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NONSTANDARD FINITE DIFFERENCE SCHEMES WITH APPLICATION TO FINANCE: OPTION PRICING

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Abstract. The paper is devoted to pricing options characterized by discontinuities in the initial conditions of the respective Black-Scholes partial differential equation. Finite difference schemes are examined to highlight how discontinuities can generate numerical drawbacks such as spurious oscillations. We analyze the drawbacks of the Crank-Nicolson scheme that is most frequently used numerical method in Finance because of its second order accuracy. We propose an alternative scheme that is free of spurious oscillations and satisfy the positivity requirement, as it is demanded for the financial solution of the Black-Scholes equation.

1. Introduction. In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option. The Black-Scholes analytic model for determining the behavior of the stock price turns out to be fundamental in option pricing, [1].

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Key words: Black-Scholes equation, finite difference schemes, Jacobi matrix; M-matrix, nonsmooth initial conditions, positivity-preserving.

In absence of a valuation formula for non-standard options (see [3]), the finite difference approach is a powerful tool for pricing. Usually the choice goes toward numerical methods with high order of accuracy (e.g. Crank-Nicolson scheme), and no attention is paid to how the financial provision of the contract can affect the reliability of the numerical solution, see Smith in [6], Tavella and Randall in [8], or Zvan *et al.* in [11]. Indeed these schemes are applied without considering the well-known problems that can arise in presence of *discontinuities* which deteriorate the numerical approximation.

In order to make our analysis concrete, we concentrate the attention on a double barrier knock-out call option with a discrete monitoring clause, but the presented analysis can be easily extended to many other exotic contracts (digital, supershare, binary and truncated payoff options, callable bonds and so on). Such option has a payoff condition equal to $\max(S-K,0)$ but the option expires worthless if before the maturity the asset price has fallen outside the corridor $[L,U]$ at the prefixed monitoring dates.

In the intermediate periods the Black Scholes equation is applied over the real positive domain. The discontinuity in the initial conditions will be *renewed at every monitoring date* and often the Crank-Nicolson numerical solution is affected by *spurious oscillations* that do not decay quickly, Tagliani *et al.* in [7]. The oscillations derive from an inaccurate approximation of the very sharp gradient produced by the knock-out clause, generating an error that is damped out very slowly.

In Section 2 we discuss the model for *discrete double barrier knock-out call options* and particularly the main drawbacks like undesired spurious oscillations, arising from centered difference discretization of the Black-Scholes PDE.

In Section 3 we propose a *suitable finite difference scheme* that enables us to solve accurately the examined PDE. An important factor for numerical schemes is the condition of positivity of the solution that must be satisfied as a consequence of the financial meaning of the involved PDE. We will demonstrate:

1. how an accurate scheme is not necessarily the best, as it could require prohibitively small time steps;

2. how a less accurate scheme could work successfully and preserve all financial requirements of the option contract such as positivity.

In Section 4 we explore examples of discrete double barrier knock-out options that are most frequently used in literature such as Wade *et al.* in [9], Zvan *et al.* in [11]. In the conclusion, we have pointed out the advantages of the proposed nonstandard finite difference scheme for the Black-Scholes partial differential equation.

2. Mathematical model. The Black-Scholes PDE. We consider as a model for the movement of the asset price under the risk-neutral measure a standard *geometric Brownian motion* diffusion process with constant coefficients r and σ:

$$dS/S = rdt + \sigma dW_t \tag{1}$$

The contract to be priced is a discretely monitored double barrier knock-out call option. If t is the time to expiry T of the contract, $0 \leq t \leq T$, the price $V(S,t)$ of the option satisfies the Black-Scholes partial differential equation

$$-\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{2}$$

endowed with initial and boundary conditions:

$$V(S,0) = \max(S-K,0)\;1_{[L,U]}(S) \tag{3}$$

$$V(S,t) \to 0 \;\text{ as }\; S \to 0 \;\text{ or }\; S \to \infty \tag{4}$$

with updating of the initial condition at the monitoring dates $t_i, i = 1, \ldots, F$:

$$V(S,t_i) = V(S,t_i^-)1_{[L,U]}(S), \qquad 0 = t_0 < t_1 < \ldots < t_F = T \tag{5}$$

where $\mathbf{1}_{[L,U]}(x)$ is the indicator function, i.e.,

$$\mathbf{1}_{[L,U]} = \begin{cases} 1 & \text{if} \quad S \in [L,U] \\ 0 & \text{if} \quad S \notin [L,U] \end{cases}$$

