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ESTIMATION OF A REGRESSION FUNCTION ON A POINT PROCESS AND ITS APPLICATION TO FINANCIAL RUIN RISK FORECAST

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ABSTRACT. We estimate a regression function on a point process by the Tukey regressogram method in a general setting and we give an application in the case of a Risk Process. We show among other things that, in classical Poisson model with parameter ρ , if W is the amount of the claim with finite expectation $E(W) = m$, S_n (resp. R_n) the *accumulated interval waiting time for successive claims* (resp. the *aggregate claims amount*) up to the n th arrival, the regression curve of R on S predicts ruin arrival time when the *premium intensity* c is less than ρm whatever be the *initial reserve*.

^{*}To the memory of A. Kone with whom we started this work in [9], Departement de Mathematiques Universite Cheick Anta Diop de Dakar, Senegal.

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Key words: Point process, regressogram, superposition, claim amount, aggregate claim amount, mean inter-arrival claim intensity, mean intensity of the claim process, ruin time.

1. General hypotheses Our study is motivated by the following real problem:

GH1: Let $(T_n)_{n \in \mathbf{N}^*}$ be a claim arrival process and $X_n = T_n - T_{n-1}$, $n = 1, 2, \dots$, be the interval arrival times we suppose i.i.d having the same distribution as a variable X with values in \mathbf{R}_+ . Denote by $F(x) = P(X < x)$ its distribution which we suppose continuous with density \hat{f} , strictly positive and continuous. We put $T_0 = 0$ and $S_n = X_1 + X_2 + \dots + X_n$.

S_n is the *accumulated claim* up to the n th arrival. We suppose that, with the i th variable X_i is associated a second variable W_i such that the (X_i, W_i) are independent. We impose X_i and W_i to be dependent. W_i is interpreted as the *claim amount* of the n th claim. Define $R_n := W_1 + W_2 + \dots + W_n$ the *aggregate claims amount* of the claims occurring up to the n th arrival.

Define $N_t := \sup\{n \in \mathbf{N} | (R_n, S_n) \in [0, t] \times \mathbf{R}_+\}$.

This paper is devoted to the study of the regression function $E(R_{N_t}/S_{N_t} = x)$. With this aim in view we give the statement in a general setting.

GH2: Let f_0 be a bidimensional point process f_0 defined on a probability space (Ω, \mathcal{A}, P) with values in $\mathbf{R}_+ \times \mathbf{R}_+$. For any Borel set A of $\mathbf{R}_+ \times \mathbf{R}_+$ denote by $f_0(A)$ the number of points falling in A . We suppose that $l = f_0(\mathbf{R}_+ \times \mathbf{R}_+)$ is finite almost surely and that the mean measure μ of f_0 is finite on bounded Borel sets and admits a Radon Nikodym derivative f^* .

Let $f_{0,1}$ be the first projection of f_0 . We denote by μ_1 its mean measure and f its Radon Nikodym density. If $l \geq 1$, let $(X_1, Y_1), \dots, (X_l, Y_l)$ be the points of the process ordered such that $X_1 < \dots < X_l$.

We define $(X_0, Y_0) = (0, 0)$. Let $\alpha = 1, 2$ and suppose $l = l_0$, $l_0 > 0$.

The model of regression we are considering satisfies the following:

a) $E\left(Y_j^\alpha / X_1 = x_1, \dots, X_j = x_j, \dots, X_{l_0} = x_{l_0}\right) = E\left(Y_j^\alpha / X_j = x_j\right)$ for $j = 1, \dots, l_0$

b) $E\left(Y_j^\alpha / X_j = x\right)$ is independent of j and l_0 for $j = 1, \dots, l_0$. We denote this function as $\Psi_\alpha(x)$

This model had been investigated by Dia [5], Dia et al. [8], Diakhaby [10], Dia et al. [9].

Consider f_i for $i = 1, \dots, n$ n i.i.d points processes having the same distribution as f_0 and $f_{(n)}$ their superposition in the sens of Cox [2]. Let $m = f_{(n)}(\mathbf{R}_+ \times \mathbf{R}_+)$ and $f_{1,(n)}$ be the first projection of $f_{(n)}$. If $\alpha=1$ we denote as Ψ the function Ψ_α .

The estimator we are dealing with is the fixed bandwidth regressogram of Tukey [18] developped later by Major [14], Geffroy [12]. It was utilized for estimating the regression function on a Poisson Process in [6].

Suppose $m \geq 1$ and let $(X_1^{(n)}, Y_1^{(n)}), \dots, (X_m^{(n)}, Y_m^{(n)})$ be the points of $f_{(n)}$. If $m = 0$ we put $(X_0^{(n)}, Y_0^{(n)}) = (0, 0)$.

Let k be a function of n . We denote

$$\begin{aligned} \Delta_{k,r} &= \left[\frac{r}{k}, \frac{r+1}{k} \right], \quad r \in N \\ \mathcal{J}_{n,r} &= \left\{ i, i \geq 1 \mid X_i^{(n)} \in \Delta_{k,r} \right\} \\ \nu_{n,r} &= \text{card } \mathcal{J}_{n,r} \\ \bar{Y}_{n,r} &= \begin{cases} \frac{1}{\nu_{n,r}} \sum_{i \in \mathcal{J}_{n,r}} Y_i^{(n)} & \text{if } \nu_{n,r} > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We then define $\Psi_{n,k}$ the estimator of Ψ by

$$(\forall r \geq 0) \quad (\forall x \in \Delta_{k,r}) \quad \Psi_{n,k}(x) := \bar{Y}_{n,r}.$$

2. The main theorems. Let $f_{0,1}$ be the first projection of f_0 and let us consider the following hypotheses:

- **H₁)** f is continuous and strictly positive.
- **H₂)** Ψ_α exists and is continuous for $\alpha = 1, 2$.

