta}\theta^n}{n!} h(y|x), \tag{3.2}$$

where $P(\cdot)$ is the *cdf* of Poisson distribution.

In the following two subsections, we use (3.2) to derive the Bayesian predictive *pdf* in the case of RSS based on both Statistic-1 and Statistic-2.

3.1.1. Prediction based on Statistic-1. Under Statistic-1 and replacing $h(y|x)$ given in (3.2) by $h_1(w|x)$ given in (2.5), then the Bayes predictive *pdf* of W when the sample size distributed as $p(n;\theta)$ is

$$\begin{aligned} (3.3)\quad & q_1(w|x,n) \\ &= \frac{RA^R cw^{c-1}e^{-\theta}}{1-P(s-1)} \sum_{n=s}^{\infty} \sum_{i=0}^{s-r-1} \frac{\theta^n \binom{s-r-1}{i}(-1)^i \left[A + (n-s+i+1)w^c\right]^{-R-1}}{n!\beta(s-r, n-s+1)}, \end{aligned}$$

with *cdf* as

$$
\begin{aligned}
Q_1(t) &= Pr(W \le t|x) \\
&= 1 - \frac{1}{1-P(s-1)} \sum_{n=s}^{\infty} \sum_{i=0}^{s-r-1} \frac{A^R e^{-\theta} \theta^n \binom{s-r-1}{i} (-1)^i}{n! \beta(s-r, n-s+1)(n-s+i+1)} \\
&\quad \times \left[\frac{1}{(A+(n-s+i+1)t^c)^R} \right], \qquad (3.4)
\end{aligned}
$$

where A and R are given in (2.4).

The percentage points of the predictive *cdf* given in (3.4) can be easily obtained by solving the following nonlinear equation

$$
Q_1(t) = 1 - \alpha. \qquad (3.5)
$$

Then the exact two sided $(1-\alpha)100\%$ Bayesian interval for the future observation $x_{s:n}$ is given in (2.8), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $Q_1(t)$.

Example 3. In this example we generate the sample size n from $p(n;2)$ to be 7. Then, we generate the first 6 order statistics based on $n=7$ from the standard Weibull *pdf* they are: .153, .417, .433, .720, .876 and .926. Next, this sample is used to predict the 90% and 95% confidence intervals for the future observations up to 10 as give below:

r	s	90% *P.I*	95% *P.I*
6	7	$(0.9520, 3.1091)$	$(0.9390, 3.7941)$
6	8	$(1.0538, 3.7831)$	$(1.0118, 4.5553)$
6	9	$(1.1603, 4.2142)$	$(1.0985, 5.0401)$
6	10	$(1.2547, 4.5361)$	$(1.1796, 5.4022)$

3.1.2. Prediction based on Statistic-2. Under Statistic-2 and replacing $h(y|x)$ given in (3.2) by $h_2(u|x)$ given in (2.12), then the Bayes predictive *pdf* of U when the sample size distributed as $p(n;\theta)$ is

$$q_2(u|x,n) = \frac{(r+2)cu^{c-1}}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\frac{e^{-\theta}\theta^n\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{Kn!\beta(s-r,n-s+1)}$$

$$\times\frac{1}{(S_r+(n-r+i+1)x^c+(n-s+j+1)u^c)^{r+3}}, \tag{3.6}$$

with *cdf* as

$$\begin{aligned} Q_2(t) &= Pr(U \le t|x) \\ &= \frac{1}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\frac{e^{-\theta}\theta^n\binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i}}{Kn!(n-s+i+1)} \\ &\quad\times\left[\frac{1}{(S_r+(n-r+i+1)x^c+(n-s+j+1)t^c)^{r+2}}\right. \\ &\quad\left.-\frac{1}{(S_r+(n-r+i+1)x^c)^{r+2}}\right], \end{aligned} \tag{3.7}$$

where S_r and K are given in (1.6) and (2.11), respectively.