The knock-out clause at the monitoring date introduces a discontinuity at the barriers (set at $L = 90$, $U = 110$), as it is illustrated in Fig. 1. We notice the presence of undesired spurious oscillations near the barriers (set at $L = 90$ and $U = 110$ respectively) and near the strike ($K = 100$), where the Delta$=\dfrac{\partial V}{\partial S}$ is discontinuous at $t = 0$.

These spikes which remain well localized, don't reflect instability but rather that the *discontinuities* that are periodically produced by the barriers at monitoring dates. The spikes can not decay fast enough in the Crank-Nicolson solution. In [4] Milev and Tagliani show that mathematically, such spurious oscillations stem from the combined effect of two factors, as

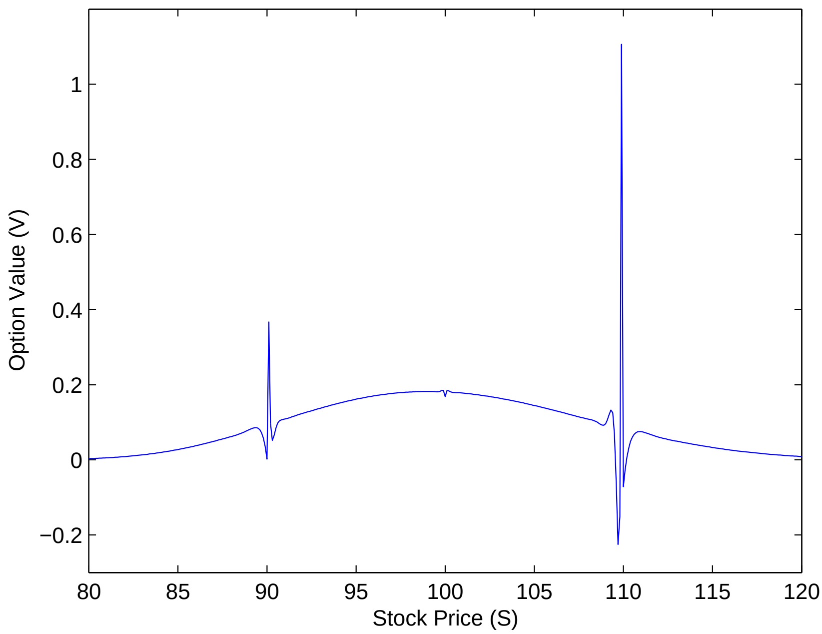

Fig. 1. Option pricing $V(S,t)$ of a discrete double barrier knock-out call option just before the last monitoring date $t_F = 12$. The Crank-Nicolson scheme is applied for steps, respectively $\Delta S = 0.01$ and $\Delta t = 0.001$, and parameters $\sigma = 0.2$, $r = 0.05$, $T = 0.5$, $K = 100$, $L = 90$, $U = 110$

1. *positivity* of the solution $V(S,t)$ not preserved;

2. presence of *negative/complex eigenvalues* in the spectrum of the corresponding matrix originating from the finite difference equation.

Then special finite difference schemes will be investigated where

1. the solution is positive;

2. the spectrum contains only positive eigenvalues.

3. Finite difference approach. As usual, in the finite difference approximation the S-domain is truncated at the value $S_{\max}$, sufficiently large such that computed values are not appreciably affected by the upper boundary. The computational domain $[0, S_{\max}] \times [0, T]$ is discretized by a uniform mesh with steps

ΔS, Δt. Therefore we obtain the nodes S_j and t_n, where $(S_j = j\Delta S, t_n = n\Delta t)$, $j = 0, \dots, M$, $n = 0, \dots, X$ so that $S_{\max} = M\Delta S$, $T = X\Delta t$, X and M integer numbers.

