Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **36** (2010), 137–148

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

TOTALLY UMBILICAL PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY COSYMPLECTIC MANIFOLD

M. A. Khan, Siraj Uddin, Khushwant Singh

Communicated by O. Muškarov

Abstract. In the present note we study totally umbilical pseudo-slant submanifolds of a nearly cosymplectic manifold. We have obtained a classification theorem for totally umbilical pseudo-slant submanifolds of a nearly cosymplectic manifold.

1. Introduction. Slant immersions in complex geometry were defined by B. Y. Chen as a natural generalization of both holomorphic and totally real immersions [4, 5]. Recently, A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [7] and slant submanifold in Sasakian manifolds have been studied by J. L. Cabrerizo et al. [2].

N. Papaghiuc [8] introduced the notion of semi-slant submanifold of an almost Hermitian manifolds, which is in fact a generalization of CR-submanifolds.

2010 *Mathematics Subject Classification:* 53C40, 53B25.

Key words: Slant submanifold, pseudo-slant submanifold, totally umbilical, nearly cosymplectic.

Recently, Carriazo [3] defined and studied bi-slant immersion in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds. Later on V. A. Khan and M. A. Khan [6] studied pseudo-slant submanifolds of a Sasakian manifold. The purpose of the present paper is to study totally umbilical pseudo-slant submanifolds of a nearly cosymplectic manifold.

In section 2, we review and collect some necessary results. In section 3, we work out integrability condition of distributions involved in this setting and in the last section 4, we obtained a classification theorem for totally umbilical pseudo-slant submanifold of a nearly cosymplectic manifold.

2. Preliminaries. In this section, we give some terminology and notations used throughout this paper. We recall some necessary facts and formulas from the theory of almost contact metric manifolds and their submanifolds.

Let $\bar{M}$ be an odd-dimensional manifold. An almost contact structure on M is an quadruple of tensor fields (ϕ, ξ, η, g), where ϕ is an endomorphism, ξ is a vector field, η is a one form and g is a Riemannian metric, respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi) = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2}$$

for any vector field $X, Y \in T\bar{M}$. It is easy to see that

$$g(\phi X, Y) = -g(X, \phi Y).$$

A $(2n+1)$-dimensional manifold $\bar{M}$ together with an almost contact metric structure (ϕ, ξ, η, g) is said to be an *almost contact metric manifold.* The fundamental 2-form Φ of $\bar{M}$ is defined by

$$\Phi(X, Y) = g(X, \phi Y), \quad \forall X, Y \in T\bar{M}.$$

The almost contact metric manifold $\bar{M}$ is called a *nearly cosymplectic* manifold if it satisfy the following condition [1]

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0. \tag{3}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential and normal to M, respectively and ∇ be the induced Levi-Civita connection on TM, then the Gauss and Weingarten formulas are given by

$$(4)\qquad (a)\ \bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad (b)\ \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X,Y), V).$$

The mean curvature vector H on M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M.

A submanifold M of a Riemannian manifold $\bar{M}$ is called *totally umbilical* if

$$(5)\qquad h(X,Y) = g(X,Y)H.$$

If $H = 0$ then the submanifold is said to be *totally geodesic* submanifold in $\bar{M}$. For any $x \in M$ and $X \in T_xM$, we write

$$(6)\qquad \phi X = TX + NX,$$

where $TX \in T_xM$ and $NX \in T_x^{\perp}M$. Similarly, for $V \in T_x^{\perp}M$, we have

$$(7)\qquad \phi V = tV + fV,$$

where tV (resp. fV) is the tangential component (resp. normal component) of ϕV. The covariant derivative of the morphisms T, N, t and f are defined respectively as

$$(8)\qquad (\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y$$

$$(9)\qquad (\bar{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y$$

$$(10)\qquad (\bar{\nabla}_X t)V = \nabla_X tN - t\nabla^{\perp}_X N$$

$$(\bar{\nabla}_X f)V = \nabla^{\perp}_X fV - f\nabla^{\perp}_X \xi \tag{11}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$.

