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UNITS OF $\mathbb{F}_{5^k} D_{10}$

Joe Gildea

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Abstract. The Structure of the Unit Group of the Group Algebra of the group D_{10} over any field of characteristic 5 is established in terms of split extensions of cyclic groups.

1. Introduction. Let KG denote the group algebra KG of the group G over the field K and $\mathcal{U}(KG)$ denote the unit group of KG. The homomorphism $\varepsilon : KG \longrightarrow K$ given by $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$ is called the augmentation mapping of KG. It is well known that $\mathcal{U}(KG) = V(KG) \times \mathcal{U}(K)$ where $V(KG)$ is the group of units of augmentation 1. See [6] for further details on group algebras.

Let $\mathbb{F}_{p^k}$ is the Galois field of p^k-elements and D_{2p^m} be the dihedral group of order $2p^m$ where p is a prime and $m \in \mathbb{N}_0$. The structure of $\mathcal{U}(\mathbb{F}_{3^k} D_6)$ is established in terms of split extensions of elementary abelian groups in [1]. In [3], the order of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})$ is determined to be $p^{2k(p^m-1)}(p^k-1)^2$.

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Let $J(KG)$ denote the Jacobson radical of KG and $Z(G)$ denote the center of G. It is shown that $\mathcal{V}_1$ and $\mathcal{V}_1/Z(\mathcal{V}_1)$ are elementary abelian 3-groups where $\mathcal{V}_1 = 1 + J(\mathbb{F}_{3^k}D_6)$ in [7]. Additionally in [5], it is shown that $\mathcal{U}(\mathbb{F}_{5^k}D_{10})/\mathcal{V}_2 \cong {C_{5^k-1}}^2$, $\mathcal{V}_2$ is nilpotent of class 4 and $Z(\mathcal{V}_2) \cong {C_5}^{3k}$ where $\mathcal{V}_2 = 1 + J(\mathbb{F}_{5^k}D_{10})$. Our main result is:

Theorem 1. $V(\mathbb{F}_{5^k}D_{10}) \cong (({C_5}^{5k} \rtimes {C_5}^{2k}) \rtimes {C_5}^{k}) \rtimes C_{5^k-1}$.

Let C_n be the cyclic group of order n and $M_n(R)$ be the ring of $n \times n$ matrices over a ring R. Define a circulant matrix over R to be

$$\mathrm{circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix}$$

where $a_i \in R$. For further details on circulant matrices see Davis [2].

If $G = \{g_1, \ldots, g_n\}$, then denote by $M(G)$ the matrix $({g_i}^{-1}g_j)$ where $i, j = 1, \ldots, n$. Similarly, if $w = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$ where R is a ring, then denote by $M(RG, w)$ the matrix $(\alpha_{{g_i}^{-1}g_j})$, which is called the RG-matrix of w.

Theorem 2 (see [4]). *Let G be a finite group of order n. There is a ring isomorphism between RG and the $n \times n$ G-matrices over R, which is given by $\sigma : w \mapsto M(RG, w)$.*

We fix the presentation of the dihedral group,

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yx = x^{-1}y \rangle.$$

Let $\kappa = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=0}^{n-1} b_j x^j y \in \mathbb{F}_{p^k}D_{2n}$ where $a_i, b_j \in \mathbb{F}_{p^k}$ and p is a prime, then

$$\sigma(\kappa) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$$

where $A = \mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ and $B = \mathrm{circ}(b_0, b_1, \ldots, b_{n-1})$.

Proposition 1 (see [3]). *Let* $A = \text{circ}\,(a_0, a_2, \dots, a_{p^m-1})$*, where* $a_i \in \mathbb{F}_{p^k}$*, p is a prime and* $m \in \mathbb{N}_0$*. Then*

$$\det(A) = \sum_{i=0}^{p^m-1} {a_i}^{p^m}.$$

2. The main result.

P r o o f. Define the group epimorphism $\theta : \mathcal{U}(\mathbb{F}_{5^k}D_{10}) \longrightarrow \mathcal{U}(\mathbb{F}_{5^k}C_2)$ given by

$$\sum_{i=0}^{4} a_i x^i + \sum_{j=0}^{4} b_j x^j y \longmapsto \sum_{i=0}^{4} a_i + \sum_{j=0}^{4} b_j\, \overline{y}.$$