1. The choice of a specific numerical scheme is based on its property of convergence. The requirement rests on the Lax equivalence theorem.
2. The parabolic nature of the Black-Scholes equation ensures that being the initial condition $V(S,0) = (S-K)^+ 1_{[L,U]}(S)$ square-integrable the solution is smooth in the sense that $V(\cdot,t) \in C^\infty(\mathbb{R}^+), \forall t \in (t_{i-1}, t_i^-]$, $i = 1, \dots, F$. Thus rough initial data give rise to smooth solutions in infinitesimal time.

In some cases, as a consequence the solution obeys the maximum principle:

$$\max_{S\in[0,S_{\max}]} \mid V(S,t_1) \mid \geq \max_{S\in[0,S_{\max}]} \mid V(S,t_2) \mid, \qquad t_1 \leq t_2 \tag{6}$$

This inequality means that the maximum value of $V(S,t)$ should not increase as t increases. If that condition is violated then the numerical solution may exhibit *spurious wiggles near sharp gradients*. As a consequence, even though the numerical method converges, it often yields approximate solutions that differ qualitatively from corresponding exact solutions.

3.1. Analysis of the Crank-Nicolson scheme. The Crank-Nicolson scheme provides the difference equation

$$PV^{n+1} = NV^n$$

with P and N the following tridiagonal matrices

$$P = \text{tridiag}\left\{\frac{r}{4}\frac{S_j}{\Delta S} - \left(\frac{\sigma}{2}\frac{S_j}{\Delta S}\right)^2; \frac{1}{\Delta t} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{r}{2}; -\frac{r}{4}\frac{S_j}{\Delta S} - \left(\frac{\sigma}{2}\frac{S_j}{\Delta S}\right)^2\right\}$$

$$N = \text{tridiag}\left\{ -\frac{r}{4}\frac{S_j}{\Delta S} + \left(\frac{\sigma}{2}\frac{S_j}{\Delta S}\right)^2; \frac{1}{\Delta t} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{r}{2}; \frac{r}{4}\frac{S_j}{\Delta S} + \left(\frac{\sigma}{2}\frac{S_j}{\Delta S}\right)^2\right\}$$

a) The case $\sigma^2 > r$

Theorem 3.1. *Under the given hypothesis, the following hold:*

1. *If* $\sigma^2 > r$, *then* $P^{-1} > 0$, *with* $\|P^{-1}\|_\infty \leq \dfrac{1}{\frac{1}{\Delta t} + \frac{r}{2}}$

2. *If* $\sigma^2 > r$ *and* $\Delta t < \dfrac{2}{r + (\sigma M)^2}$, *then*

   i) $N \geq 0$ *with* $\|N\|_\infty = \dfrac{1}{\Delta t} - \dfrac{r}{2} > 0$*;*

   ii) *both positivity and discrete maximum principle are satisfied;*

3. *If* $\sigma^2 > r$ *and* $\Delta t < \dfrac{2}{r + 2(\sigma M)^2}$, *then* $\lambda_i(P^{-1}N) \in (0, 1)$ *are distinct.*

P r o o f. 1) P is an irreducible row diagonally dominant matrix. Then P is an M-matrix and thus $P^{-1} > 0$ with $\|P^{-1}\|_\infty \leq \dfrac{1}{\frac{1}{\Delta t} + \frac{r}{2}}$, see Windisch in [10].

We will state the main result of Windisch (1989). Let $P = [p_{ij}]$ and $r_i = |p_{ii}| - \sum_{j \neq i} |p_{ij}| > 0 \ \forall i$ then the bound $\|P^{-1}\|_\infty \leq \max_i \dfrac{1}{r_i}$ is strict. If $\Delta t \to 0$ then P is strongly row diagonally dominant and equality holds.

2) The matrix N has positive entries with $\|N\|_\infty = \dfrac{1}{\Delta t} - \dfrac{r}{2} > 0$. Then $V^{n+1} = P^{-1}NV^n > 0$ and the numerical solution is positive.

$$\|V_{n+1}\|_\infty = \|P^{-1}N\, V_n\|_\infty = \|P^{-1}\|_\infty \|N\|_\infty \|V_n\|_\infty \leq \frac{\frac{1}{\Delta t} - \frac{r}{2}}{\frac{1}{\Delta t} + \frac{r}{2}} \|V_n\|_\infty \leq \|V_n\|_\infty$$

so that the scheme satisfies the discrete maximum principle.

