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SELECTIONS, PARACOMPACTNESS AND COMPACTNESS

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Abstract. In the present paper the Lindelöf number and the degree of compactness of spaces and of the cozero-dimensional kernel of paracompact spaces are characterized in terms of selections of lower semi-continuous closed-valued mappings into complete metrizable (or discrete) spaces.

Introduction. All considered spaces are assumed to be T_1-spaces. Our terminology comes, as a rule, from ([10], [11], [12], [16]).

A topological space X is called *paracompact* if X is Hausdorff and every open cover of X has a locally finite open refinement (see [10, 11, 14]).

One of the main results of the theory of continuous selections is the following theorem:

Theorem 0.1 (E. Michael [13]). *For any lower semi-continuous closed-valued mapping $\theta : X \to Y$ of a paracompact space X into a complete metrizable space Y there exist a compact-valued lower semi-continuous mapping $\varphi : X \to Y$*

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and a compact-valued upper semi-continuous mapping $\psi : X \to Y$ *such that* $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ *for any* $x \in X$.

Moreover, if $\dim X = 0$, *then the selections* φ, ψ *of* θ *are single-valued and continuous.*

It will be shown that the existence of upper semi-continuous selections for lower semi-continuous closed-valued mappings into a discrete spaces implies the paracompactness of the domain (see [1, 2, 3, 4, 8, 15, 16]). The papers [12, 16, 17, 15] contain some appications of the theory of selections.

A family γ of subsets of a space X is *star-finite (star-countable)* if for every element $\Gamma \in \gamma$ the set $\{L \in \gamma : L \bigcap \Gamma \neq \emptyset\}$ is finite (countable).

A topological space X is called *strongly paracompact or hypocompact* if X is Hausdorff and every open cover of X has a star-finite open refinement.

The cardinal number $l(X) = min\{m :$ *every open cover of* X *has an open refinement of cardinality* $\leq m\}$ is the *Lindelöf number* of X.

The cardinal number $k(X) = min\{m :$ *every open cover of* X *has an open refinement of cardinality* $< m\}$ is *the degree of compactness* of X.

Denote by τ^+ the least cardinal number larger than the cardinal number τ. It is obvious that $l(X) \leq k(X) \leq l(X)^+$.

For a space X put $\omega(X) = \bigcup\{U :\ U$ *is open in* X *and* $dimU = 0\}$ and let $c\omega(X) = X \setminus \omega(X)$ be *the cozero-dimensional kernel* of X (see [4]).

Lemma 0.2. *Let* X *be a paracompact space,* U *be an open subset of* X *and* $U \cap c\omega(X) \neq \emptyset$. *Then* $dim\ cl_X(U \cap c\omega(X)) \neq 0$.

Proof. See [4]. □

A family ξ of subsets of X is called τ*-centered* if $\cap\eta \neq \emptyset$ provided $\eta \subseteq \xi$ and $|\eta| < \tau$.

Lemma 0.3. *Let* X *be a paracompact space and* τ *be an infinite cardinal. Then:*

1. $l(X) \leq \tau$ *if and only if any discrete closed subset of* X *has cardinality* $\leq \tau$.
2. *The following assertions are equivalent:*
   a) $k(X) \leq \tau$.
   b) *Any discrete closed subset of* X *has cardinality* $< \tau$;
   c) $\cap\xi \neq \emptyset$ *for any* τ*-centered filter of closed subsets of* X.

Proof. It is obvious. □

Assertions $2a$ and $2c$ are equivalent and implication $2a \to 2b$ is true for every space X.

Lemma 0.4. *Let X be a metrizable space and τ be an infinite non-sequential cardinal. Then:*

1. $l(X) \leq \tau$ *if and only if* $w(X) \leq \tau$.
2. $k(X) \leq \tau$ *if and only if* $w(X) < \tau$.

Proof. It is obvious. □

The aim of the present article is to determine the conditions on a space X under which for any lower semi-continuous closed-valued mapping $\theta : X \to Y$ of the space X into a complete metrizable (or discrete) space Y there exists a selection $\varphi : X \to Y$ for which the image $\varphi(X)$ is "small" in a given sense.

In Section 2 we study the mutual relations between the following properties of topological spaces:

$K1$. $k(X) \leq \tau$.

$K2$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a lower semi-continuous selection $\phi : X \to Y$ of θ such that $k(cl_Y \phi(X)) \leq \tau$.*

$K3$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a set-valued selection $g : X \to Y$ of θ such that $k(cl_Y g(X)) \leq \tau$.*

$K4$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a single-valued selection $g : X \to Y$ of θ such that $k(cl_Y g(X)) \leq \tau$.*

$K5$. *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists a lower semi-continuous selection $\phi : X \to Y$ of θ such that $|\phi(X)| < \tau$.*

$K6$. *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists a set-valued selection $g : X \to Y$ of θ such that $\mid g(X) \mid < \tau$.*

$K7$. *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists a single-valued selection $g : X \to Y$ of θ such that $\mid g(X) \mid < \tau$.*

$K8$. *Every open cover of X has a subcover of cardinality $< \tau$.*

$K9$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exist a compact-valued lower semi-continuous mapping $\varphi : X \to Y$ and a compact-valued upper semi-continuous mapping $\psi : X \to Y$ such that $k(cl_Y(\psi(X))) \leq \tau$ and $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in X$.*

$K10$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists an upper semi-continuous selection*

$\phi : X \to Y$ of θ such that $k(cl_Y \phi(X)) \leq \tau$.