Now, for any $X, Y \in TM$, let us denote the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$ by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ respectively. Then by an easy computation, we obtain the following formulae

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{NY}X - th(X, Y), \tag{12}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X N)Y + h(X, TY) - fh(X, Y). \tag{13}$$

Similarly, for $V \in T^{\perp}M$, denoting the tangential and normal parts of $(\bar{\nabla}_X J)V$ by $\mathcal{P}_X V$ and $\mathcal{Q}_X V$ respectively, we find that

$$\mathcal{P}_X V = (\bar{\nabla}_X t)V + TA_V X - A_{fV}X, \tag{14}$$

$$\mathcal{Q}_X V = (\bar{\nabla}_X f)V + h(tV, X) + NA_V X. \tag{15}$$

Now, from equation (3) we have

$$(a)\ \ \mathcal{P}_X Y + \mathcal{P}_Y X = 0 \qquad (b)\ \ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \tag{16}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a *slant submanifold* if for any $x \in M$ and $X \in T_xM - \langle \xi \rangle$, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M in $\bar{M}$. The tangent bundle TM of M is decomposed as

$$TM = D \oplus \langle \xi \rangle$$

where the orthogonal complementary distribution D of $\langle \xi \rangle$ is known as the *slant distribution* on M.

We say that M is a *pseudo-slant submanifold* of an almost contact metric manifold $\bar{M}$, if there exist two orthogonal distributions $D^{\perp}$ and D_{θ} on M such that

(i) $TM = D^{\perp} \oplus D_{\theta} \oplus \langle \xi \rangle$.

(ii) The distribution $D^{\perp}$ is anti-invariant *i.e.*, $\phi D^{\perp} \subseteq T^{\perp}M$.

(iii) The distribution D_{θ} is slant with slant angle $\theta \neq \pi/2$.

From the above definition it is clear that if $\theta = 0$, the pseudo-slant submanifold become *semi-invariant* submanifold.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$ then, in the case of pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \mu \oplus ND^{\perp} \oplus ND_{\theta}. \tag{17}$$

We have the following result in the setting of almost contact manifolds given by Cabrerizo et.al [2].

Theorem 1 [2]**.** *Let M be a submanifold of an almost contact metric manifold $\bar{M}$. Then, M is slant submanifold of $\bar{M}$ if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{18}$$

Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Hence, for a slant submanifold, we have

$$g(TX, TX) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{19}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{20}$$

for any $X, Y \in TM$.

3. Integrability conditions of involved distributions.

In this section we shall see the integrability of involved distributions.

Proposition 2. *Let M be a pseudo-slant submanifold of a nearly cosymplectic manifold then the anti-invariant distribution $D^{\perp} \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(A_{\phi X}Y - A_{\phi Y}X, \phi Z) = 2g((\bar{\nabla}_X\phi)Y, \phi Z)$$

for all $X, Y \in D^{\perp} \oplus \langle\xi\rangle$ and $Z \in D_{\theta}$.

P r o o f. For any $X, Y \in D^{\perp} \oplus \langle\xi\rangle$, we have

$$\bar{\nabla}_X\phi Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y. \tag{21}$$

Also, we know that

$$\bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi\bar{\nabla}_X Y = (\bar{\nabla}_X\phi)Y + \phi\nabla_X Y + \phi h(X,Y) \tag{22}$$

From equations (21) and (22), we obtain

$$-A_{\phi Y}X + \nabla^{\perp}_X\phi Y = (\bar{\nabla}_X\phi)Y + \phi\nabla_X Y + \phi h(X,Y),$$

or

$$(\bar{\nabla}_X\phi)Y + \phi\nabla_X Y + \phi h(X,Y) + A_{\phi Y}X - \nabla^{\perp}_X\phi Y = 0. \tag{23}$$

Interchanging X and Y, we get

$$(\bar{\nabla}_Y\phi)X + \phi\nabla_Y X + \phi h(Y,X) + A_{\phi X}Y - \nabla^{\perp}_Y\phi X = 0. \tag{24}$$

Then from equations (23) and (24), we obtain

$$2(\bar{\nabla}_X\phi)Y + \phi[X,Y] - A_{\phi X}Y + A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla^{\perp}_Y\phi X = 0.$$

That is

$$\phi[X,Y] = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - 2(\bar{\nabla}_X\phi)Y.$$

Taking the inner product with ϕZ, where $Z \in D_\theta$

$$g(\phi[X,Y],\phi Z) = g(A_{\phi X}Y - A_{\phi Y}X, \phi Z) - 2g((\bar{\nabla}_X\phi)Y, \phi Z).$$