3) P and N may be written as $P = \dfrac{1}{\Delta t} I + C$ and $N = \dfrac{1}{\Delta t} I - C$, where

$$C = \text{tridiag}\left\{ \frac{r}{4} \frac{S_j}{\Delta S} - \left(\frac{\sigma}{2} \frac{S_j}{\Delta S}\right)^2 ; \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{r}{2} ; -\frac{r}{4} \frac{S_j}{\Delta S} - \left(\frac{\sigma}{2} \frac{S_j}{\Delta S}\right)^2 \right\}$$

Then the matrix C admits M distinct real eigenvalues $\lambda_i(C) \in \left[\dfrac{r}{2}; \dfrac{r}{2} + (\sigma M)^2\right]$, being similar to a Jacobi matrix[1], and $\lambda_i(P^{-1}N) = \dfrac{1 - \Delta t \lambda_i(C)}{1 + \Delta t \lambda_i(C)}$. And from the condition $\Delta t < \dfrac{2}{r + 2(\sigma M)^2}$ it follows $\lambda_i(P^{-1}N) \in (0, 1)$. $\square$

[1]The term *Jacobi matrix* unfortunately has more than one meaning in the literature. Here, we mean a tridiagonal real square matrix $A = \text{tridiag}\{c_i,\ a_i,\ b_i\}$ of order n with off-diagonal elements that have the same sign, i.e. $c_i\, b_{i-1} > 0$, for $i = 2, 3, \ldots, n$. For every Jacobi matrix A there exists a nonsingular diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ such that DAD^{-1} is a Jacobi symmetric matrix. All eigenvalues of a Jacobi matrix are real and distinct, [5].

If the discretization scheme does not preserve the above properties, i.e. the financial condition $\sigma^2 > r$ and the Δt restrictions, then the solution may take *negative values* and *oscillate*. Let us explore the case $\sigma^2 < r$.

Indeed, for a fixed Δt, when $M \to \infty$ (equivalently, $\Delta S \to 0$) we have that the matrix N could have negative elements, and then the numerical solution V^{n+1} could take negative values. This follows from the finite difference equation $V^{n+1} = (P^{-1}N)^n V^0$, where V^0 is a vector representing the initial condition (3) in the the Black-Scholes equation. (In fact V^0 is the payoff for the original problem at T). Such is the case illustrated on Fig. 1 where $\sigma^2 < r$ and the Crank-Nicolson numerical solutions takes negatives values for values of the underlying asset close to the barriers $L = 90$ and $U = 110$.

Now, we will prove that if $\sigma^2 < r$, then the numerical solution could oscillate. Let us denote the spectrum of eigenvalues of the matrix C with $\rho(C) := \max(\lambda_i(C))$. Diminishing the space step ΔS, i.e. $M \to \infty$, having in mind that $\lambda_i(C) \in \left[\frac{r}{2}; \frac{r}{2} + (\sigma M)^2\right]$, then the spectrum $\rho(C) \to \infty$ and $\lambda_i(P^{-1}N) = \dfrac{1 - \Delta t \lambda_i(C)}{1 + \Delta t \lambda_i(C)} \to -1$. We will demonstrate that if $\lambda_i(P^{-1}N) \to -1$ then the Crank-Nicolson numerical solution V^{n+1} may oscillate.

Further, to the M distinct eigenvalues $\lambda_i(P^{-1}N)$ are associated M linearly independent eigenvectors v_i. Such eigenvectors can be used as a basis for the M dimensional space of the initial condition $V^0 = \sum_{j=1}^{M} c_j v_j$, where c_j are appropriate weights. Then

$$V^{n+1} = (P^{-1}N)^n V^0 = (P^{-1}N)^n \sum c_j v_j = \sum c_j (P^{-1}N)^n v_j = \sum c_j \lambda_j^n v_j$$

From $\lambda_j \to -1$ then V^{n+1} oscillates. The Crank-Nicolson solution is affected by spurious oscillations that do not decay quickly as it is shown in Fig. 1. The scheme is applied to the Black-Scholes equation (2) that has discontinuous initial conditions (3)-(5) in case of a *discrete double barrier knock-out call option* with parameters $\sigma = 0.2$ and $r = 0.05$, i.e. the case $\sigma^2 < r$.

b) For the case $\sigma^2 < r$, in [4] Milev and Tagliani have proposed a variant of the Crank-Nicolson scheme that works successfully, i.e. the numerical solution is positive and does not suffer from spurious oscillations.