- **H₃**) for any x in \mathbf{R}_+

$$P\left(f_{0,1}([x, x + \Delta x[) \geq 2\right) = o(\Delta x).$$

- **H₄**) $f_{0,1}$ satisfies the approximation (see [3])

$$P\left(f_{0,1}(I) = 1\right) \cong \mu_1(I),$$

whenever I is an interval with arbitrarily small length.

- **H₅**) the second factorial moment of $f_{0,1}(I)$ exists for every bounded interval I .

Remark 2.1. It results from the hypothesis **H₃** that $f_{0,1}$ is without double points, that is

$$(\forall i, j) \ (1 \leq i < j) \ P\left(\varpi : X_i(\varpi) = X_j(\varpi), \quad l > 1\right) = 0.$$

Therefore the points X_1, \dots, X_l can be strictly ordered with probability one (see [3]).

Theorem 2.1. *If for $l = k \geq 1$ $E(Y_j^\alpha / X_j = x)$ is finite and independent of k and j , $j = 1, \dots, k$, then*

$$\Psi_\alpha(x) = \frac{1}{f(x)} \int_{\mathbf{R}} y^\alpha f^*(x, y) dy,$$

where $f(x) = \int_{\mathbf{R}} f^*(x, y) dy$.

Theorem 2.2. *Suppose that the hypotheses **H₁**, **H₂**, **H₃**, **H₄**, **H₅** are satisfied. If $\frac{n}{k^2} \rightarrow +\infty$ and $k = o\left(\frac{n}{\text{Log } n}\right)$ as $n \rightarrow +\infty$ then*

$$(\forall x \in \mathbf{R}) \quad \lim_{n \rightarrow \infty} E\left[(\Psi_{n,k}(x) - \Psi(x))^2\right] = 0$$

i.e $\Psi_{n,k}(x)$ converges in quadratic mean to $\Psi(x)$.

3. Preliminary results.

Lemma 3.1. *If $l = k$, then the variables $(X_1, Y_1), \dots, (X_k, Y_k)$ are absolutely continuous with respect to the Lebesgue measure with conditional density $[P(l = k)]^{-1} h_k^i f^*$, $i = 1, \dots, k$, say. Moreover*

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^k h_k^i \right) = 1.$$

Proof. Let $\Phi = \chi_A$, be the indicator function of a Borel set A . We have

$$\begin{aligned} E \left(\sum_{i=1}^l \Phi((X_i, Y_i)) \right) &= \mu(A) \\ (3.1) \qquad \qquad \qquad &= \sum_{k=1}^{\infty} \sum_{i=1}^k P((X_i, Y_i) \in A, l = k). \end{aligned}$$

Since $P((X_i, Y_i) \in A, l = k) \leq \mu(A)$, there exists Borel measurable function h_k^i such that

$$P((X_i, Y_i) \in A, l = k) = \int_A h_k^i(x, y) d\mu(x, y) = \int_A h_k^i(x, y) f^*(x, y) dx dy.$$

Therefore

$$(3.2) \qquad \sum_{k=1}^{\infty} \left(\int_A \sum_{i=1}^k h_k^i(x, y) dx dy \right) = \int_A f^*(x, y) dx dy.$$

The Beppo-Levi theorem implies that

$$(3.3) \qquad \sum_{k=1}^{\infty} \left(\sum_{i=1}^k h_k^i(x, y) f^*(x, y) \right) = f^*(x, y)$$

and so we have established the lemma. \square

A similar result was obtained for the one dimensional process $f_{0,1}$. The variables $X_i, i = 1, \dots, k$ are absolutely continuous with respect to Lebesgue measure with conditional density $[P(l = k)]^{-1} g_k^i(x) f(x)$ and $\sum_{k=1}^{\infty} \left[\sum_{i=1}^k g_k^i \right] = 1$.

Remark 3.2.

- (X_i, Y_i) , $i \geq 1$ exists only if $l \geq i$. The event $((X_i, Y_i), l < i)$ is an empty set.
- Suppose $(X_i, Y_i) = (X_{i,1}, Y_{i,1}) + \cdots + (X_{i,s}, Y_{i,s})$ is a sum of s independent random variables with density then, permuting the summation in (3.1), (3.2) and (3.3) we have:

$$P((X_i, Y_i) \in A) = \int_A \sum_{k=i}^{+\infty} h_k^i(x, y) f^*(x, y) dx dy.$$

Hence the density $f^{(i)}$ of (X_i, Y_i) , which is the s -convolution of the density of $(X_{i,j}, Y_{i,j})$, $j = 1, \dots, s$, can be expressed formally as:

$f^{(i)}(x, y) = \sum_{k=i}^{+\infty} h_k^i(x, y) f^*(x, y)$. (see e.g. [15, p. 128] for convolution of multivariate functions). Conversely if such decomposition of (X_i, Y_i) exists then equality (3.1) and the remark just above give:

$$\sum_{i=1}^{\infty} P((X_i, Y_i) \in A) = \mu(A).$$

Hence μ admits a derivative $f^*(x, y) = \sum_{i=1}^{+\infty} f^{(i)}(x, y)$ almost everywhere.

An analogous remark holds in the one dimensional case (see [4, p. 84] for the density of a renewal process).