Using (2), the above equation takes the form

$$g([X,Y],Z) = g(A_{\phi X}Y - A_{\phi Y}X, \phi Z) - 2g((\bar{\nabla}_X\phi)Y, \phi Z) + \eta([X,Y])\eta(Z). \tag{25}$$

Thus the assertion follows from (25). □

Proposition 3. *Let M be a pseudo-slant submanifold of a nearly cosymplectic manifold then the slant distribution $D_\theta \oplus \langle\xi\rangle$ is integrable if and only if*

$$h(Z,TW) - h(W,TZ) - \nabla^{\perp}_W NZ + \nabla^{\perp}_Z NW + 2g((\bar{\nabla}_W\phi)Z \in ND_\theta$$

for any $Z, W \in D_\theta \oplus \langle\xi\rangle$.

Proof. For any $Z, W \in D_\theta \oplus \langle\xi\rangle$ and $X \in D^{\perp}$, we have

$$\phi[Z,W] = \phi\bar{\nabla}_Z W - \phi\bar{\nabla}_W Z.$$

Then from (8), we obtain

$$\phi[Z,W] = \bar{\nabla}_Z \phi W - (\bar{\nabla}_Z \phi)W + (\bar{\nabla}_W \phi)Z - \bar{\nabla}_W \phi Z.$$

Using (6), we get

$$\phi[Z,W] = 2(\bar{\nabla}_W \phi)Z + \bar{\nabla}_Z TW + \bar{\nabla}_Z NW - \bar{\nabla}_W TZ - \bar{\nabla}_W NZ.$$

Thus by Gauss and Wiengarten formulae we obtain

$$\phi[Z,W] = 2(\bar{\nabla}_W \phi)Z + \nabla_Z TW + h(Z,TW) - A_{NW}Z + \nabla^{\perp}_Z NW$$
$$-\nabla_W TZ - h(W,TZ) + A_{NZ}W - \nabla^{\perp}_W NZ.$$

Taking the inner product with ϕX, for any $X \in D^{\perp}$

$$g(\phi[Z,W], \phi X) = g(2(\bar{\nabla}_W \phi)Z, \phi X)$$
$$+g(h(Z,TW) - h(W,TZ) - \nabla^{\perp}_W NZ + \nabla^{\perp}_Z NW, \phi X)$$

Using (2), we get

$$g([Z,W], X) = 2g((\bar{\nabla}_W \phi)Z, \phi X) + \eta([Z,W])\eta(X)$$
$$+g(h(Z,TW) - h(W,TZ) - \nabla^{\perp}_W NZ + \nabla^{\perp}_Z NW, \phi X)$$

Since ξ is tangential to D_θ then the second term of right hand side of the above equation is zero, hence we obtain

$$g([Z,W], X) = 2g((\bar{\nabla}_W \phi)Z, \phi X) \tag{26}$$
$$+g(h(Z,TW) - h(W,TZ) - \nabla^{\perp}_W NZ + \nabla^{\perp}_Z NW, \phi X).$$

Thus the assertion follows from the above equation. □

4. Totally umbilical pseudo-slant submanifolds. Through out the section we consider M as a totally umbilical pseudo-slant submanifold of nearly cosymplectic manifold $\bar{M}$. We shall prove that the totally umbilical pseudo-slant submanifold is totally geodesic under some conditions. Finally, we obtain a characterization result for totally umbilical pseudo-slant submanifold of nearly cosymplectic manifold.

Theorem 4. *Let M be a totally umbilical pseudo-slant submanifold of a nearly cosymplectic manifold, then the following conditions are equivalent*

(a) *M has a nearly cosymplectic character (T, g) on slant distribution, i.e. $(\bar{\nabla}_X T)X = 0$, for all $X \in D_\theta \oplus \langle\xi\rangle$.*

(b) *The mean curvature vector $H \in \mu$.*

P r o o f. For any $X \in D_\theta \oplus \langle\xi\rangle$

$$h(X, TX) = 0$$

and

$$h(X, TY) + h(Y, TX) = 0.$$

From (4) the above equation reduced to

$$0 = \phi(\nabla_X Y + h(X, Y)) + A_{NY}X - \nabla^{\perp}_X NY - \nabla_X TY$$
$$+\phi(\nabla_Y X + h(X, Y)) + A_{NX}Y - \nabla^{\perp}_Y NX - \nabla_Y TX.$$