$K11$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a lower semi-continuous selection $\phi : X \to Y$ of θ such that $w(\phi(X)) < \tau$.*

$K12$. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exist a closed-valued lower semi-continuous selection $\phi : X \to Y$ of θ, a selection $\mu : X \to Y$ of θ and a closed G_δ subset F of the space X such that:*

- *$\phi(x) \subseteq \mu(x)$ for any $x \in X$;*
- *$\phi(x) = \mu(x)$ for any $x \in X \setminus F$;*
- *the mapping $\mu|F : F \to Y$ is upper semi-continuous and closed-valued;*
- *$c(X, \tau) \subseteq F$, $\Phi = \mu(F)$ is a compact subset of Y and $k(Z \cap \mu(X)) < \tau$ provided $Z \subseteq Y \setminus \Phi$ and Z is a closed subspace of the space Y (Here $k(X, \tau) = \bigcup\{U : U$ is open in X and $k(cl_X U) < \tau\}$ and $c(X, \tau) = X \setminus k(X, \tau)$.);*
- *$k(cl_Y \phi(X)) \leq k(\mu(X)) \leq \tau$.*

Let us mention that, in the conditions of $K12$:

- $k(\mu(X)) \leq \tau$ provided the set Φ is compact and $k(Z \cap \mu(X)) < \tau$ for a closed subset $Z \subseteq Y \setminus \Phi$ of the space Y;
- the mapping $\mu : X \to Y$ is closed-valued and the mapping $\mu|F : F \to Y$ is compact-valued;
- the mapping $\mu|(X \setminus F) : X \setminus F \to Y$ is lower semi-continuous;
- the mapping $\mu : X \to Y$ is Borel measurable, i.e. $\mu^{-1}(H)$ is a Borel subset of the space X for any open or closed subset H of Y.

Theorems 2.3, 2.5, 2.7 and their corollaries contain the conditions under which some of Properties $K1$ - $K12$ are equivalent.

Let Π be the class of all paracompact spaces.

For every infinite cardinal number τ we denote by $\Pi(\tau)$ the class $\{X \in \Pi : k(c\omega(X)) \leq \tau\}$. We put $\Pi_l(\tau) = \{X \in \Pi : l(c\omega(X)) \leq \tau\}$.

It is obvious that $\Pi(\tau) \subseteq \Pi_l(\tau) \subseteq \Pi(\tau^+)$.

We consider that $\Pi(n) = \{X \in \Pi : \dim X = 0\}$ for any $n \in \{0\} \cup \mathbb{N}$.

In Section 3 we establish that the classes $\Pi(\tau)$, $\Pi_l(\tau)$ may be characterized in terms of selections (Theorems 3.1 and 3.2). In this section we continue the investigations beginning in [4].

1. On the degree of compactness of spaces. A subset L of a completely regular space X is *bounded* in X if for every continuous function $f : X \to \mathbb{R}$ the set $f(L)$ is bounded.

A space X is called *μ-complete* if it is completely regular and the closure $cl_X L$ of every bounded subset L of X is compact.

Every paracompact space is μ-complete. Moreover, every Dieudonné complete space is μ-complete (see [10]).

Definition 1.1. *Let X be a space and τ be an infinite cardinal. Put $k(X, \tau) = \bigcup\{U : U$ is open in X and $k(cl_X U) < \tau\}$ and $c(X, \tau) = X \setminus k(X, \tau)$. For every $x \in X$ put $k(x, X) = \min\{k(cl_X U) : U$ is an open in X neighborhood of $x\}$.*

By definition, $k(X, \tau) = \{x \in X : k(x, X) < \tau\}$.

Lemma 1.2. *Let X be a space, τ be an infinite cardinal, $\{U_\alpha : \alpha \in A\}$ be an open discrete family in X and $k(X) \le \tau$. Then:*

1. *$| A |< \tau$;*
2. *If $x_\alpha \in U_\alpha \cap k(X, \tau)$ for every $\alpha \in A$, then $\sup\{k(x_\alpha, X) : \alpha \in A\} < \tau$;*
3. *If $x_\alpha \in U_\alpha \cap c(X, \tau)$ for every $\alpha \in A$, then $| A |< cf(\tau)$.*

Proof. Since $k(X) \le \tau$, every discrete family in X has cardinality $< \tau$.

Suppose that $x_\alpha \in U_\alpha \cap k(X, \tau)$ for every $\alpha \in A$ and $\sup\{k(x_\alpha, X) : \alpha \in A\} = \tau$. In this case τ is a non-regular limit cardinal and $cf(\tau) \le| A |< \tau$. From our assumption it follows that there exists a family of cardinals $\{\tau_\alpha : \alpha \in A\}$ such that $\tau_\alpha < k(x_\alpha, X)$ for every $\alpha \in A$ and $\sup\{\tau_\alpha : \alpha \in A\} = \tau$. For every $\alpha \in A$ there exists an open family γ_α of X such that $cl_X U_\alpha \subseteq \bigcup \gamma_\alpha$ and $| \xi |\ge \tau_\alpha$ provided $\xi \subseteq \gamma_\alpha$ and $cl_X U_\alpha \subseteq \bigcup \xi$. One can assume that $U_\beta \cap V = \emptyset$ for every $\alpha,\ \beta \in A, \alpha \ne \beta$ and $V \in \gamma_\alpha$. Let $\gamma = (X \setminus \bigcup\{cl_X U_\alpha : \alpha \in A\}) \cup (\bigcup\{\gamma_\alpha : \alpha \in A\})$. Then γ is an open cover of X and every subcover of γ has a cardinality $\ge \sup\{\tau_\alpha : \alpha \in A\} = \tau$, which is a contradiction.

If $x_\alpha \in U_\alpha \cap c(X, \tau)$ for every $\alpha \in A$ and $| A |\ge cf(\tau)$, then there exists a family of cardinals $\{\tau_\alpha : \alpha \in A\}$ such that $\tau_\alpha < \tau$ for every $\alpha \in A$ and $\sup\{\tau_\alpha : \alpha \in A\} = \tau$. Since $k(x_\alpha, X) = \tau \ge \tau_\alpha$ for every $\alpha \in A$, one can obtain a contradiction as in the previous case. $\square$

Lemma 1.3. *Let X be a completely regular space, τ be a sequential cardinal and $k(X) \le \tau$. Then the set $c(X, \tau)$ is closed and bounded. Moreover, if X is a μ-complete space, then:*

1. *$c(X, \tau)$ is a compact subset;*
2. *If $Y \subseteq k(X, \tau)$ is a closed subset of X, then $k(Y) < \tau$.*

Proof. If $\tau = \aleph_0$, then the space X is compact and $k(X,\tau)$ is the subset of all isolated in X points. Thus the set $c(X,\tau)$ is compact and every closed in X subset of $k(X,\tau)$ is finite.