Let $\psi : \mathcal{U}(\mathbb{F}_{5^k}C_2) \longrightarrow \mathcal{U}(\mathbb{F}_{5^k}D_{10})$ be the group homomorphism defined by $a+b\overline{y} \mapsto a + by$. Then $\theta \circ \psi(a + b\overline{y}) = \theta\,(a + by) = a + b\overline{y}$. Therefore $\mathcal{U}(\mathbb{F}_{5^k}D_{10})$ is a split extension of $\mathcal{U}(\mathbb{F}_{5^k}C_2)$ by $\ker(\theta)$. Thus $\mathcal{U}(\mathbb{F}_{5^k}D_{10}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{5^k}C_2)$ where $H = \ker(\theta)$. Let $\alpha = \sum_{i=0}^{4} a_i x^i + \sum_{j=0}^{4} b_j x^j y \in \mathcal{U}(\mathbb{F}_{5^k}D_{10})$ where $a_i, b_j \in \mathbb{F}_{5^k}$. Now $\alpha \in H$ if and only if $\sum_{i=0}^{4} a_i = 1$ and $\sum_{j=0}^{4} b_j = 0$. Thus $|H| = (5^{4k})^2 = 5^{8k}$. We shall split the proof in several lemmas.

Lemma 1. *H has exponent 5.*

P r o o f. If $\alpha = 1 + \sum_{i=1}^{4}[(-a_i) + a_i x^i] + \sum_{j=1}^{4}[(-b_j)y + b_j x^j y] \in H$, then

$$(\sigma(\alpha))^5 = \begin{pmatrix} A^5 & 0 \\ 0 & (A^T)^5 \end{pmatrix}$$

where $A = \text{circ}\left(1 + \sum_{i=1}^{4}(-a_i), a_1, a_2, a_3, a_4\right)$ and $a_i, b_j \in \mathbb{F}_{5^k}$. Using Proposition 1,

$$A^5 = \left(\left(1 + \sum_{i=1}^{4}(-a_i)\right)^5 + \sum_{i=1}^{4}(a_i)^5\right) I_5 = I_5. \qquad \square$$

Lemma 2. *Let T be the set of elements H of the form $1 + r\sum_{i=0}^{4} ix^i y$ where $r \in \mathbb{F}_{5^k}$. Then $T \cong {C_5}^k$.*

P r o o f. Let $\alpha = 1 + r\sum_{i=0}^{4} ix^i y \in T$ and $\beta = 1 + s\sum_{i=0}^{4} ix^i y \in T$ where $r, s \in \mathbb{F}_{5^k}$. Then

$$\alpha\beta = 1 + (r+s)\sum_{i=0}^{4} ix^i y + rs\left(\sum_{i=0}^{4} ix^i y\right)^2.$$

Now

$$\begin{aligned}
\left(\sum_{i=0}^{4} ix^i y\right)^2 &= \left(\sum_{i=0}^{4} ix^i y\right)\left(-y\sum_{i=0}^{4} ix^i\right) \\
&= -\left(\sum_{i=0}^{4} ix^i\right)^2 \\
&= -\left[(1-x)(x)(1+3x+x^2)\right]^2 \\
&= -(1-x)^6 x^2 \\
&= 0 \quad \text{since } (1-x)^5 = 0.
\end{aligned}$$

Thus T is closed under multiplication and clearly T is abelian. □

Lemma 3. $|N_H(T)| = 5^{7k}$.

P r o o f. $N_H(T) = \{h \in H \,|\, T^h = T\}$. Let $t = 1 + r\sum_{i=0}^{4} ix^i \in T$ and $h = 1 + \sum_{i=1}^{4}[(-a_i) + a_i x^i] + \sum_{j=1}^{4}[(-b_j)y + b_j x^j y] \in H$ where $a_i, b_j, r \in \mathbb{F}_{5^k}$.

$$\sigma(t^h) = \begin{pmatrix} I & D \\ D^T & I \end{pmatrix}$$

where $D = \text{circ}(r\tau, r(1+\tau), r(2+\tau), r(3+\tau), r(4+\tau))$ where $\tau = 2(a_1 + 2a_2 + 3a_3 + 4a_4)$. Then $h \in N_H(T)$ iff $a_4 = a_1 + 2a_2 + 3a_3$. Thus every element of