4. The proofs of the theorems.

Proof of Theorem 2.1. Since Y_j and X_j are only defined if $l \geq j$ we have from the Lemma 3.1 and hypothesis **GH2 b)**

$$\Psi_\alpha(x) = E(Y_j^\alpha / X_j = x) = \frac{\sum_{k=j}^{+\infty} \int_{\mathbf{R}^+} y^\alpha h_k^j(x, y) f^*(x, y) dx dy}{\sum_{k=j}^{+\infty} g_k^j(x) f(x)}$$

for each j . We deduce that for all $s \geq 1$

$$\Psi_\alpha(x) = \frac{\sum_{j=1}^s \sum_{k=j}^{+\infty} \int_{\mathbf{R}_+} y^\alpha h_k^j(x, y) f^*(x, y) dx dy}{\sum_{j=1}^s \sum_{k=j}^{+\infty} g_k^j(x) f(x)}.$$

Hence, letting s tend to $+\infty$ and by using Fubini's theorem we obtain

$$\Psi_\alpha(x) = \frac{\sum_{k=1}^{+\infty} \sum_{j=1}^k \int_{\mathbf{R}_+} y^\alpha h_k^j(x, y) f^*(x, y) dx dy}{\sum_{k=1}^{+\infty} \sum_{j=1}^k g_k^j(x) f(x)}$$

the series in the numerator being convergent.

Lemma 3.1 and the Beppo-Levi Theorem complete the proof of the theorem. \square

Let $r := [kx]$ for fixed x in \mathbf{R}_+^* where $[z]$ stands for the integer part of a real number z . $\Delta_{k,r}^j := (\nu_{n,r} = j)$.

Consider the following partition of the set $\mathcal{J}_{n,r}$ defined by

$$\mathcal{J}_{n,r} := \bigcup_{s=1}^n \mathcal{J}_{n,r}^{(s)}$$

where $\mathcal{J}_{n,r}^{(s)}$ stands for the set of indexes i such that $X_i^{(n)}$, element of the s -th component of $f_{1,(n)}$ denoted by $f_{1,s}$, belongs to $\Delta_{k,r}$.

Let $\nu_{n,r}^{(s)} := \text{card } \mathcal{J}_{n,r}^{(s)}$ where card denotes the cardinal number of a set.

Lemma 4.1. *Suppose that \mathbf{H}_1 , \mathbf{H}_2 , \mathbf{H}_3 are satisfied. Then there exists a point $\zeta_{k,r,\alpha}$ in the closure of $\Delta_{k,r}$ such that*

$$(\forall j \geq 1), \quad (\forall i \in \mathcal{J}_{n,r}), \quad E((Y_i^{(n)})^\alpha / \Delta_{k,r}^j) = \Psi_\alpha(\zeta_{k,r,\alpha}).$$

Proof.

$$\int_{\nu_{n,r}=j} \left(Y_i^{(n)}\right)^\alpha dP = \sum_{\substack{j_1, j_2, \dots, j_s, \dots, j_n \\ j_1 + j_2 + \dots + j_n = j}} \int_{\nu_{n,r}^{(s)}=j_s} \left(Y_i^{(n)}\right)^\alpha dP.$$

We make the convention that the integral in the right hand side is nul if $\mathcal{J}_{n,r}^{(s)} = \emptyset$ or if $i \notin \mathcal{J}_{n,r}^{(s)}$.

Therefore if $\mathcal{J}_{n,r}^{(s)} \neq \emptyset$ and $i \in \mathcal{J}_{n,r}^{(s)}$ by the hypothesis **GH2 a)** we have

$$\int_{\nu_{n,r}^{(s)}=j_s, i \in \mathcal{J}_{n,r}^{(s)}} (Y_i^{(n)})^\alpha dP = \int_{X_i^{(n)} \in \Delta_{k,r}, \dots, X_{i_{j_s}}^{(n)} \in \Delta_{k,r}} \Psi_\alpha(x_i) dF_{X_i^{(n)}, \dots, X_{i_{j_s}}^{(n)}}(x_i, \dots, x_{i_{j_s}}),$$

where $(X_i^{(n)}, \dots, X_{i_{j_s}}^{(n)})$ stands for the j_s variables of the set $(\nu_{n,r}^{(s)} = j_s)$.

Since Ψ_α is continuous, we obtain

$$\int_{\nu_{n,r}^{(s)}=j_s} (Y_i^{(n)})^\alpha dP = P(\nu_{n,r}^{(s)} = j_s) \Psi_\alpha(\zeta_s)$$

with ζ_s belonging to the closure of $\Delta_{k,r}$. Hence

$$\int_{\nu_{n,r}=j} (Y_i^{(n)})^\alpha dP = \sum_{\substack{j_1, j_2, \dots, j_s, \dots, j_n \\ j_1 + j_2 + \dots + j_n = j}} P(\nu_{n,r}^{(s)} = j_s) \Psi_\alpha(\zeta_s).$$

$$E\left((Y_i^{(n)})^\alpha / \Delta_{k,r}^j\right) = \frac{1}{P(\Delta_{k,r}^j)} \int_{\nu_{n,r}=j} (Y_i^{(n)})^\alpha dP \text{ is then between } \min_{x \in \Delta_{k,r}} \Psi_\alpha(x)$$

and $\max_{x \in \Delta_{k,r}} \Psi_\alpha(x)$. Since Ψ_α is continuous, the lemma is proved. \square

Proposition 4.1. *If $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5$ are satisfied and $\frac{n}{k^2} \rightarrow \infty$, $n \rightarrow \infty$ then*

- 1) $\lim_{n \rightarrow +\infty} \sum_{j=1}^{+\infty} P(\Delta_{k,r}^j) = 1.$
- 2) $\lim_{n \rightarrow +\infty} \sum_{j=1}^{+\infty} \frac{1}{j} P(\Delta_{k,r}^j) = 0.$
- 3) $\lim_{n \rightarrow +\infty} n \sum_{j=1}^{+\infty} \frac{1}{j^2} P(\Delta_{k,r}^j) = 0.$