Suppose that τ is uncountable. There exists a family of infinite cardinal numbers $\{\tau_n : n \in \mathbb{N}\}$ such that $\tau_n < \tau_{n+1} < \tau$ for every $n \in \mathbb{N}$ and $\sup\{\tau_n : n \in \mathbb{N}\} = \tau$. Suppose that the set $c(X,\tau)$ is unbounded in X. Then there exist a continuous function $f : X \to \mathbb{R}$ and a sequence $\{x_n \in c(X,\tau) : n \in \mathbb{N}\}$ such that $f(x_1) = 1$ and $f(x_{n+1}) \geq 3 + f(x_n)$ for every $n \in \mathbb{N}$. The family $\xi = \{U_n = f^{-1}((f(x_n) - 1, f(x_n) + 1)) : n \in \mathbb{N}\}$ is discrete in X and $x_n \in U_n$ for every $n \in \mathbb{N}$. Then, by virtue of Lemma 1.2, $|\ \xi\ | < cf(\tau) = \aleph_0$, which is a contradiction. Thus the set $c(X,\tau)$ is closed and bounded in X.

Assume now that X is a μ-complete space. In this case the set $c(X,\tau)$ is compact.

Suppose that $Y \subseteq k(X,\tau)$ is a closed subset of X and $k(Y) = \tau$. We affirm that $\sup\{k(y,X) : y \in Y\} < \tau$. For every $x \in k(X,\tau)$ fix a neighborhood U_x in X such that $k(cl_X U_x) = k(x,X)$. Suppose that $\sup\{k(y,X) : y \in Y\} = \tau$. For every $n \in \mathbb{N}$ fix a point $y_n \in Y$ such that $k(y_n,X) \geq \tau_n$. Put $L = \{y_n : n \in \mathbb{N}\}$. If the set L is unbounded in X, then there exists a continuous function $f : X \to \mathbb{R}$ such that $\sup\{f(y_n) : n \in \mathbb{N}\} = \infty$. One can assume that $f(y_{n+1}) > 3 + f(y_n)$. The family $\xi = \{U_n = f^{-1}((f(y_n) - 1, f(y_n) + 1)) : n \in \mathbb{N}\}$ is discrete in X and $y_n \in U_n \cap k(X,\tau)$ for every $n \in \mathbb{N}$. Then, by virtue of Lemma 1.2, $\sup\{k(y_n,X) : n \in \mathbb{N}\} < \tau$, which is a contradiction. Thus the set L is bounded in X. Hence $cl_X L$ is a compact subset of Y and there exists an accumulation point $y \in cl_X L \setminus (L \setminus \{y\})$. In this case $y \in k(X,\tau)$, $k(y,X) < \tau$ and $k(y,X) = \sup\{k(y_n,X) : n \in \mathbb{N}\} = \tau$, which is a contradiction. Hence $\sup\{k(y,X) : y \in Y\} \leq \tau' < \tau$.

Since $k(X) \leq \tau$, then there exists a subset $Y' \subseteq Y$ such that $|\ Y'\ | \leq \tau''$, $\tau' \leq \tau'' < \tau$ and $Y \subseteq \bigcup\{U_y : y \in Y'\}$. Since $k(cl_X U_y) \leq \tau' \leq \tau''$ for every $y \in Y'$ and $|\ Y'\ | \leq \tau''$, then $l(\bigcup\{cl_X U_y : y \in Y'\}) \leq \tau'' < \tau$. Thus $l(Y) \leq \tau'' < \tau$ and $k(Y) < \tau$. □

A subspace Z of a space X *is paracompact in* X if for every open family $\gamma = \{W_\mu : \mu \in M\}$ of X for which $Z \subseteq \cup\gamma$ there exists an open locally finite family $\eta = \{W'_\mu : \mu \in M\}$ of X such that $Z \subseteq \cup\eta$ and $W'_\mu \subseteq W_\mu$ for any $\mu \in M$.

Lemma 1.4. *Let X be a regular space, Z be a paracompact in X subspace, τ be a limit cardinal number, $k(Z) \leq \tau$ and $k(Y) < \tau$ for every closed subspace $Y \subseteq X \setminus Z$ of the space X. Then:*

1. *$k(X) \leq \tau$, $c(X,\tau) \subseteq Z$ and $k(c(X,\tau)) \leq cf(\tau)$;*
2. *If $Y \subseteq k(X,\tau)$ is a closed subset of X, then $k(Y) < \tau$.*

3. *Z is a closed subspace of X.*

P r o o f. Assertion 3 is obvious.

Let $x \in X \setminus Z$. Fix an open subset U of X such that $x \in U \subseteq cl_X U \subseteq X \setminus Z$. Then $k(cl_X U) < \tau$ and $c(X, \tau) \subseteq Z$.

Let γ be an open cover of X. Since $k(Z) \leq \tau$, there exists a subsystem ξ of γ such that $|\xi| < \tau$ and $Z \subseteq \cup\xi$. Let $Y = X \setminus \cup\xi$. Since $k(Y) < \tau$, there exists a subsystem ζ of γ such that $|\zeta| < \tau$ and $Y \subseteq \cup\zeta$. Put $\eta = \zeta \cup \xi$. Then η is a subcover of γ and $|\eta| < \tau$. Thus $k(X) \leq \tau$.

Suppose that $k(c(X, \tau)) > cf(\tau)$. Since Z is paracompact in X, the subspace $c(X, \tau)$ is paracompact in X and there exists an open locally-finite family $\{V_\alpha : \alpha \in A\}$ of X such that $c(X, \tau) \subseteq \bigcup\{V_\alpha : \alpha \in A\}, |A| \geq cf(\tau)$ and $c(X, \tau) \setminus \bigcup\{V_\beta : \beta \in A \setminus \{\alpha\}\} \neq \emptyset$ for every $\alpha \in A$. For every $\alpha \in A$ fix $y_\alpha \in c(X, \tau) \setminus \bigcup\{V_\beta : \beta \in A \setminus \{\alpha\}\} \neq \emptyset$. Then $\{y_\alpha : \alpha \in A\}$ is a closed discrete subset of X. There exists an open discrete family $\{W_\alpha : \alpha \in A\}$ such that $y_\alpha \in W_\alpha \subseteq V_\alpha$ for every $\alpha \in A$. By virtue of Lemma 1.2, $| A | < cf(\tau)$, which is a contradiction. Thus $k(c(X, \tau)) \leq cf(\tau)$.