$N_H(T)$ has the form

$$1 + \sum_{i=1}^{3} \left[(4-i)a_i + a_i x^i + (ia_i)x^4\right] + \sum_{j=1}^{4} \left[(-b_j)y + b_j x^j y\right].$$

Therefore $|N_H(T)| = 5^{7k}$. □

Lemma 4. *Let S be the set of elements of H of the form $1+r(x+x^2)(1-x^2)(1+y)$ where $r \in \mathbb{F}_{5^k}$. Then $S \cong {C_5}^k$.*

Proof. Let $\alpha = 1 + r(x+x^2)(1-x^2)(1+y) \in S$ and $\beta = 1 + s(x+x^2)(1-x^2)(1+y) \in S$ where $r, s \in \mathbb{F}_{5^k}$. Then

$$\alpha\beta = 1 + (r+s)(x+x^2)(1-x^2)(1+y)$$

since $((x+x^2)(1-x^2)(1+y))^2 = 0$. Thus S is closed under multiplication. It can easily be shown that S is abelian. □

Lemma 5. *$H \cong N_H(T) \rtimes S$.*

Proof. Let

$$n = 1 + \sum_{i=1}^{3} \left[(4-i)a_i + a_i x^i + (ia_i)x^4\right] + \sum_{j=1}^{4} \left[(-b_j)y + b_j x^j y\right] \in N_H(T)$$

and $s = 1 + r(x+x^2)(1-x^2)(1+y) \in S$ where $a_i, b_j, r \in \mathbb{F}_{5^k}$. Then

$$\sigma(n^s) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$$

where

$$A = \operatorname{circ}\left[1 + \sum_{i=1}^{3}(4-i)a_i, a_1 + \gamma+\delta, a_2 + 2(\gamma+\delta), a_3 + 3(\gamma+\delta), \sum_{i=1}^{3} ia_i + 4(\gamma+\delta)\right],$$

$$B = \operatorname{circ}\left[\sum_{j=1}^{4}(-b_j - \delta_j), b_1 + \delta_1, b_2 + \delta_2, \beta_3 + \delta_3, \beta_4 + \delta_4\right],$$

$$\delta_1 = \alpha + 2r(b_1 + 2b_2 + 2b_3) + \delta,$$
$$\delta_2 = \alpha + 2r(b_2 + 2b_3 + 2b_4) + 2\delta,$$

$$\delta_3 = \alpha + r(b_1 + b_2 + 3b_3) + 3\delta,$$
$$\delta_4 = \alpha + r(b_2 + b_3 + 3b_4) + 4\delta,$$
$$\delta = 3r^2 \sum_{i=1}^{4} ib_i,$$
$$\gamma = r(b_1 + 4b_2 + 4b_3 + b_4),$$
$$\alpha = 2r(a_2 - a_3) \text{ and } a_i, b_j \in \mathbb{F}_{5^k}.$$

Clearly $n^s \in N_H(T)$, S normalizes $N_H(T)$ and $\langle N_H(T), S\rangle = N_H(T)S$. By the Second Isomorphism Theorem $N_H(T)S/S \cong S/N_H(T) \cap S$. Now $N_H(T) \cap S = \{1\}$, therefore $H = N_H(T)S = N_H(T) \rtimes S$. □

Consider the set

$$U = \left\{ 1 + \sum_{i=1}^{3} \left[(4-i)a_i + a_i x^i + (ia_i)x^4\right] + 3(b_1 + b_2)y + b_1(x + x^4)y + b_2(x^2 + x^3)y \right\},$$

where $a_i, b_j \in \mathbb{F}_{5^k}$. It can easily be shown that R is a group and $R \cong {C_5}^{5k}$. Also it can be shown that the set

$$V = \left\{1 + 3(a_1 + a_2) + a_1(x + x^4) + a_2(x^2 + x^3) + b_1(1 - x^4)y + b_2(x - x^3)y\right\},$$

is a group and $V \cong {C_5}^{4k}$.

Lemma 6. $N_H(T) \cong {C_5}^{5k} \rtimes {C_5}^{2k}$.