Proof. 1. Write

$$\sum_{j=1}^{+\infty} P(\Delta_{k,r}^j) = 1 - P(\Delta_{k,r}^0).$$

We have

$$\begin{aligned} P(\Delta_{k,r}^0) &= (P(f_{0,1}(\Delta_{k,r}) = 0))^n \\ &= (1 - (P(f_{0,1}(\Delta_{k,r}) = 1) + P(f_{0,1}(\Delta_{k,r}) \geq 2)))^n \end{aligned}$$

and

$$(4.1) \quad P(f_{0,1}(\Delta_{k,r}) = 1) = \mu_1(\Delta_{k,r}) + o\left(\frac{1}{k}\right)$$

$$(4.2) \quad = \int_{\frac{\tau}{k}}^{\frac{\tau+1}{k}} f(x) dx + o\left(\frac{1}{k}\right) = \frac{1}{k} f(\tau) + o\left(\frac{1}{k}\right)$$

where $\tau \in \Delta_{k,r}$ because of **H₄** and the continuity of f .

By **H₃** we have $P(f_{0,1}(\Delta_{k,r}) \geq 2) = o\left(\frac{1}{k}\right)$.

Hence

$$(4.3) \quad P(\Delta_{k,r}^0) = e^{n \text{Log}\left(1 - \left(\frac{1}{k} f(\tau) + o\left(\frac{1}{k}\right)\right)\right)}$$

$$(4.4) \quad \cong e^{\frac{-n}{k} (f(\tau) + o\left(\frac{1}{k}\right))}.$$

Since $\frac{n}{k} \rightarrow \infty$ as $n \rightarrow +\infty$ and $f(\tau) \rightarrow f(x) > 0$ by continuity of f , the part 1) of the proposition is proved.

2. This equality can be written in the form $\sum_{j=1}^{\infty} \frac{1}{j} P\left(\Delta_{k,r}^j\right) = E\left(\frac{1}{\nu_{n,r}}\right)$.

Let us show that $\nu_{n,r} \rightarrow +\infty$ with probability one as $n \rightarrow +\infty$.

Let $0 < \epsilon < 1$. We have from (4.2)

$$P(f_{0,1}(\Delta_{k,r}) > \epsilon) \geq P(f_{0,1}(\Delta_{k,r}) = 1) = \frac{1}{k} f(\tau) + o\left(\frac{1}{k}\right).$$

It follows that it exists $\delta > 0$ such that $P(\nu_{n,r} > \epsilon) > \frac{\delta}{k}$. Since $\frac{n}{k} \rightarrow +\infty$ as

$n \rightarrow +\infty$ the series $\sum_{s=1}^{+\infty} P(f_{1,s}(\Delta_{k,r}) > \epsilon) = +\infty$. Therefore by the Borel-Cantelli lemma infinitely many events $(f_{1,s}(\Delta_{k,r}) > \epsilon)$ occur with probability one. Hence

$\nu_{n,r} = \sum_{s=1}^n f_{1,s}(\Delta_{k,r}) \rightarrow +\infty$ with probability one.

The Lebesgue dominated convergence theorem completes the proof.

3. It is equivalent to show that $\lim_{n \rightarrow +\infty} E \left(\frac{n}{\nu_{n,r}^2} \right) = 0$. Write

$$(4.5) \quad \frac{n}{\nu_{n,r}^2} = \frac{1}{n\mu_1^2(\Delta_{k,r})} \left(\frac{n\mu_1(\Delta_{k,r})}{\nu_{n,r}} \right)^2.$$

$$(4.6) \quad \frac{n\mu_1(\Delta_{k,r})}{\nu_{n,r}} = \frac{n\mu_1(\Delta_{k,r})}{\sum_{s=1}^n \nu_{n,r}^{(s)}}.$$

But $E \left(\frac{\nu_{n,r}^{(s)}}{\mu_1(\Delta_{k,r})} \right) = 1$ and the random variables $\frac{\nu_{n,r}^{(s)}}{\mu_1(\Delta_{k,r})}$, $s = 1, \dots, n$ are independent and identically distributed. Hence $\frac{n\mu_1(\Delta_{k,r})}{\nu_{n,r}}$ tends to 1 with probability one as $n \rightarrow +\infty$ by the strong law of large numbers; therefore it is bounded with probability one. Since $\mu_1(\Delta_{k,r}) = \int_{\frac{r}{k}}^{\frac{r+1}{k}} f(x) dx = \frac{f(\tau)}{k}$ with $\tau \in \Delta_{k,r}$ and $f(\tau) \rightarrow f(x) > 0$ as $n \rightarrow +\infty$ by the continuity of f , we then obtain $\frac{n}{\nu_{n,r}^2} = O \left(\frac{k^2}{n} \right)$ a.s.

The Lebesgue dominated convergence theorem completes the proof. \square

Proposition 4.2. *Under the conditions of Theorem 2.2, if **H4** is satisfied then*

$$\lim_{n \rightarrow +\infty} \sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{i \neq i'} \text{cov}(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j) P(\Delta_{k,r}^j) = 0.$$

Proof. We suppose that i and i' belong to the same $\mathcal{J}_{n,r}^{(s)}$ and $\text{card } \mathcal{J}_{n,r}^{(s)} \geq 2$ otherwise the covariance is nul.

Let s be fixed and i, i' belong to $\mathcal{J}_{n,r}^{(s)}$.