Fix now a closed subset Y of the space X such that $Y \subseteq k(X, \tau)$. We put $S = Y \cap Z$ and $\tau' = \sup\{k(y, X) : y \in S\}$.

Suppose that $\tau' = \tau$. There exists a family of cardinals $\{\tau_\alpha : \alpha \in A\}$ such that $| A | = cf(\tau)$, $\sup\{\tau_\alpha : \alpha \in A\} = \tau$ and $\tau_\alpha < \tau$ for every $\alpha \in A$. One can assume that A is well ordered and $\tau_\alpha < \tau_\beta$ for every $\alpha, \beta \in A$ and $\alpha < \beta$. For every $\alpha \in A$ there exists $y_\alpha \in S$ such that $k(y_\alpha, X) > \tau_\alpha$. Let $L = \{y_\alpha : \alpha \in A\}$. The cardinal $cf(\tau)$ is regular. If $y \in X$ and $|W \cap L| = |A|$ for every neighborhood W of y in X, then $y \in Y \subseteq k(X, \tau), k(y, X) < \tau$ and $k(y, X) \geq \sup\{k(y_\alpha, X) : \alpha \in A\} = \sup\{\tau_\alpha : \alpha \in A\} = \tau$, which is a contradiction. Thus for every $y \in X$ there exists an open neighborhood W_y of y in X such that $|cl_X W_y \cap L| < |A| = cf(\tau)$. There exists an open locally-finite family $\{H_z : z \in Z\}$ of X such that $Z \subseteq \cup\{H_z : z \in Z\}$ and $H_z \subseteq W_z$ for every $z \in Z$. Let $Z' = \{z \in Z : H_z \cap L \neq \emptyset\}$. The set Z' is discrete and closed in X. Since Z is a paracompact space, we have $|Z'| = \tau'' < \tau$ and $|H_z \cap L| = \tau(z) < cf(\tau)$ for any $z \in Z'$. Thus $|L| =| \cup\{H_z \cap L : z \in Z'\} |< \tau$, a contradiction. Therefore $\tau' < \tau$.

Since S is paracompact in X and $S \subseteq k(X, \tau)$, there exist a set $M \subseteq S$ and a locally finite open in X family $\{U_\mu : \mu \in M\}$ such that $k(cl_X U_\mu) \leq \tau'$, $|M| = \tau_1 < \tau$ and $S \subseteq \cup\{U_\mu : \mu \in M\}$. Then $k(S_1) = \tau_2 < \tau$, where $S_1 = \cup\{cl_X U_\mu : \mu \in M\}$. Let $Y_1 = Y \setminus \cup\{U_\mu : \mu \in M\}$. Since Y_1 is a closed subset of X and $Y_1 \subseteq X \setminus Z$, $k(Y_1) = \tau_3 < \tau$. Thus $k(Y) \leq k(Y_1) + k(S_1) < \tau$. □

Corollary 1.5. *Let X be a paracompact space, τ be a limit cardinal and $k(X) \leq \tau$. Then:*

1. $k(c(X,\tau)) \leq cf(\tau)$*;*
2. *If $Y \subseteq k(X,\tau)$ is a closed subset of X, then $k(Y) < \tau$.*

A *shrinking* of a cover $\xi = \{U_\alpha : \alpha \in A\}$ of the space X is a cover $\gamma = \{V_\alpha : \alpha \in A\}$ such that $V_\alpha \subseteq U_\alpha$ for every $\alpha \in A$ (see [10], [11]). The operation of shrinking preserves the properties of local finiteness, star-finiteness and star-countableness.

Let τ be an infinite cardinal number. A family γ of subsets of a space X is called τ-*star* (τ^--*star*) if $\mid \{H \in \gamma : H \cap L \neq \emptyset\} \mid \leq \tau$ ($\mid \{H \in \gamma : H \cap L \neq \emptyset\} \mid < \tau$) for every $L \in \gamma$.

A family $\{H_\alpha : \alpha \in A\}$ of subsets of a space X is *closure-preserving* if $\bigcup\{cl_X H_\beta : \beta \in B\} = cl_X(\bigcup\{H_\beta : \beta \in B\})$ for every $B \subseteq A$ (see [14]).

Proposition 1.6. *Let τ be an infinite cardinal and X be a paracompact space. Then the following assertions are equivalent:*

1. $k(c\omega(X)) \leq \tau$.
2. *For every open cover of X there exists an open τ^--star shrinking.*
3. *For every open cover of X there exists a closed closure-preserving τ^--star shrinking.*
4. *For every open cover of X there exists a closed τ^--star shrinking.*

P r o o f. ($1 \Rightarrow 2$) and ($1 \Rightarrow 3$) Let $\xi = \{U_\alpha : \alpha \in A\}$ be an open cover of X. There exist a subset B of A and an open-and-closed subset H of X such that $c\omega(X) \subseteq H \subseteq \bigcup\{U_\alpha : \alpha \in B\}$ and $\mid B \mid < \tau$ (see the proof of Proposition 4 [4]). Since $\dim(X \setminus H) = 0$ (unless $X \setminus H$ is empty) there exists a discrete family $\{W_\alpha : \alpha \in A\}$ of open-and-closed subsets of X such that $\bigcup\{W_\alpha : \alpha \in A\} = X \setminus H$ and $W_\alpha \subseteq U_\alpha$ for every $\alpha \in A$. Let $V_\alpha = (U_\alpha \cap H) \cup W_\alpha$ for $\alpha \in B$ and $V_\alpha = W_\alpha$ for $\alpha \in A \setminus B$. Obviously $\gamma = \{V_\alpha : \alpha \in A\}$ is an open τ^--star shrinking of ξ.

Since X is paracompact, there exists a closed locally finite family $\{H_\alpha : \alpha \in B\}$ such that $H = \cup\{H_\alpha : \alpha \in B\}$ and $H_\alpha \subseteq U_\alpha$ for any $\alpha \in B$. Put $H_\alpha = W_\alpha$ for any $\alpha \in A \setminus B$. Obviously $\lambda = \{H_\alpha : \alpha \in A\}$ is a closed locally finite τ^--star shrinking of ξ. Every locally finite family is closure-preserving. Implications ($1 \Rightarrow 2$) and ($1 \Rightarrow 3$) are proved.