Proof. Let

$$u = \left\{ 1 + \sum_{i=1}^{3} \left[(4-i)a_i + a_i x^i + (ia_i)x^4\right] + 3(b_1 + b_2)y + b_1(x + x^4)y + b_2(x^2 + x^3)y \right\} \in U$$

and

$$v = \{1 + 3(c_1 + c_2) + c_1(x + x^4) + c_2(x^2 + x^3) + d_1(1 - x^4)y + d_2(x - x^3)y\} \in V$$

where $a_i, b_j, c_l, d_m \in \mathbb{F}_{5^k}$. Then

$$\sigma(u^v) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$$

where

$$A = \operatorname{circ}\left(1 + \sum_{i=1}^{3}(4-i)a_i, a_1 + \delta, a_2 + 2\delta, a_3 + 3\delta, \sum_{i=1}^{3} ia_i + 4\delta\right),$$

$$B = \operatorname{circ}(3b_1 + 3b_2 + \gamma, b_1 + \gamma, b_2 + \gamma, b_2 + \gamma, b_1 + \gamma),$$
$$\delta = (b_1 - b_2)(2d_1 + d_2),$$
$$\gamma = (2d_1 + d_2)(a_2 - a_3) + (3b_1 + 2b_2)({d_1}^2 + d_1d_2 + 4{d_2}^2).$$

Clearly $u^v \in U$ and V normalizes U. Let

$$R = U \cap V = \left\{1 + 3(a + b) + a(x + x^4) + b(x^2 + x^3)\right\}$$

where $a, b \in \mathbb{F}_{5^k}$. By the second Isomorphism Theorem $N_H(T) = UV$. Clearly V is an elementary abelian 5-group and therefore V completely reduces. Let $V \cong R \times W \cong {C_5}^{2k} \times {C_5}^{2k}$. Now $W \cap U = \{1\}$ and W normalizes V. Thus $N_H(T) \cong U \rtimes W \cong {C_5}^{5k} \rtimes {C_5}^{2k}$. □

Now we make the final step of the proof. Recall that $\mathcal{U}(\mathbb{F}_{5^k}D_{10}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{5^k}C_2)$. Also $H \cong N_H(T) \rtimes S \cong ({C_5}^{5k} \rtimes {C_5}^{2k}) \rtimes {C_5}^k$.

$$\begin{aligned} \mathcal{U}(\mathbb{F}_{5^k}D_{10}) &\cong \left(({C_5}^{5k} \rtimes {C_5}^{2k}) \rtimes {C_5}^k\right) \rtimes (C_{5^k-1} \times C_{5^k-1}) \\ &\cong \left[(({C_5}^{5k} \rtimes {C_5}^{2k}) \rtimes {C_5}^k) \rtimes C_{5^k-1}\right] \times \mathcal{U}(\mathbb{F}_{5^k}). \end{aligned}$$

□

REFERENCES

[1] Leo Creedon, Joe Gildea. The structure of the unit group of the group algebra $\mathbb{F}_{3^k}D_6$. *Int. J. Pure Appl. Math.* **45**, *2* (2008), 315–320.

[2] P. J. Davis. Circulant Matrices. Chelsea Publishing, New York, 1979.

[3] J. Gildea. On the Order of $\mathcal{U}(\mathbf{F}_{p^k} D_{2p^m})$. *Int. J. Pure Appl. Math.* **46**, *2* (2008), 267–272.

[4] T. Hurley. Group Rings and Rings of Matrices. *Int. J. Pure Appl. Math.* **31** (2006), 319–335.

[5] M. Khan. Structure of the unit group of FD_{10}. *Serdica Math. J.* **35**, *1* (2009), 15–24.

[6] César Polcino Milies, Sudarshan K. Sehgal. An introduction to group rings. Algebras and Applications, vol. 1, Kluwer Academic Publishers, Dordrecht, 2002, xii+371 pp.

[7] R. K. Sharma, J. B. Srivastava, M. Khan. The unit group of FS_3. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **23**, *2* (2007), 129–142.

School of Engineering
Institute of Technology Sligo
Sligo, Ireland
e-mail: gildea.joe@itsligo.ie
URL: http://www.itsligo.ie/staff/jgildea

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