The percentage points of the predictive *cdf* given in (3.7) can be easily obtained by solving the following nonlinear equation

$$Q_2(t) = 1-\alpha. \tag{3.8}$$

Then the exact two sided $(1-\alpha)100\%$ Bayesian interval for the future observation $x_{s:n}$ is given in (2.8), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $Q_2(t)$.

Example 4. By using the same data in Example 3, we predict the 90% and 95% confidence intervals for the future observations up to 10 based on Statistic-2 when the sample size is $p(n;2)$ as give below:

r	s	90% $P.I$	95% $P.I$
6	7	$(0.9494, 2.6086)$	$(0.9379, 3.0815)$
6	8	$(1.0591, 3.3841)$	$(1.0157, 3.9823)$
6	9	$(1.1972, 4.0648)$	$(1.1264, 4.7787)$
6	10	$(1.3485, 4.6963)$	$(1.2596, 5.4864)$

3.2. Sample size has Binomial distribution. In this section, we assume that the sample size n is distributed as binomial pmf, $b(n; M, p)$ as

$$b(n; M, p) = \binom{M}{n} p^n q^{M-n}, \qquad q = 1 - p, \quad n = 0, 1, 2 \ldots, M. \tag{3.9}$$

Replacing $r(n)$ in (1.9) by (3.9), we get

$$v(y|x, n) = \frac{1}{1 - B(s-1)} \sum_{n=s}^{M} \binom{M}{n} p^n q^{M-n} h(y|x), \tag{3.10}$$

where $B(\cdot)$ is the *cdf* of binomial distribution. In the following two subsections, we use (3.10) to derive the Bayesian predictive *pdf* in the case of RSS based on both Statistic-1 and Statistic-2.

3.2.1. Prediction based on Statistic-1. Under Statistic-1 and replacing $h(y|x)$ given in (3.10) by $h_1(w|x)$ given in (2.5), then the Bayes predictive *pdf* of W when the sample size distributed as $b(n; M, p)$ is

$$\begin{aligned} &v_1(w|x, n) \\ &= \frac{Rcw^{c-1}A^R}{1 - B(s-1)} \sum_{n=s}^{M} \sum_{j=0}^{s-r-1} \frac{(-1)^i \binom{M}{n} \binom{s-r-1}{j} \left[A + (n-s+j+1)w^c\right]^{-R-1}}{p^{-n} q^{n-M} \beta(s-r, n-s+1)}, \end{aligned} \tag{3.11}$$

with *cdf* as

$$\begin{aligned} V_1(t) &= Pr(W \le t) \\ &= 1 - \frac{1}{1 - B(s-1)} \sum_{n=s}^{M} \sum_{j=0}^{s-r-1} \frac{A^R \binom{M}{n} p^n q^{M-n} \binom{s-r-1}{j} (-1)^j}{(n-s+j+1)\beta(s-r, n-s+1)} \\ &\quad \times \frac{1}{(A + (n-s+j+1)t^c)^R}, \end{aligned} \tag{3.12}$$

where A, R are given in (2.4).

Example 5. In this example, we generate the sample size n from $b(n; 15, 0.3)$, then based on the generated n, we generate the first 6 order statistics form the standard Weibull as: .177, .195, .493, 1.262, 1.376 and 1.444. By using this data we predict the 90% and 95% confidence intervals for the future observations up to 10 based on Statistic-1 when the sample size is $b(n; 15; 0.3)$ as give below:

r	s	90% $P.I$	95% $P.I$
6	7	$(1.4724, 4.0975)$	$(1.4581, 5.0142)$
6	8	$(1.5975, 5.0538)$	$(1.5473, 6.0555)$
6	9	$(1.7426, 5.6908)$	$(1.6646, 6.7684)$
6	10	$(1.8745, 6.1762)$	$(1.7738, 7.3189)$