3.2. Efficient nonstandard semi-implicit scheme. In this section we introduce a less accurate scheme, i.e. $O(\Delta S^2, \Delta t)$, which allows us to choose a more acceptable Δt time step. In the meantime, the scheme prevents spurious oscillations and guarantees a positive solution.

First of all, we discretize the reaction term $-rV$ in equation (2) through a bivariate approximation which involves the values of V_j^{n+1}, V_{j-1}^n and V_{j+1}^n.

By the standard procedure we have

$$V(S,t) = b\,(V_{j-1}^n + V_{j+1}^n) + (1-2b)V_j^{n+1}$$

with discretization error $O(\Delta S^2, \Delta t)$ and b an arbitrary constant to be fixed below. The term $\frac{\partial V}{\partial S}$ is discretized through an explicit centered difference, while $\frac{\partial^2 V}{\partial S^2}$ through an implicit scheme, in order to split the contribution of $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ in two different matrices. The stencil of the scheme nodes is shown on Fig. 2. The scheme is then obtained in a semi-implicit form.

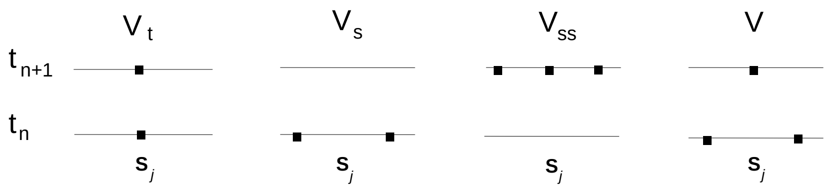

Fig. 2. Involved nodes in the semi-implicit scheme

The corresponding finite difference equation is $PV^{n+1} = NV^n$, with

$$P = \text{tridiag}\Big\{ -\frac{\Delta t}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2;\ 1+\Delta t\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r(1-2b)\right];\ -\frac{\Delta t}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\Big\}$$

$$N = \text{tridiag}\Big\{\Delta t\,\frac{r}{2}(-\frac{S_j}{\Delta S} - 2b)\ ;\ \ 1\ \ ;\ \Delta t\,\frac{r}{2}\,(\frac{S_j}{\Delta S} - 2b)\Big\}$$

By choosing $b = -\frac{1}{2}M$ then

- N has non-negative entries, with $\|N\|_\infty = 1 + r\Delta t M$;
- P is symmetric row diagonally dominant. From Gerschgorin theorem follows $\lambda_i(P) > 0$. Thus, P is positive definite, so that P^{-1} exists.

P is an irreducible diagonally dominant matrix and then P is an M-matrix, so that $P^{-1} > 0$. In addition $\|P^{-1}\|_\infty \leq \dfrac{1}{1+r\Delta t(M+1)}$, (Windisch, [10]).

- By combining $N \geq 0$ and $P^{-1} > 0$, then the solution $V^{n+1} = P^{-1}NV^n$ is positive since $V^0 \geq 0$.
- For the spectral radius $\rho(P^{-1}N)$ of the iteration matrix $P^{-1}N$, we have

$$\rho(P^{-1}N) \leq \|P^{-1}N\|_\infty \leq \|P^{-1}\|_\infty \|N\|_\infty \leq \frac{1 + r\Delta tM}{1 + r\Delta t(M+1)} < 1, \quad \forall\, \Delta t$$

Equivalently,
• the scheme is *unconditionally stable* and via Lax-equivalence theorem,
• convergent being consistent with a local truncation error $O(\Delta S^2, \Delta t)$.
Moreover, we have

$$\|V^{n+1}\|_\infty = \|P^{-1}NV^n\|_\infty \leq \frac{1 + r\Delta tM}{1 + r\Delta t(M+1)}\, \|V^n\|_\infty < \|V^n\|_\infty$$

Then the numerical solution satisfies *unconditionally* the discrete version of the maximum principle and the scheme is unconditionally monotone.