The inequality

$$|\text{cov}(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j)| \leq (E((Y_i^{(n)})^2 / \Delta_{k,r}^j))^{\frac{1}{2}} (E((Y_{i'}^{(n)})^2 / \Delta_{k,r}^j))^{\frac{1}{2}}$$

implies

$$\begin{aligned}
 & \sum_{i \neq i'} \left| \text{cov} \left(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j \right) \right| P \left(\Delta_{k,r}^j \right) \\
 & \leq \sum_{i \neq i'} E \left(\chi_{\Delta_{k,r}^j} \left(E \left(\left(Y_i^{(n)} \right)^2 / \Delta_{k,r}^j \right) \right)^{\frac{1}{2}} \left(E \left(\left(Y_{i'}^{(n)} \right)^2 / \Delta_{k,r}^j \right) \right)^{\frac{1}{2}} \right) \\
 & \leq \sum_{\beta=2}^j E \left(\sum_{\substack{i \neq i' \\ \mathcal{J}_{n,r}^{(s)}}} \chi_{\Delta_{k,r}^j} \Psi_2(\zeta_{k,r,2}) / \text{card } \mathcal{J}_{n,r}^{(s)} = \beta \right) P \left(\text{card } \mathcal{J}_{n,r}^{(s)} = \beta \right) \\
 & \leq \sum_{\beta=2}^j \beta(\beta-1) \Psi_2(\zeta_{k,r,2}) P(\nu_{n-1,r} = j - \beta) P \left(\nu_{n,r}^{(s)} = \beta \right).
 \end{aligned}$$

We have for such i and i'

$$\begin{aligned}
 & \sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{i \neq i'} \left| \text{cov} \left(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j \right) \right| P \left(\Delta_{k,r}^j \right) \\
 & \leq \sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{\beta=2}^j \beta(\beta-1) \Psi_2(\zeta_{k,r,2}) P(\nu_{n-1,r} = j - \beta) P \left(\nu_{n,r}^{(s)} = \beta \right) \\
 & \leq \sum_{\beta=1}^{+\infty} \beta(\beta-1) P \left(\nu_{n,r}^{(s)} = \beta \right) \sum_{j=\beta}^{+\infty} \frac{1}{j^2} P(\nu_{n-1,r} = j - \beta) \\
 & \leq \Psi_2(\zeta_{k,r,2}) \eta_{k,r}^{(2)} \left(\sum_{j=1}^{+\infty} \frac{1}{j^2} P(\nu_{n-1,r} = j) + P(\nu_{n-1,r} = 0) \right)
 \end{aligned}$$

where $\eta_{k,r}^{(2)}$ stands for the second factorial moment of $f_{0,1}(\Delta_{k,r})$.

Hence for i and i' belonging to $\mathcal{J}_{n,r}$ we have

$$(4.7) \quad \sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{i \neq i'} \left| \text{cov} \left(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j \right) \right| P \left(\Delta_{k,r}^j \right) \\ \leq n \Psi_2(\zeta_{k,r,2}) \eta_{k,r}^{(2)} \left(\sum_{j=1}^{+\infty} \frac{1}{j^2} P(\nu_{n-1,r} = j) + P(\nu_{n-1,r} = 0) \right).$$

We have from (4.3) and (4.4)

$$\text{Log} (nP(\nu_{n-1,r} = 0)) \cong \text{Log} n - \frac{(n-1)}{k} \left(f(\tau) + \epsilon \left(\frac{1}{k} \right) \right)$$

which tends to $-\infty$ as $n \rightarrow +\infty$.

Thus part **2.** and **3.** of Proposition 4.1 then complete the proof of the proposition. \square

Proof of Theorem 2.2. By Lemma 4.1 we have

$$(4.8) \quad E(\Psi_{n,k}(x)) = E(E(\Psi_{n,k}(x)/\nu_{n,r})) = \Psi(\zeta_{k,r,1}) \sum_{j=1}^{+\infty} P(\Delta_{k,r}^j).$$

In the same way

$$E(\Psi_{n,k}^2(x)) = \sum_{j=1}^{+\infty} E \left(\Psi_{n,k}^2(x) / \Delta_{k,r}^j \right) P \left(\Delta_{k,r}^j \right). \\ E \left(\Psi_{n,k}^2 / \Delta_{k,r}^j \right) = \frac{1}{j^2} \sum_i E \left(\left(Y_i^{(n)} \right)^2 / \Delta_{k,r}^j \right) + \frac{1}{j^2} \sum_{i \neq i'} E \left(Y_{i'}^{(n)} Y_i^{(n)} / \Delta_{k,r}^j \right).$$

Express

$$\sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{i \neq i'} E \left(Y_{i'}^{(n)} Y_i^{(n)} / \Delta_{k,r}^j \right) P \left(\Delta_{k,r}^j \right)$$

as

$$\sum_{j=1}^{+\infty} \frac{1}{j^2} \sum_{i \neq i'} \text{cov} \left(Y_{i'}^{(n)}, Y_i^{(n)} / \Delta_{k,r}^j \right) P \left(\Delta_{k,r}^j \right) \\ + \Psi^2(\zeta_{k,r,1}) \sum_{j=1}^{+\infty} P \left(\Delta_{k,r}^j \right) - \Psi^2(\zeta_{k,r,1}) \sum_{j=1}^{+\infty} \frac{1}{j} P \left(\Delta_{k,r}^j \right).$$

Proposition 4.1 and Proposition 4.2 imply that this last expression tends to $\Psi^2(x)$ as $n \rightarrow +\infty$.

On the other hand

$$\sum_1^{+\infty} \frac{1}{j^2} \sum_i E \left(\left(Y_i^{(n)} \right)^2 / \Delta_{k,r}^j \right) P \left(\Delta_{k,r}^j \right) \leq \Psi_2(\zeta_{k,r,2}) \sum_{j=1}^{+\infty} \frac{1}{j} P \left(\Delta_{k,r}^j \right).$$

The right hand side of this inequality tends to 0 as $n \rightarrow +\infty$ by Proposition 4.1. It follows that $E(\Psi_{n,k}^2(x)) \rightarrow \Psi^2(x)$ as $n \rightarrow +\infty$.