Implication ($3 \Rightarrow 4$) is obvious.

($2 \Rightarrow 1$) and ($4 \Rightarrow 1$) Suppose $k(c\omega(X)) > \tau$. There exists a locally finite open cover $\xi = \{U_\alpha : \alpha \in A\}$ of $c\omega(X)$ such that $c\omega(X) \setminus \bigcup\{U_\alpha : \alpha \in B\} \neq \emptyset$ provided $B \subseteq A$ and $\mid B \mid < \tau$. One can assume that $c\omega(X) \setminus \bigcup\{U_\alpha : \alpha \in B\} \neq \emptyset$

for every proper subset B of A. Fix a point $x_\alpha \in c\omega(X) \setminus \bigcup\{U_\beta : \beta \in A \setminus \{\alpha\}\}$ for every $\alpha \in A$. The set $\{x_\alpha : \alpha \in A\}$ is discrete in X. There exists a discrete family $\{V_\alpha : \alpha \in A\}$ of open subsets of X such that $x_\alpha \in V_\alpha \subseteq cl_X V_\alpha \subseteq U_\alpha$ for every $\alpha \in A$. Let $X_\alpha = cl_X V_\alpha$. Then $\dim X_\alpha > 0$ and there exist two closed disjoint subsets F_α and P_α of X_α such that if W_α and O_α are open in X and $F_\alpha \subseteq W_\alpha \subseteq X \setminus P_\alpha$, $P_\alpha \subseteq O_\alpha \subseteq X \setminus F_\alpha$ and $X_\alpha \subseteq W_\alpha \cup O_\alpha$, then $X_\alpha \cap W_\alpha \cap O_\alpha \neq \emptyset$. The family $\{F_\alpha : \alpha \in A\}$ and the family $\{P_\alpha : \alpha \in A\}$ are discrete in X. There exists a discrete family $\{Q_\alpha : \alpha \in A\}$ of open subsets of X such that $(\bigcup\{Q_\alpha : \alpha \in A\}) \cap (\bigcup\{F_\alpha : \alpha \in A\}) = \emptyset$, $P_\alpha \subseteq Q_\alpha$ and $Q_\alpha \cap (\bigcup\{X_\beta : \beta \in A \setminus \{\alpha\}\}) = \emptyset$ for every $\alpha \in A$. Let $\mu \notin A$, $M = A \cup \{\mu\}$ and $Q_\mu = X \setminus \bigcup\{P_\alpha : \alpha \in A\})$. Then $\zeta = \{Q_m : m \in M\}$ is an open cover of X. If $\gamma = \{H_m : m \in M\}$ is an open shrinking of ζ, then $H_\mu \cap H_\alpha \neq \emptyset$ for every $\alpha \in A$. The last contradicts 2. Suppose now that $\gamma = \{H_m : m \in M\}$ is a closed shrinking of ζ. Let $\alpha \in A$ and $H_\alpha \cap H_\mu = \emptyset$. There exist two disjoint open subsets W_α and O_α of X such that $H_\alpha \subseteq W_\alpha$ and $H_\mu \subseteq O_\alpha$. Then $X_\alpha \subseteq H_\alpha \cup H_\mu \subseteq O_\alpha \cup W_\alpha$, $P_\alpha \subseteq W_\alpha \subseteq X \setminus F_\alpha$, $F_\alpha \subseteq O_\alpha \subseteq X \setminus P_\alpha$, $X_\alpha \subseteq W_\alpha \cup O_\alpha$ and $X_\alpha \cap W_\alpha \cap O_\alpha = \emptyset$. The last contradicts 4. Implications $(2 \Rightarrow 1)$ and $(4 \Rightarrow 1)$ are proved. □

2. The degree of compactness and selections.

Let X and Y be non-empty topological spaces. A *set-valued mapping* $\theta : X \to Y$ assigns to every $x \in X$ a non-empty subset $\theta(x)$ of Y. If ϕ, $\psi : X \to Y$ are set-valued mappings and $\phi(x) \subseteq \psi(x)$ for every $x \in X$, then ϕ is called *a selection* of ψ.

Let $\theta : X \to Y$ be a set-valued mapping and let $A \subseteq X$ and $B \subseteq Y$. The set $\theta^{-1}(B) = \{x \in X : \theta(x) \bigcap B \neq \emptyset\}$ is *the inverse image* of the set B, $\theta(A) = \theta^1(A) = \bigcup\{\theta(x) : x \in A\}$ is *the image* of the set A and $\theta^{n+1}(A) = \theta(\theta^{-1}(\theta^n(A)))$ is *the* $n+1$*-image* of the set A. The set $\theta^\infty(A) = \bigcup\{\theta^n(A) : n \in \mathbb{N}\}$ is *the largest image* of the set A.

A set-valued mapping $\theta : X \to Y$ is called *lower (upper) semi-continuous* if for every open (closed) subset H of Y the set $\theta^{-1}(H)$ is open (closed) in X.

In the present section we study the mutual relations between the properties $K1 - K12$ of topological spaces.

The σ-algebra generated by the open subsets of the space X is the algebra of Borel subsets of the space X.

Lemma 2.1. *Let X be a space and τ be an infinite cardinal. Then the following implications* $(K9 \to K2 \to K3 \to K4 \to K3 \to K6 \to K7 \to K8 \to K1 \to K5 \to K6$, $K10 \to K3)$ *and* $(K12 \to K11 \to K2 \to K5 \to K6)$ *are true.*

P r o o f. Implications ($K12 \to K11 \to K2 \to K5 \to K6$, $K9 \to K2 \to K3 \to K6$), ($K4 \to K7$, $K4 \to K3$), ($K7 \to K6$, $K8 \to K1 \to K8$) and ($K10 \to K3$) are obvious.

Let $\phi : X \to Y$ be a set-valued selection of the mapping $\theta : X \to Y$ and $k(cl_Y \phi(X)) \leq \tau$. For every $x \in X$ fix a point $f(x) \in \phi(x)$. Then $f : X \to Y$ is a single-valued selection of θ and ϕ, $f(X) \subseteq \phi(X)$ and $k(cl_Y f(X)) \leq k(cl_Y \phi(X)) \leq \tau$. The implications ($K3 \to K4$) and ($K6 \to K7$) are proved.