For the eigenvalues of the iteration matrix $P^{-1}N$, the following result holds:

Theorem 3.2. *Under the condition* $\Delta t < \dfrac{1}{rM}$*, then* $P^{-1}N$ *admits* M *real positive and distinct eigenvalues* $\lambda_i(P^{-1}N) \in (0, 1)$.

Proof. The matrix N is diagonally dominant if $1 > -\Delta t\, \dfrac{r}{2}\, 4b$, where $b = -\dfrac{1}{M}$ or, equivalently, $\Delta t < \dfrac{1}{rM}$. Then N is similar to a symmetric positive definite matrix N^{spd}, (Windisch, [10], with $N = D^{-1}N^{spd}D$ and D a diagonal matrix, whose entries are obtained by the off-diagonal entries of N.

From $P^{-1}Nu = \lambda u \Rightarrow DP^{-1}D^{-1}N^{spd}Du = \lambda Du$. Putting $u_1 = Du$, it follows that $N^{spd}u_1 = \lambda(DPD^{-1})u_1$ and

$$\lambda = \frac{u_1^H N^{spd} u_1}{u_1^H (DPD^{-1}) u_1}$$

where H denotes conjugate transpose.

From $u_1^H N^{spd} u_1 > 0$, $u_1^H (DPD^{-1}) u_1 > 0$, being DPD^{-1} similar to P, then λ is real and positive. From which $\lambda_i(P^{-1}N) > 0$, $\forall i$ holds.

The eigenvalue problem $P^{-1}Nu = \lambda u$ admits M eigenvalues. Now we prove such eigenvalues are distinct. We have that $N - \lambda P$ is a tridiagonal matrix, i.e:

$$N - \lambda P = \text{tridiag}\{b_i, a_i, c_i\}$$

with b_i, $c_i > 0$, $\forall i$. If $D = \text{diag}\left\{1, \dfrac{b_1}{c_1}, \dfrac{b_1 b_2}{c_1 c_2}, \dots, \dfrac{b_1 \dots b_{M-1}}{c_1 \dots c_{M-1}}\right\}$ then

$$N - \lambda P = D^{-1}\,\text{tridiag}\{\gamma_i, a_i, \gamma_i\} D$$

with $\gamma_i = \sqrt{b_i c_i}$, $i = 1, \dots, M-1$, (see Ortega, p. 113, [5]).

The matrix $J = \text{tridiag}\{\gamma_i, a_i, \gamma_i\}$ is a Jacobi matrix which admits M distinct eigenvalues $\lambda_1^{(J)}(\lambda), \dots, \lambda_M^{(J)}(\lambda)$. Then J may be diagonalized so that

$$J = S^{-1}\text{diag}\{\lambda_1^{(J)}(\lambda), \dots, \lambda_M^{(J)}(\lambda)\} S$$

where S is the eigenvectors matrix. Then

$$0 = \det(N - \lambda P) = \det D^{-1} S^{-1}\,\text{diag}\{\lambda_1^{(J)}(\lambda), \dots, \lambda_M^{(J)}(\lambda)\} S D = \prod_{j=1}^{M} \lambda_j^{(J)}(\lambda)$$

leads to a set of M equations

$$\lambda_j^{(J)}(\lambda) = 0, \qquad j = 1, \dots, M$$

The set of equations admits M distinct solutions $\lambda_1, \dots, \lambda_M$, being $\lambda_j^{(J)}(\lambda)$ distinct. So $\lambda_1, \dots, \lambda_M$ are the distinct eigenvalues of $P^{-1}N$. $\square$

The solution accuracy is defined by analyzing the error component introduced by the bivariate approximation of the reaction term $-rV$. By Taylor expansion about the time level $n\Delta t$ we have

$$(1 - 2b)\, V_j^{n+1} + b\,(V_{j-1}^n + V_{j+1}^n) = V_j^n + \Delta t(1 + M)\,\frac{\partial V}{\partial t} + O(\Delta S^2) + O(\Delta t^2)$$

Then the scheme is consistent with equation (2) with a local truncation error $O(\Delta t) + O(\Delta S^2)$.