The Proposition 4.2 again implies, by equality (4.3), that $E(\Psi_{n,k}(x)) \rightarrow \Psi(x)$ as $n \rightarrow +\infty$. Hence $\text{Var}(\Psi_{n,k}(x)) \rightarrow 0$ as $n \rightarrow +\infty$.

Since $\lim_{n \rightarrow +\infty} (\text{Bias} \Psi_{n,k}(x))^2 = 0$ the proof of the theorem is complete. \square

Remark 4.2. If there exists Y independent of the process such that $Y_i < Y$ for $i = 1, 2, \dots$ then $E \left(\left(Y_i^{(n)} \right)^2 / \Delta_{k,r}^j \right) < \Psi_1(\zeta_{k,r,1}) E(Y)$ and the theorem remains valid if Y has a finite moment. Therefore, in this case, we shall restrict ourself to processes for which in the general hypotheses **GH2** $\alpha = 1$.

5. Application. We suppose the hypothesis in the preceding Remark 4.2 satisfied. The risk process introduced earlier in Paragraph 1 is considered in this section.

Let $Z_n := (R_n, S_n)_{n \in \mathbb{N}}$. We suppose that (W_n, X_n) admits a continuous density $f_n(w, x)$. Therefore the random vector $Z_2 = (W_1 + W_2, X_1 + X_2)$ admits a density given by

$$f^{(2)}(w, x) := \int_0^{+\infty} \int_0^{+\infty} f_2(w - u, x - v) f_1(u, v) du dv$$

where $f^{(2)}$ stands for the two-fold convolution of f_2 and f_1 (e.g. [15, p. 128]). In a iterative manner the density of f_n is expressed as $f^{(n)}(w, x) = f^{(n-1)} * f_n(w, x)$, $f_0 = 1$, $f^1 = f_1$. It follows from the Remark 3.2 that the process $Z_{N_t} := \left(\sum_{i=1}^{N_t} W_i, \sum_{i=1}^{N_t} X_i \right)$ admits mean measure μ with density f^* defined by

$$f^*(w, x) := \sum_1^{+\infty} f^{(n)}(w, x).$$

We suppose that f^* is continuous.

Consider the marginal process denoted by $f_{0,1} := S_{N_t}$. It admits a mean measure μ_1 defined by

$$\forall B \in \mathcal{B}(\mathbf{R}_+), \quad \mu_1(B) := \sum_{n=1}^{+\infty} P(S_n \in B).$$

Define $\mu_1([0, x]) := \mu_1(x) = \sum_{k=1}^{+\infty} F_k^*(x)$ where $F_k^*(x)$ stands for the k -convolution of the distribution F and $f(x) := \frac{d\mu_1}{dx}(x)$. We suppose f strictly positive.

We have

$$\Psi(x) = E(R_{N_t}/S_{N_t} = x) = \frac{\int_0^{+\infty} w f^*(w, x) dw}{f(x)}.$$

H₂ is verified. It is well-known that hypothesis **H₅** is satisfied.

Let us now show that $f_{0,1}$ satisfies also **H₃** and **H₄**.

1) For **H₃** we have

$$\{f_{0,1}([x, x + \Delta x]) \geq 2\} \subset \bigcup_{k=1}^{+\infty} (S_k \in [x, x + \Delta x], S_{k+1} \in [x, x + \Delta x])$$

and

$$\begin{aligned} P(S_k \in [x, x + \Delta x], S_{k+1} \in [x, x + \Delta x]) \\ &= \int_x^{x+\Delta x} P(S_{k+1} \in [x, x + \Delta x] / S_k = u) dF_{S_k}(u) \\ &= \int_x^{x+\Delta x} P(X_{k+1} + u \in [x, x + \Delta x] / S_k = u) dF_{S_k}(u) \\ &= \int_x^{x+\Delta x} P(X_{k+1} + u \in [x, x + \Delta x]) dF_{S_k} \end{aligned}$$

because the variables X_k are independant.

We now express the term in the integral as

$$P(X_{k+1} + u \in [x, x + \Delta x]) = \int_{x-u}^{x+\Delta x-u} \widehat{f}(t) dt = \Delta x \widehat{f}(\zeta)$$

where $\zeta \in [x - u, x + \Delta x - u]$. Hence

$$\begin{aligned}
 P(f_{0,1}([x, x + \Delta x]) \geq 2) &\leq \sum_{k=1}^{+\infty} \Delta x \hat{f}(\zeta) \int_x^{x+\Delta x} dF_{S_k}(u) \\
 &\leq \Delta x \hat{f}(\zeta) \int_x^{x+\Delta x} d \left(\sum_{k=1}^{+\infty} F_{S_k}(u) \right) \\
 &\leq \Delta x \hat{f}(\zeta) \int_x^{x+\Delta x} d\mu_1(x) = \Delta x \hat{f}(\zeta) \mu_1(\Delta x).
 \end{aligned}$$

Since μ_1 is continuous we get

$$P(f_{0,1}([x, x + \Delta x]) \geq 2) = o(\Delta x).$$

2) For \mathbf{H}_4 we have on the one hand

$$(5.1) \quad P(f_{0,1}([x, x + \Delta x]) = 1) = \sum_{n=0}^{+\infty} P(f_{1,0}([x, x + \Delta x]) = 1, N(x) = n).$$

But

$$\begin{aligned}
 P(f_{0,1}([x, x + \Delta x]) = 1/N(x) = n) &= P(S_{n+1} - S_n \in [x, x + \Delta x]/S_n = x) \\
 &= P(X_{n+1} \in [x, x + \Delta x]).
 \end{aligned}$$