Let $\gamma = \{U_\alpha : \alpha \in A\}$ be an open cover of X. One may assume that A is a discrete space. For every $x \in X$ put $\theta_\gamma(x) = \{\alpha \in A : x \in U_\alpha\}$. Since $\theta^{-1}(\{\alpha\}) = U_\alpha$, the mapping θ_γ is lower semi-continuous. Let $\phi : X \to Y$ be a set-valued selection of θ_γ and $\mid \phi(X) \mid < \tau$. Put $B = \phi(X)$ and $H_\alpha = \phi^{-1}(\alpha)$ for every $\alpha \in B$. Then $H_\alpha \subseteq \theta^{-1}(\{\alpha\}) = U_\alpha$ for every $\alpha \in B$, $X = \bigcup\{H_\alpha : \alpha \in B\}$, $\xi = \{H_\alpha : \alpha \in B\}$ is a refinement of γ and $\mid B \mid < \tau$. Implications ($K3 \to K8$) and ($K6 \to K8$) are proved.

Let $k(X) \leq \tau$ and $\theta : X \to Y$ be a lower semi-continuous mapping into a discrete space Y. Then $\{U_y = \theta^{-1}(y) : y \in Y\}$ is an open cover of X. There exists a subset $Z \subseteq Y$ such that $\mid Z \mid < \tau$ and $X = \bigcup\{U_y : y \in Z\}$. Now we put $\phi(x) = \{y \in Z : x \in U_y\}$. Then $\phi : X \to Y$ is a lower semi-continuous selection of θ, $\phi(x) = Z \cap \theta(x)$ for every $x \in X$ and $\mid \phi(X) \mid = \mid Z \mid < \tau$. Implication ($K1 \to K5$) is proved. The proof is complete.

Proposition 2.2. *Let X be a space, τ be an infinite cardinal and $\theta : X \to Y$ be an upper semi-continuous mapping onto Y. Then:*

1. *If $l(X) \leq \tau$ and $l(\theta(x)) \leq \tau$ for every $x \in X$, then $l(Y) \leq \tau$;*
2. *If $k(X) \leq \tau$ and $k(\theta(x)) \leq cf(\tau)$ for every $x \in X$, then $k(Y) \leq \tau$;*
3. *If θ is compact-valued, then $l(Y) \leq l(X)$ and $k(Y) \leq k(X)$.*
4. *If X is a μ-complete space, τ is a sequential cardinal number and θ is compact-valued, then $c(Y,\tau) \subseteq \theta(c(X,\tau))$ and $k(Z) < \tau$ provided $Z \subseteq Y \setminus c(Y,\tau)$ and Z is closed in the space Y.*

P r o o f. If V is an open subset of Y, then $\theta^*(V) = \{x \in X : \theta(x) \subseteq V\}$ is open in X.

1. Let τ be an infinite cardinal, $l(X) \leq \tau$ and $l(\theta(x)) \leq \tau$ for every $x \in X$. Let $\gamma = \{V_\alpha : \alpha \in A\}$ be an open cover of Y. If $x \in X$, then $l(\theta(x)) \leq \tau$. Thus every open family in Y, which covers $\theta(x)$, has a subfamily of cardinality $\leq \tau$ covering $\theta(x)$. Hence there exists a subset $A_x \subseteq A$ such that $\mid A_x \mid = \tau_x \leq \tau$ and $\theta(x) \subseteq \bigcup\{V_\alpha : \alpha \in A_x\}$. We put $W_x = \cup\{V_\alpha : \alpha \in A_x\}$ and $U_x = \{z \in X : \theta(z) \subseteq W_x\}$.

Obviously $\lambda = \{U_x : x \in X\}$ is an open cover of X. Since $l(X) \leq \tau$, there exists an open subcover $\zeta = \{U_x : x \in X'\}$ of λ such that $\mid X' \mid \leq \tau$ and $X' \subseteq X$.

Let $B = \cup\{A_x : x \in X'\}$. Obviously $|B| \leq \tau$. Since $\theta(U_x) \subseteq W_x$ for any $x \in X$, we have $Y = \theta(X) = \theta(\cup\{U_x : x \in X'\}) = \cup\{\theta(U_x) : x \in X'\} \subseteq \cup\{W_x : x \in X'\} = \cup\{V_\alpha : \alpha \in B\}$. Hence $\gamma' = \{V_\alpha : \alpha \in B\}$ is a subcover of γ of cardinality $\leq \tau$. Assertion 1 is proved.

2. One can follow the proof of the previous assertion 1. Let τ be an infinite cardinal, $k(X) \leq \tau$ and $k(\theta(x)) \leq cf(\tau)$ for every $x \in X$. Let $\gamma = \{V_\alpha : \alpha \in A\}$ be an open cover of Y. For any $x \in X$ there exists a subset $A_x \subseteq A$ such that $\mid A_x \mid = \tau_x < cf(\tau)$ and $\theta(x) \subseteq \bigcup\{V_\alpha : \alpha \in A_x\}$. We put $W_x = \cup\{V_\alpha : \alpha \in A_x\}$ and $U_x = \{z \in X : \theta(z) \subseteq W_x\}$.

Obviously $\lambda = \{U_x : x \in X\}$ is an open cover of X. Since $k(X) \leq \tau$, there exists an open subcover $\zeta = \{U_x : x \in X'\}$ of λ such that $\mid X' \mid = \tau_0 < \tau$ and $X' \subseteq X$. Let $B = \cup\{A_x : x \in X'\}$. Since $\theta(U_x) \subseteq W_x$ for any $x \in X$, we have $Y = \theta(X) = \theta(\cup\{U_x : x \in X'\}) = \cup\{\theta(U_x) : x \in X'\} \subseteq \cup\{W_x : x \in X'\} = \cup\{V_\alpha : \alpha \in B\}$. Hence $\gamma' = \{V_\alpha : \alpha \in B\}$ is a subcover of γ.

We affirm that $|B| < \tau$.

Consider the following cases:

Case 1. τ is regular, i.e. $cf(\tau) = \tau$.

Since $\mid X' \mid = \tau_0 < \tau = cf(\tau)$ and $\mid A_x \mid < \tau$ for every $x \in X$, it follows that $|B| \leq \Sigma\{\tau_x : x \in X'\} = \tau' < \tau$.