When M assumes large values the error term $r\Delta t(1+M)$ becomes influent and under the only constraint $\Delta t < \dfrac{1}{rM}$ of Theorem 3.2 the scheme provides a poor solution. Then an accurate solution requires

$$\Delta t \ll \frac{1}{r\,(1 + M)} \tag{7}$$

Then, under (7), the proposed scheme guarantees an accurate solution being positivity-preserving and free of spurious oscillations.

4. Numerical results. We will present numerical results for some of the most explored examples in literature for discrete barrier knock-out options that are discretely monitored, see Wade *et al.* in [9] and Zvan *et al.* in [11].

Example 4.1. Let us price a discrete double barrier knock-out call option having a discontinuous payoff defined by conditions (3)-(5) and for which the strike price is 100, the volatility is 25% per annum, the option has six months remaining to maturity, the risk-free rate is 5% per annum (compounded continuously), the two barriers are placed, respectively at 95 and at 110.

The numerical results and parameters are presented in Table 1 and Fig. 3.

Values of underlying asset S_0	Standard implicit scheme	Crank-Nicolson scheme	Duffy implicit scheme	Semi-implicit scheme	Monte Carlo method (standard error) 10^8 – asset paths
95	0.17564	0.17561	0.17315	0.17398	0.17359 (0.00054)
95.0001	0.17904	0.17963	0.17395	0.17412	0.17486 (0.00064)
95.5	0.18322	0.18324	0.18109	0.18152	0.18291 (0.00066)
99.5	0.22818	0.22813	0.22819	0.22902	0.22923 (0.00073)
100	0.23123	0.23122	0.23137	0.23171	0.23263 (0.00036)
100.5	0.23359	0.23361	0.23386	0.23246	0.23410 (0.00073)
109.5	0.17582	0.17583	0.17323	0.17326	0.17426 (0.00063)
109.9999	0.16982	0.16989	0.16656	0.16719	0.16732 (0.00062)
110	0.16906	0.16912	0.16616	0.16703	0.16712 (0.00042)

Table 1. Prices of a discrete double knock-out call option monitored 5 times. The price of the underlying asset is S_0, $K = 100$, $\sigma = 0.25$, $T = 0.5$, $r = 0.05$, $L = 95$, $U = 110$. The Crank-Nicolson, fully implicit and Duffy scheme are applied with $\Delta S = 0.05$, $\Delta t = 0.00001$, while the semi-implicit scheme for $\Delta S = 0.05$, $\Delta t = 0.001$, $S_{max} = 200$

We have applied the Crank-Nicolson scheme and the proposed semi-implicit scheme, presented, respectively in Sections 3.1 and 3.2. The results are compared with those obtained by other standard numerical methods in Finance such as standard fully implicit scheme, the Crank-Nicolson method, the exponentially fitted finite difference scheme of Duffy in [2], and finally to the Monte Carlo simulations. The results in Table 1 and Fig. 3 show that the proposed semi-implicit scheme gives results that are in very good agreement with the Monte Carlo method which could be taken as a bench mark.

The computational results are as expected, because the full implicit scheme and the Duffy implicit schemes are first order accurate, i.e. $O(\Delta S, \Delta t)$.