Thus we obtain from (5.1) the equality

$$(5.2) \quad P(f_{0,1}([x, x + \Delta x]) = 1) = \int_x^{x+\Delta x} dF(u).$$

On the other hand, the renewal equation

$$\mu_1(t) = F(t) + \int_0^t \mu_1(t - u) dF(u)$$

gives

$$\begin{aligned}
 \mu_1([x, x + \Delta x]) &= \int_x^{x+\Delta x} dF(u) + \int_x^{x+\Delta x} (\mu_1(x + \Delta x - u) - \mu_1(x - u)) dF(u) \\
 (5.3) \quad &= \int_x^{x+\Delta x} dF(u) + \mu_1(\Delta x) \int_x^{x+\Delta x} dF(u).
 \end{aligned}$$

Thus equalities (5.2) and (5.3) complete the proof. \square

It remains to establish that the hypotheses **a)** and **b)** in **GH2** are satisfied. For this aim we need the following hypothesis:

H₆) : There exists an integrable function g such that

$$E(W_i/S_{i-1} = x_{i-1}, S_i = x_i) = \int_{x_{i-1}}^{x_i} g(u) du.$$

The points x_i , $i \geq 1$ stand for the jumps points of the process.

Theorem 5.1. *Suppose that the hypothesis **H₆** is satisfied and the conditions **GH1** in the general hypotheses are verified. If $N_t = r$, then*

1) $E(R_k/S_1 = x_1, S_2 = x_2, \dots, S_k = x_k, \dots, S_r = x_r) = E(R_k/S_k = x_k)$, $k = 1, \dots, r$,

2) $E(R_k/S_k = x_k)$ is independant of r and k , $1 \leq k \leq r$. Moreover, Ψ is differentiable with $\Psi'(x) = g(x)$ for almost all $x \in [0, t]$.

Proof. We have for $i = 1, \dots, r$ and all r , using the independance of the variables

$$\begin{aligned} & E(W_i/S_1 = x_1, \dots, S_i = x_i, \dots, S_r = x_r) \\ &= \int_0^{+\infty} \frac{w f_{(W_i, X_1, X_2, \dots, X_i, \dots, X_r)}(w, x_1, x_2 - x_1, \dots, x_i - x_{i-1}, \dots, x_r - x_{r-1})}{f_{(X_1, X_2, \dots, X_i, \dots, X_r)}(x_1, x_2 - x_1, \dots, x_i - x_{i-1}, \dots, x_r - x_{r-1})} dw \\ (5.4) \quad &= \int_0^{+\infty} \frac{w f_{(X_i, W_i)}(x_i - x_{i-1}, w)}{f_{X_i}(x_i - x_{i-1})} dw = E(W_i/X_i = x_i - x_{i-1}). \end{aligned}$$

But we have also

$$(5.5) \quad E(W_i/S_{i-1} = x_{i-1}, S_i = x_i) = E(W_i/X_i = x_i - x_{i-1}).$$

Because of **H₆**) we have:

$$(5.6) \quad E(W_i/S_{i-1} = x_{i-1}, S_i = x_i) = \int_{x_{i-1}}^{x_i} g(u) du.$$

Hence for $1 \leq k \leq r$ we get:

$$\begin{aligned}
 E(R_k/S_1 = x_1, S_2 = x_2, \dots, S_k = x_k, \dots, S_r = x_r) &= \sum_{i=1}^k \int_{x_{i-1}}^{x_i} g(u) du \\
 (5.7) \qquad \qquad \qquad &= \int_0^{x_k} g(u) du.
 \end{aligned}$$

By integrating this equality with respect to the distribution of $(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_r)$ we obtain

$$(5.8) \qquad E(R_k/S_k = x_k) = \int_0^{x_k} g(u) du.$$

By Theorem 2.1 we have $\Psi(x) = \int_0^x g(u) du$. Theorem 18.17 [11, p. 286] leads to the completion of the proof. \square

Remark 5.1.

1. If $E(W_i/X_i = x) = \lambda x$ for all i then hypothesis **H₆** is verified because of equalities (5.4) and (5.5) by summing the terms for $i = 1$ to $i = k$. Therefore the theorem is coarsely verified with $g = \lambda$.
2. Consider the following function

$$E(W_i/X_i = u - x_{i-1}) = (e^{\lambda u} - e^{\lambda x_{i-1}})\chi_{[x_{i-1}, x_i]}(u).$$

H₆ is also verified. Moreover for all $k \geq 1$ we have on $[0, x_k]$:

$$g(u) = \lambda e^{\lambda u}.$$

g satisfies **H₆**).

Equation (5.8) gives

$$(5.9) \qquad E(R_k/S_k = x_k) = e^{\lambda x_k} - 1.$$

Remark 5.2. Suppose now the conditions of the theorem fulfilled.