Hence $\gamma' = \{V_\alpha : \alpha \in B\}$ has cardinality $< \tau$.

Case 2. τ is not regular, i.e. $cf(\tau) = m < \tau$.

In this case τ is a limit cardinal, $\tau_0 < \tau$ and $m < \tau$. Hence $\tau' = sup\{m, \tau_0\} < \tau$.

Since $\mid A_x \mid = \tau_x < m$ for every $x \in X$, it follows that $|B| \leq \Sigma\{\tau_x : x \in X'\} \leq \tau' < \tau$.

Hence $\gamma' = \{V_\alpha : \alpha \in B\}$ has cardinality $< \tau$.

Assertion 2 is proved.

3. Assertion 3 follows easily from assertions 1 and 2.

4.Obviously, $\Phi = \theta(c(X, \tau))$ and $c(Y, \tau)$ are compact subsets of the space Y. Let $Z \subseteq Y \setminus \Phi$ be a closed subspace of the space Y. Then $X_1 = \theta^{-1}(Z)$ is a closed subspace of the space X and $X_1 \cap c(X, \tau) = \emptyset$. By virtue of Lemma 1.3, $k(X_1) < \tau$. Let $Y_1 = \theta(X_1)$. Then $\theta_1 = \theta|X_1 : X_1 \to Y_1$ is an upper semi-continuous mapping onto Y_1. From assertion 2 it follows that $k(Y_1) \leq k(X_1) < \tau$. Since Z is a closed subspace of the space Y_1, we have $k(Z) \leq k(Y_1) < \tau$. In particular, $Y \setminus \Phi \subseteq k(Y, \tau)$ and $c(Y, \tau) \subseteq \Phi$. Since Φ is a compact subset of Y, $k(Z) < \tau$ provided $Z \subseteq Y \setminus c(Y, \tau)$ and Z is closed in the space Y. $\square$

Theorem 2.3. *Let X be a regular space and τ be a regular cardinal number. Then Properties $K1 - K8$ and $K12$ are equivalent. Moreover, if the cardinal number τ is regular and uncountable, then assertions $K1 - K8$, $K11$ and $K12$ are equivalent.*

Proof. Let $k(X) \leq \tau$ and $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping into a complete metric space (Y, ρ).

Case 1. $\tau = \aleph_0$.

In this case the space X is compact. Thus, from E.Michael's Theorem [13] (see Theorem 0.1), it follows that there exist a lower semi-continuous compact-valued mapping $\varphi : X \to Y$ and an upper semi-continuous compact-valued mapping $\psi : X \to Y$ such that $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in X$. The set $\psi(X)$ is compact and $\varphi(X) \subseteq \psi(X)$. The implication $(K1 \Rightarrow K9)$ is proved.

Case 2. $\tau > \aleph_0$.

There exists a sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open covers of the space X, a sequence $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open families of the space Y and a sequence $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\}$ of mappings such that:

— $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ for any $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $\cup\{cl_Y V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ and $\mathrm{diam}(V_\alpha) < 2^{-n}$ for any $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $|A_n| < \tau$ for any $n \in \mathbb{N}$.

Let $\eta = \{V : V$ *is open in* Y *and* $\mathrm{diam}(V) < 2^{-1}\}$. Let $\gamma' = \{U : U$ *is open in* X *and* $cl_X U \subseteq \theta^{-1}(V)$ *for some* $V \in \eta\}$. Since $k(X) \leq \tau$, there exists an open subcover $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$ of γ' such that $|A_1| < \tau$. For any $\alpha \in A_1$ fix $V_\alpha \in \eta$ such that $cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$.

Consider that the objects $\{\gamma_i, \xi_i, \pi_{i-1} : i \leq n\}$ are constructed. Fix $\alpha \in A_n$. Let $\eta_\alpha = \{V : V$ *is open in* Y, $cl_Y V \subseteq V_\alpha$ *and* $\mathrm{diam}(V) < 2^{-n-1}\}$. Let $\gamma'_\alpha = \{W : W$ *is open in* X *and* $cl_X W \subseteq \theta^{-1}(V)$ *for some* $V \in \eta_\alpha\}$. Since $k(cl_X U_\alpha) \leq \tau$ and $cl_X U_\alpha \subseteq \cup\gamma'_\alpha$, there exists an open subfamily $\gamma_\alpha = \{W_\beta : \beta \in A_\alpha\}$ of γ'_α such that $|A_\alpha| < \tau$ and $cl_X U_\alpha \subseteq \cup\{W_\beta : \beta \in A_\alpha\}$. For any $\beta \in A_\alpha$ fix $V_\beta \in \eta_\alpha$ such that $cl_X W_\beta \subseteq \theta^{-1}(V_\beta)$. Let $A_{n+1} = \cup\{A_\alpha : \alpha \in A_n\}$, $\pi_n^{-1}(\alpha) = A_\alpha$ and $U_\beta = U_\alpha \cap W_\beta$ for all $\alpha \in A_n$ and $\beta \in A_\alpha$. Since τ is regular and uncountable, then $|A_{n+1}| < \tau$.

The objects $\{\gamma_n, \xi_n, \pi_n : n \in \mathbb{N}\}$ are constructed.

Let $x \in X$. Denote by $A(x)$ the set of all sequences $\alpha = (\alpha_n : n \in \mathbb{N})$ for which $\alpha_n \in A_n$ and $x \in U_{\alpha_n}$ for any $n \in \mathbb{N}$. For any $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$ there exists a unique point $y(\alpha) \in Y$ such that $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$. It is obvious that $y(\alpha) \in \theta(x)$. Let $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$. Then ϕ is a selection of θ. By construction:

— $U_\alpha \subseteq \phi^{-1}(V_\alpha)$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$;

— the mapping ϕ is lower semi-continuous;

— if $Z = \phi(X)$, then $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup\{A_n : n \in \mathbb{N}\}\}$ is an open base of the subspace Z.

We affirm that $w(Z) < \tau$.

Subcase 2.1. τ is a limit cardinal.

In this subcase $m = sup\{|A_n| : n \in \mathbb{N}\} < \tau$ and $w(Z) \leq |A| \leq m < \tau$.

Subcase 2.2. τ is not a limit cardinal.