3.2.2. Prediction based on Statistic-2. Again under Statistic-2 and replacing $h(y|x)$ given in (3.10) by $h_2(u|x)$ given in (2.12), then the Bayes predictive *pdf* of U when the sample size distributed as $b(n; M, p)$ is

$$
\begin{aligned}
v_2(u|x,n) &= \frac{(r+2)cu^c}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{B\beta(s-r, n-s+1)} \\
&\quad \times \frac{1}{S_r + (n-r+i+1)x^c + (n-s+j+1)u^c)^{r+3}},
\end{aligned} \tag{3.13}
$$

with *cdf* as

$$
\begin{aligned}
V_2(t) &= Pr(U \leq t|x) \\
&= \frac{1}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n} p^n q^{M-n} \binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i}}{K\beta(s-r, n-s+1)(n-s+i+1)} \\
&\quad \times \left[\frac{1}{(S_r + (n-r+i+1)x^c + (n-s+j+1)t^c)^{r+2}} \right. \\
&\quad \left. - \frac{1}{(S_r + (n-r+i+1)x^c)^{r+2}} \right],
\end{aligned}
$$

where S_r and K are given in (1.6) and (2.11), respectively.

Example 6. By using the same data in Example 5, we predict the 90% and 95% confidence intervals for the future observations up to 10 based on Statistic-2 when the sample size is $b(n; 15, 0.3)$ as give below:

r	s	90% $P.I$	95% $P.I$
6	7	$(1.4742, 3.7642)$	$(1.4589, 4.4340)$
6	8	$(1.6288, 4.9745)$	$(1.5701, 5.8308)$
6	9	$(1.8394, 6.0728)$	$(1.7394, 7.1167)$
6	10	$(2.0763, 7.0983)$	$(1.9372, 8.2842)$

4. Application. In this section, we apply our technique to some real data that follow Weibull distribution as presented in [16] in which the test is terminated after 4 failures, they are 30, 90, 120 and 170 hours. By using these four times, we calculate the percentage points for the predictive functions presented in sections 2 and 3 based on FSS and RSS up to 10 as given below:

Table 4.1. Percentage Points Based on Statistic-1 when n is Fixed

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.2	2.5	5.1	139.4	195.8	260.0	360.3
4	6	7.2	11.8	17.5	26.7	273.6	365.3	469.9	630.4
4	7	22.1	31.9	43.0	59.5	435.4	570.2	722.7	957.9
4	8	45.5	61.7	79.4	105.0	650.3	843.2	1061.1	1396.1
4	9	80.8	106.0	133.0	171.6	976.6	1261.6	1582.1	2077.8
4	10	141.6	183.0	227.2	290.6	1655.0	2146.4	2703.3	3565.9

Table 4.2. Percentage Points Based on Statistic-2 when n is Fixed

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.3	2.7	5.6	138.6	187.8	241.0	317.9
4	6	8.2	13.5	19.9	30.0	266.0	341.0	420.8	534.4
4	7	25.9	37.1	49.6	67.9	417.8	523.4	634.9	793.4
4	8	54.0	72.7	92.6	120.8	618.8	766.0	921.0	1141.0
4	9	97.0	125.9	156.1	198.5	925.1	1139.6	1365.6	1687.0
4	10	171.4	218.3	267.5	336.4	1569.7	1946.4	2346.8	2919.6

Table 4.3. Percentage Points Based on Statistic-1 when $n \sim p(n; 2)$

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	2.0	5.1	10.4	21.7	707.7	1014.8	1370.0	1926.7
4	6	21.0	34.9	52.4	81.3	1002.3	1367.5	1786.0	2436.0
4	7	48.4	70.9	97.0	137.1	1196.8	1599.3	2058.6	2770.3
4	8	75.8	104.6	136.8	184.8	1343.7	1773.5	2263.3	3022.2
4	9	101.2	134.7	171.5	225.5	1461.7	1915.4	2430.6	3224.2
4	10	124.1	161.6	201.9	260.5	1560.6	2033.2	2569.4	3397.1