- We then have $E(W_i/X_i = u - x_{i-1}) = (\Psi(u) - \Psi(x_{i-1}))\chi_{[x_{i-1}, x_i]}(u)$ and $\Psi(x) = \int_0^{x_k} g(u) du$.
- Define Λ by $\Lambda([x_{i-1}, x_i]) := \int_{x_{i-1}}^{x_i} g(u) du$. This function can be thought of as the **mean inter-arrival claim intensity measure** and g the **mean intensity of the claim process**.
- Suppose that Ψ admits an asymptotic line with $\lambda > 0$ as slope. Then for any arbitrarily small $\epsilon > 0$, if x is large enough we have $E(W_1/X_1 = x) = \Psi(x) > (\lambda - \epsilon)x$. Hence $E(W_1) > (\lambda - \epsilon)E(X_1)$. Consequently, any line having a slope c such that $c < \frac{E(W_1)}{E(X_1)}$ will intersect the regression curve.
- By analogous reasoning, the same conclusion is evidently valid if $\lim_{x \rightarrow +\infty} \frac{\Psi(x)}{x} = +\infty$ by considering $\frac{\Psi(x)}{x} > c$ if x is large enough.
- If $\liminf_{x \rightarrow +\infty} g(x) > 0$, then $\lim_{x \rightarrow +\infty} \frac{\Psi(x)}{x} = 0$ is impossible because $\Psi(x) - xg(x)$ must be positive for all x .
- If X is exponentially distributed with density $\rho e^{-\rho x}$, then N_t is a Poisson process. Consequently, under the conditions of the preceding remarks, the intersection of the line $y = R_0 + cx$ in the classical ruin problem and the curve $\Psi(x)$ will necessarily occur if $c < m\rho$ whatever be the initial value R_0 (here $E(W_i) = m, i = 1, \dots$) (see [17, Corollary 7.1.4, p. 160] for another result). Therefore the ruin time in the futur can be predicted.

Note. The limit here is thought of as $t \rightarrow +\infty$ with N_t .

Remark 5.3.

- The ruin problem is predicted by this model for any deterministic premium function.
- It remains to investigate the case of the stochastic premium function. This case was studied by V. Kalashnikov [13]. The solution of the ruin problem ceases then to be analytic. The risk model takes the form:

$$(5.10) \quad B(t) = R_0 + C(t) - R(t)$$

with $B(0) = R_0$ the initial capital. The directions of further research are the following.

Suppose we can write $B_{N(t)} = U_{N(t)} - R_{N(t)}$ where $U_{N(t)} = R_0 + C_{N(t)}$. Then $E(B_{N(t)}/S_{N(t)} = x) = E(U_{N(t)}/S_{N(t)} = x) - E(R_{N(t)}/S_{N(t)} = x)$. Defining $\Phi(x) = E(U_{N(t)}/S_{N(t)} = x)$, the problem to solve is whether stochastic dominance holds or not between Φ and Ψ (see e.g. [1, 16]).

Recall that, under some conditions (see [7]), we have for all $y \in \mathbf{R}$ and all $b > 0$,

$$(5.11) \quad \lim_{n \rightarrow +\infty} P \left(\sup_{x \in [0, b]} (v(x))^{-1/2} ((n/k)f(x))^{1/2} (\Psi_{n,k}(x) - \Psi(x)) \right. \\ \left. < (2 \text{Log } k - \text{Log Log } k + y)^{1/2} \right) \\ = \exp \left(-\frac{e^{-y/2}}{2\sqrt{\pi}} \right)$$

where $v(x) = \text{Var}(Y_1/X_1 = x)$, the right-hand side of (5.11) being the Gumbel's distribution.

Suppose we have at our disposal $\Phi_{n,k}$, the regression estimation of Φ by the same method as in paragraph I. Let $\sigma_{n,k}(x)$ be a convergent estimation of $w(x) := \frac{\text{Var}(B_1/X_1 = x)}{f(x)}$. The statistical testing hypothesis dominance we are going to resolve is then

$$\begin{aligned} \mathbf{H}_0 &: \Psi(x) \leq \Phi(x) \quad \text{for all } x \in [0, b], \\ \overline{\mathbf{H}}_0 &: \Psi(x) > \Phi(x) \quad \text{for some } x \in [0, b]. \end{aligned}$$

Consider any constant c_0 . The test statistic

$$T_{n,k} = \sup_{x \in [0, b]} (\sigma_{n,k}(x))^{-1/2} ((n/k))^{1/2} (\Psi_{n,k}(x) - \Phi_{n,k}(x))$$

which rejects \mathbf{H}_0 if $T_{n,k} > (2\text{Log } k - \text{Log Log } k + c_0)^{1/2}$ satisfies:

A) if \mathbf{H}_0 is true.

$$\lim_{n \rightarrow +\infty} P(\text{reject } \mathbf{H}_0) \leq 1 - \exp \left(-\frac{e^{-c_0/2}}{2\sqrt{\pi}} \right).$$

This inequality results from the equality $\Psi_{n,k} - \Phi_{n,k} = ((\Psi_{n,k} - \Phi_{n,k}) - (\Psi - \Phi)) + (\Psi - \Phi)$.

B) If \mathbf{H}_0 is false.

Then there exists $\delta > 0$ and x_0 such that $\Psi(x_0) - \Phi(x_0) = \delta > 0$. We have

$$T_{n,k} \geq \sigma_{n,k}(x_0))^{-1/2}((n/k))^{1/2}(\Psi_{n,k}(x_0) - \Phi_{n,k}(x_0)).$$

Hence

$$\begin{aligned} P(\text{reject } \mathbf{H}_0) &\geq P(\sigma_{n,k}(x_0))^{-1/2}((n/k))^{1/2}(\Psi_{n,k}(x_0) - \Phi_{n,k}(x_0)) \\ &> (2\text{Log } k - \text{Log Log } k + c_0)^{1/2}. \end{aligned}$$

Since $k = o\left(\frac{n}{\text{Log } n}\right)$, the conditions

- i) $\inf_{x \in [0, b]} w(x) = d > 0$,
- ii) $\lim_{n \rightarrow +\infty} \sup_{x \in [0, b]} ((\sigma_{n,k}(x))^{1/2}(\Psi_{n,k}(x) - \Phi_{n,k}(x)) - (w(x))^{1/2}(\Psi(x) - \Phi(x))) = 0$ a.s imply

$$\lim_{n \rightarrow +\infty} P(\text{reject } \mathbf{H}_0) = 1.$$

These previous lines are the framework of ideas which we can make more precise later in an another paper.

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