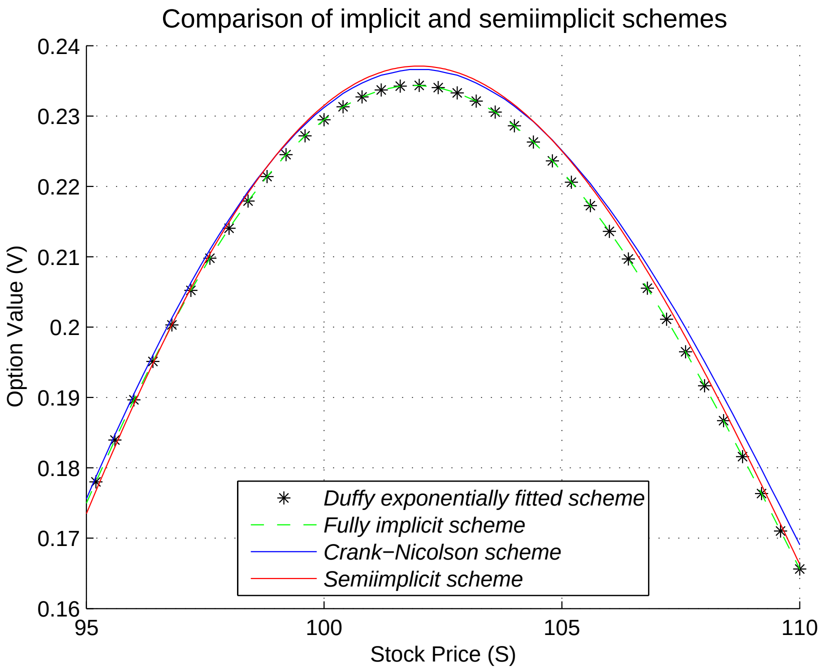

Fig. 3. A comparison of different implicit and semi-implicit finite difference schemes applied to the Black-Scholes equation in case of a discrete double barrier knock-out call option monitored 5 times, $K = 100$, $\sigma = 0.25$, $T = 0.5$, $r = 0.05$, $L = 95$, $U = 110$

Although the Crank-Nicolson scheme is second order accurate both in time and space, i.e. $O(\Delta S^2, \Delta t^2)$, in order to be used, the scheme should be applied with a *prohibitively small time step*, as it is showed in Section 3.1.

We can note the following advantages of the proposed *semi-implicit scheme* over the other applied finite difference schemes in Table 1:

• The proposed semi-implicit scheme is *second order accurate in space*, i.e. $O(\Delta S^2, \Delta t)$, and it is more accurate than standard first order accurate schemes in case the schemes are applied for a fixed time step.

• The semi-implicit scheme is *positivity-preserving* and *free of oscillations.*

• Moreover, to preserve these properties, it is sufficient to be applied for $\Delta t < \dfrac{1}{rM}$ that is much bigger than the *prohibitively small* time step restriction $\Delta t < \dfrac{2}{r + 2(\sigma M)^2}$ for the Crank Nicolson scheme, i.e. M-times smaller, as we have proved, respectively in Theorem 3.2 and Theorem 3.1.

• The semi-implicit scheme works successfully both for $\sigma^2 < r$ and $\sigma^2 > r$.

• The semi-implicit scheme is as *fast* as the other first order accurate schemes, but it is *much faster* than the Crank-Nicolson scheme.

• We can note that the close distance of each of the barriers to the strike price is not an obstacle for the presented *semi-implicit scheme* for obtaining a smooth numerical solution. This is one of the frequently met practical problems applying finite difference schemes in Finance because usually *oscillations* derive from an inaccurate approximation of the *very sharp gradient produced by the knock-out clause*, generating an error that is damped out very slowly, (Tagliani *et al.*, [7]), (Wade *et al.*, [9].).

Thus, the semi-implicit scheme turns out to be very efficient because it gives both highly accurate results, satisfies all financial requirements of the option contract and differs with a quick computational time.

5. Discussion and conclusions. We have presented an alternative scheme to the Crank-Nicolson one that do not suffer from spurious oscillations originating from *discontinuous* boundary conditions. The proposed *semi-implicit scheme* has lower accuracy, i.e. $O(\Delta S^2, \Delta t)$, but requires *less restrictive* conditions on the time step.

We have used a *non-standard discretization technique of the reaction term.* The scheme is conditionally stable but nevertheless it gives highly accurate results and guarantees the absence of spurious oscillations close to discontinuities due to the fact that the scheme has an iteration matrix characterized by real and positive spectrum which allows a fast damping of errors of any order. Moreover, in contrast to most frequently used schemes in computational Finance such as the Crank-Nicolson method, the successful application of the proposed scheme is independent of the financial parameters such as volatility and interest rate, i.e. it is unaffected for *low values of the volatility.*

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