Using equation (6) and equating the tangential components

$$0 = T(\nabla_X Y + \nabla_Y X) - \nabla_X TY - \nabla_Y TX + 2th(X, Y) + A_{NY}X + A_{NX}Y.$$

In particular,

$$0 = T\nabla_X X - \nabla_X TX + g(X, Y)tH + A_{NX}X.$$

As M is totally umbilical then the term $A_{NX}X$ becomes $Xg(H, NX)$, then above equation reduced to

$$0 = (\bar{\nabla}_X T)X + g(X, Y)tH + Xg(H, NX). \tag{27}$$

Thus the assertion follows from above equation. □

Theorem 5. *Let M be a totally umbilical pseudo-slant submanifold of nearly cosymplectic manifold $\bar{M}$ with parallelism of the endomorphism f. Then M is totally geodesic submanifold of $\bar{M}$.*

P r o o f. For any $X, Y \in D_\theta\langle\xi\rangle$, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi)Y + \phi\bar{\nabla}_X Y.$$

Now, by equations (4), (5) and (6) the above equation become

$$\nabla_X \phi Y + g(X, TY)H - A_{NY}X + \nabla^{\perp}_X NY = \mathcal{P}_X Y + \mathcal{Q}_X Y + T\nabla_X Y + N\nabla_X Y + g(X, Y)\phi H.$$

Taking the inner product with ϕH, we obtain

$$g(\nabla^{\perp}_X NY, \phi H) = g(\mathcal{Q}_X Y, \phi H) + g(N\nabla_X Y, \phi H) + g(X,Y)g(H,H).$$

The above equation takes the form

$$g(\bar{\nabla}_X NY, \phi H) = g(\mathcal{Q}_X Y, \phi H) + g(X,Y)g(H,H). \tag{28}$$

Similarly, we have

$$g(\bar{\nabla}_Y NX, \phi H) = g(\mathcal{Q}_Y X, \phi H) + g(X,Y)g(H,H). \tag{29}$$

Adding equations (28) and (29) and applying (16) (b) which yields

$$g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, \phi H) = 2g(X,Y)g(H,H). \tag{30}$$

Now for any $X \in D_\theta \oplus \langle\xi\rangle$, we have

$$\bar{\nabla}_X \phi H = (\bar{\nabla}_X \phi)H + \phi\bar{\nabla}_X H.$$

That is

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = \mathcal{P}_X H + \mathcal{Q}_X H - TA_H X - NA_H X + t\nabla^{\perp}_X H + f\nabla^{\perp}_X H.$$

Taking the inner product with NY for any $Y \in D_\theta \oplus \langle\xi\rangle$ and in view of the fact that $f\nabla^{\perp}_X H \in \mu$ the above equation takes the form

$$g(\nabla^{\perp}_X \phi H, NY) = -g(NA_H X, NY) + g(\mathcal{Q}_X H, NY). \tag{31}$$

Then from (20), we get

$$g(\nabla^{\perp}_X \phi H, NY) = -\sin^2\theta(g(X,Y)g(H,H) - \eta(A_H X)\eta(Y)) + g(\mathcal{Q}_X H, NY).$$

Since $\bar{\nabla}$ is a metric connection and NY and ϕH are orthogonal, we derive

$$g(\bar{\nabla}_X NY, \phi H) = \sin^2\theta g(X,Y)g(H,H) - g(\mathcal{Q}_X H, NY). \tag{32}$$

Similarly, we obtain

$$g(\bar{\nabla}_Y NX, \phi H) = \sin^2\theta g(X,Y)g(H,H) - g(\mathcal{Q}_Y H, NX). \tag{33}$$

Thus from (32) and (33), we get

$$g(\bar{\nabla}_X NY + \bar{\nabla}_Y NX, \phi H) = 2\sin^2\theta g(X,Y)g(H,H) - g(\mathcal{Q}_X H, NY) - g(\mathcal{Q}_Y H, NX). \tag{34}$$

From equations (30) and (34), we derive

$$2g(X,Y)g(H,H) = 2\sin^2\theta g(X,Y)g(H,H) - g(\mathcal{Q}_X H, NY) - g(\mathcal{Q}_Y H, NX).$$

That is,

$$2\cos^2\theta g(X,Y)g(H,H) + g(\mathcal{Q}_X H, NY) + g(\mathcal{Q}_Y H, NX) = 0.$$