Table 4.4. Percentage Points Based on Statistic-2 when $n \sim p(n; 2)$

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.6	4.2	8.5	17.6	501.5	690.6	899.2	1223.8
4	6	19.2	31.8	47.2	72.6	753.2	982.9	1233.7	1589.4
4	7	48.0	69.7	95.6	132.3	953.9	1226.1	1498.9	1974.0
4	8	81.0	113.1	144.8	192.0	1141.6	1446.4	1767.7	2255.2
4	9	118.3	153.4	194.3	251.3	1318.3	1646.4	2016.7	2555.4
4	10	152.4	198.2	244.2	308.0	1490.6	1846.5	2241.9	2815.6

Table 4.5. Percentage Points Based on Statistic-1 when $n \sim b(n; 15, 0.3)$

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.4	3.7	7.5	15.6	569.3	844.9	1148.2	1602.3
4	6	16.9	28.3	42.3	66.3	887.3	1229.1	1613.0	2235.4
4	7	42.3	62.4	86.5	121.5	1115.2	1506.7	1930.8	2548.4
4	8	70.3	98.2	128.0	172.7	1288.2	1712.5	2163.9	2901.1
4	9	98.5	132.7	167.1	219.8	1434.4	1886.4	2395.7	3281.3
4	10	124.6	160.5	202.3	260.8	1556.2	2044.7	2590.3	3487.2

Table 4.6. Percentage Points Based on Statistic-2 when $n \sim b(n; 15, 0.3)$

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.2	3.2	6.5	13.5	411.2	582.4	776.2	1056.9
4	6	16.3	27.0	40.0	61.7	674.6	895.4	1139.4	1492.5
4	7	43.6	63.3	86.8	120.3	895.9	1159.1	1439.1	1831.7
4	8	76.9	107.3	137.4	182.4	1103.4	1402.5	1720.6	2193.2
4	9	115.8	150.2	190.2	246.1	1299.0	1629.9	1965.5	2506.7
4	10	155.1	198.5	245.4	308.1	1487.4	1854.9	2254.8	2822.2

From the above Tables 4.1–4.6, we find the 90% and 95% predictive confidence intervals for the fifth failure FSS and RSS when $n \sim p(n;2)$ and $n \sim b(n;15,0.3)$ through (2.8) as given below

$1-\alpha$	Sample size	Statistic-1	Statistic-2
90%	Fixed ($n=10$)	(172.5 , 365.8)	(172.7 , 357.8)
	$p(n;2)$	(180.4 , 1184.8)	(178.5 , 860.6)
	$b(n;15,0.3)$	(177.5 , 1115)	(176.5 , 752.4)
95%	Fixed ($n=10$)	(171.2 , 430.0)	(171.3 , 411.0)
	$p(n;2)$	(175.1 , 1540.0)	(174.2 , 1069.2)
	$b(n;15,0.3)$	(173.7 , 1318.2)	(173.2 , 946.2)

From the above table we see that the width of the Bayesian predictive intervals based on random and fixed sample sizes are close.

To show the efficiency of our Bayesian technique compare with the results of the classical technique of Lawless [16] and Lingappaiah [18], we calculate the width of the 95% predictive intervals given in the above table analogue with those of Lawless [16] and Lingappaiah [18] as given below

	Classical Approach	Bayesian Approach
Statistic-1	262.6	258.8
Statistic-2	261.9	239.7

It is clear that the width of the Bayesian and classical approaches are close, but the Bayesian approach gives narrower predictive intervals.

5. Concluding remarks. The Bayesian predictive function for future observations from Weibull distribution based on fixed and random sample size n are investigated. For the finite population Binomial distribution is considered to be suitable model for the sample size, while for the large populations Poisson distribution can be considered as a suitable model. To show the usefulness of the proposed procedure Simulation experiments are carried out. Finally application is discussed. During the simulation some consideration are taken into accounts they are:

1. The random samples from Weibull distribution are generated by using the double precision subroutine RNEXP from the IMSL library.