In view of the equation (15), Theorem 4 and the assumption that f is parallel the above equation reduces to

$$g(X,Y)g(H,H) = 0. \tag{35}$$

Since g is Riemannian metric then from equation (35), it follows that $H = 0$, i.e., M is totally geodesic submanifold. Hence theorem is proved. □

Let us consider M as a pseudo-slant submanifold of nearly cosymplectic manifold $\bar{M}$, then for any $U \in TM$, we have $(\bar{\nabla}_U \phi)U = 0$, using this fact, we get

$$(\bar{\nabla}_Z \phi)Z = 0 \tag{36}$$

for any $Z \in D^\perp$. Therefore the tangential and normal parts of the above equation are $\mathcal{P}_Z Z = 0$ and $\mathcal{Q}_Z Z = 0$, respectively. From equation (12) and tangential components of equation (36), we obtain

$$\mathcal{P}_Z Z = 0 = (\bar{\nabla}_Z T)Z - A_{NZ}Z - th(Z,Z)$$

or,

$$(\bar{\nabla}_Z T)Z = A_{NZ}Z + th(Z,Z).$$

That is

$$T\nabla_Z Z = -g(H, NZ)Z - |Z|^2 tH.$$

Taking the inner product with $W \in D^\perp$, we get

$$g(H, NZ)g(Z,W) + |Z|^2 g(tH, W) = 0. \tag{37}$$

Thus the equation (37) has solution if one of the following holds:

(a) $dim D^\perp = 1$,

(b) $H \in \mu$.

Now, we are in position to prove our main result.

Theorem 6. *Let M be a totally umbilical pseudo-slant submanifold of a nearly cosymplectic manifold $\bar{M}$ with slant and anti-invariant distributions D_θ and $D^\perp$, respectively. Then at least one of the following statement is true*

(i) *M is totally real submanifold,*

(ii) *M has nearly cosymplectic character (T, g) on the slant distribution D_θ,*

(iii) *M is totally geodesic when f is parallel,*

(iv) $\dim D^\perp = 1$.

Proof. If $D_\theta = \{0\}$, then by definition M is totally real which is case (i). If $D_\theta \neq \{0\}$ and $H \in \mu$, then by Theorem 4, (T, g) has nearly cosymplectic character on D_θ, if moreover $H \in \mu$ and f is parallel then by Theorem 5, M is totally geodesic. Finally, if $H \notin \mu$, then equation (37) has solution if either $dim D^\perp = 1$ which is the case (iv), this proves the theorem completely. □

Acknowledgement. The authors are thankful to referees and professor Angelina Chin Yan Mui for their valuable suggestions and comments.

REFERENCES

[1] D. E. Blair. Contact manifolds in Riemannian geometry. Lecture Notes in Mathematics, vol. **509**, Berlin–Heidelberg–New York, Springer-Verlag. 1976.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, M. Fernández. Slant submanifolds in Sasakian manifolds. *Glasg. Math. J.* **42**, *1* (2000), 125–138.

[3] A. Carriazo. New developments in slant submanifolds theory. Narosa Publishing House, New Delhi, India, 2002.

[4] B. Y. Chen. Geometry of slant submanifolds. Katholieke Universiteit Leuven, 1990.

[5] B. Y. Chen. Slant immersions. *Bull. Austral. Math. Soc.* **41** (1990), 135–147.

[6] V. A. Khan, M. A. Khan. Pseudo-Slant Submanifolds of a Sasakian Manifold. *Indian J. Pure Appl. Math.* **38** (2007), 31–42.

[7] A. Lotta. Slant Submanifolds in Contact Geometry. *Bull. Math. Soc. Romania* **39** (1996), 183–198.

[8] N. Papaghuic. Semi-slant submanifolds of Kahlerian manifold. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* vol. **XL**, *1* (1994), 55–61.

M. A. Khan, Khushwant Singh
School of Mathematics and Computer Applications
Thapar University
Patiala-147 004, India
e-mails: meraj79@gmail.com, khushwantchahil@gmail.com

Siraj Uddin
Institute of Mathematical Sciences
Faculty of Sciences
University of Malaya
50603 Kuala Lumpur, Malaysia
e-mail: siraj.ch@gmail.com

Received May 5, 2010