2. The nonlinear equations are solved by using the double precision subroutine ZREAL from the IMSL library.

3. The parameters of the priors are selected randomly positive values and the calculations do not depends on the change of them.

4. The sample size n is generated randomly from binomial and Poisson distribution by using the subroutines RNBIN and RNPOI, respectively, from the IMSL library.

5. The proportion of future responses that can be predicted using the proposed predictive intervals is investigated in the sense of probability coverage that gives probabilities close to their level of significance.

6. The Bayesian approach give better results than the classical approach based on the sense of the predictive average width.

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Appendices

A. Proof of equation (2.5). The *pdf* of W given in (1.4) can be derived by using the transformation $W^c = Z^c/S_r$ as

$$\begin{aligned} f_W(w|\lambda) &= f_Z(wS_r|\lambda)\left|\frac{\partial Z}{\partial W}\right| \\ &= \frac{S_r c w^{c-1}\lambda(e^{-\lambda w^c S_r})^{n-s+1}(1-e^{-\lambda w^c S_r})^{s-r-1}}{\beta(s-r,n-s+1)}. \end{aligned} \tag{A.1}$$

From (1.8), the predictive function $h_1(w|x)$ can be written as

$$h_1(w|x) = \int_0^\infty f_W(w|\lambda)\pi(\lambda|x)\,d\lambda, \tag{A.2}$$

where $f_W(.|\lambda)$ and $\pi(\lambda|x)$ are given in (A.1) and (2.3), respectively.

By using (A.1) in (A.2), expand the term $(1-e^{-\lambda w^c S_r})^{s-r-1}$ binomially and simplifying we get (2.5).

B. Proof of equation (2.12). The *pdf* of U given in (1.5) can be derived by using the transformation $U = Z^c/x^c_{r:n}$ as

$$\begin{aligned} f_U(u|\lambda) &= f_Z(ux_{r:n}|\lambda)\left|\frac{\partial Z}{\partial U}\right| \\ &= \frac{x^c\lambda(e^{-\lambda u^c x_{r:n}^c})^{n-s+1}(1-e^{-\lambda u^c x_{r:n}^c})^{s-r-1}}{\beta(s-r,n-s+1)}. \end{aligned} \tag{B.1}$$

From (1.8), the predictive function $h_2(u|x)$ can be written as

$$h_2(u|x) = \int_0^\infty f_U(u|\lambda)\pi(\lambda|x)\,d\lambda, \tag{B.2}$$

where $f_U(.|\lambda)$ and $\pi(\lambda|x)$ are given in (B.1) and (2.10), respectively.

By using (B.1) in (B.2), expand the terms $(1-e^{-\lambda x_{r:n}^c})^{r-1}$ and $(1-e^{-\lambda u^c x_{r:n}^c})^{s-r-1}$ binomially and simplifying we get (2.12).

C. Proof of equation (3.3). Starting from the predictive function in the of RSS given in (1.9), we can write the predictive function of Statistic-1 based on RSS as

$$q_1(w|x,n) = \frac{1}{1-P(s-1)}\sum_{n=s}^{\infty} p(n)h_1(w|x), \tag{C.1}$$

where $P(\cdot)$ and $p(\cdot)$ are the *pmf* and *cdf* of Poisson distribution, respectively, and $h_1(\cdot|x)$ is given in (2.5). Upon using (2.5), (3.1) and simplifying we get (3.3).

D. Proof of equation (3.6). Following the same technique as in Appendix C and replace $h_1(\cdot|\lambda)$ in (C.1) by $h_2(\cdot|\lambda)$ in (2.12) and simplifying we get (3.6).

A. H. Abd Ellah
Department of Mathematics
Faculty of Science
Sohag University
Sohag P.O.Box 82524, Egypt
e-mail: ahmhamed@hotmail.com

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