In this subcase there exists a cardinal number m such that $m^+ = \tau$ and $|A| \leq m$. Thus $w(Z) < \tau$.

In this case we have proved the implication ($K1 \rightarrow K11$).

Lemma 2.1 completes the proof of the theorem. $\square$

Corollary 2.4. *Let X be a regular space and τ be a cardinal number. Then the following assertions are equivalent:*

L1. $l(X) \leq \tau$.

L2. For every lower semi-continuous closed-valued mapping $\theta : X \rightarrow Y$ into a complete metrizable space Y there exists a lower semi-continuous selection $\phi : X \rightarrow Y$ of θ such that $l(cl_Y \phi(X)) \leq \tau$.

L3. For every lower semi-continuous closed-valued mapping $\theta : X \rightarrow Y$ into a complete metrizable space Y there exists a set-valued selection $g : X \rightarrow Y$ of θ such that $l(cl_Y g(X)) \leq \tau$.

L4. For every lower semi-continuous closed-valued mapping $\theta : X \rightarrow Y$ into a complete metrizable space Y there exists a single-valued selection $g : X \rightarrow Y$ of θ such that $l(cl_Y g(X)) \leq \tau$.

L5. For every lower semi-continuous mapping $\theta : X \rightarrow Y$ into a discrete space Y there exists a lower semi-continuous selection $\phi : X \rightarrow Y$ of θ such that $|\phi(X)| \leq \tau$.

L6. For every lower semi-continuous mapping $\theta : X \rightarrow Y$ into a discrete space Y there exists a set-valued selection $g : X \rightarrow Y$ of θ such that $| g(X) | \leq \tau$.

L7. For every lower semi-continuous mapping $\theta : X \rightarrow Y$ into a discrete space Y there exists a single-valued selection $g : X \rightarrow Y$ of θ such that $| g(X) | \leq \tau$.

L8. Every open cover of X has a subcover of cardinality $\leq \tau$.

Proof. Let $l(X) \leq \tau$. Then $k(X) \leq \tau^+$ and τ^+ is a regular cardinal. Theorem 2.3 completes the proof. $\square$

Theorem 2.5. *Let X be a regular space, F be a compact subset of X, τ be a cardinal number and $k(X') < \tau$ for any closed subset $X' \subseteq X \setminus F$ of X. Then assertions $K1 - K8$ and $K12$ are equivalent. Moreover, if the cardinal number τ is not sequential, then Properties $K1 - K8$, $K11$ and $K12$ are equivalent.*

Proof. Let $k(X) \leq \tau$, F be a compact subset of X, τ be a cardinal number and $k(X') < \tau$ for any closed subset $X' \subseteq X \setminus F$ of X and $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping into a complete metric space (Y, ρ).

Case 1. $\tau = \aleph_0$.

In this case the space X is compact. Thus, from Theorem 0.1, it follows that there exist a lower semi-continuous compact-valued mapping $\varphi : X \to Y$ and an upper semi-continuous compact-valued mapping $\psi : X \to Y$ such that $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in X$. The set $\psi(X)$ is compact and $\varphi(X) \subseteq \psi(X)$. The implications $(K1 \Rightarrow K9)$ and $(K1 \Rightarrow K12)$ are proved.

Case 2. τ is a regular cardinal number.

In this case Theorem 2.3 completes the proof.

Case 3. τ is an uncountable limit cardinal.

Let $\tau' = cf(\tau)$.

The subspace F is compact. Thus, from the E.Michael's Theorem 0.1, it follows that there exists an upper semi-continuous compact-valued mapping $\psi : F \to Y$ such that $\psi(x) \subseteq \theta(x)$ for any $x \in F$. The set $\Phi = \psi(F)$ is compact. There exists a sequence $\{H_n : n \in \mathbb{N}\}$ of open subsets of Y such that:

— $\Phi \subseteq H_{n+1} \subseteq cl_Y H_{n+1} \subseteq H_n$ for any $n \in \mathbb{N}$;

— for every open subset $V \supseteq \Phi$ of Y there exists $n \in \mathbb{N}$ such that $H_n \subseteq V$.

There exist a sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open covers of the space X, a sequence $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open families of the space Y, a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, a sequence $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\}$ of mappings and a sequence $\{\tau_n : n \in \mathbb{N}\}$ of cardinal numbers such that:

— $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ for any $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $\cup\{cl_Y V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ and $\mathrm{diam}(V_\alpha) < 2^{-n}$ for any $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $|A_n| < \tau$ for any $n \in \mathbb{N}$;

— if $A'_n = \{\alpha \in A_n : F \cap cl_X U_\alpha = \emptyset\}$ and $A''_n = A_n \setminus A'_n$, then the set A''_n is finite and $F \subseteq U_n \subseteq cl_X U_n \subseteq \cup\{U_\alpha : \alpha \in A''_n\}$;

— $\tau_n \leq \tau_{n+1} < \tau$ for any $n \in \mathbb{N}$;

— $cl_X U_n \subseteq \theta^{-1}(H_n)$ and $|\{\alpha \in A_n : U_\alpha \setminus U_m \neq \emptyset\}| \leq \tau_m$ for all $n, m \in \mathbb{N}$;

— $cl_X U_n \cap cl_X U_\alpha = \emptyset$ for any $n \in \mathbb{N}$ and $\alpha \in A'_n$.

Let $\eta = \{V : V$ *is open in* Y *and* $\operatorname{diam}(V) < 2^{-1}\}$. There exists a finite subfamily $\{V_\beta : \beta \in B_1\}$ of η such that $\Phi \subseteq \cup\{V_\beta : \beta \in B_1\} \subseteq H_1$. Let W_1 be an open subset of Y and $\Phi \subseteq W_1 \subseteq cl_Y W_1 \subseteq \cup\{V_\alpha : \alpha \in B_1\}$.

Let $\gamma' = \{U : U$ *is open in* X *and* $cl_X U \subseteq \theta^{-1}(V)$ *for some* $V \in \eta$ and $U \subseteq X \setminus U_1\}$ and $\gamma'' = \{U : U$ *is open in* X *and* $cl_X U \subseteq \theta^{-1}(V_\beta)$ *for some* $\beta \in B_1\}$.

Since F is compact, there exist a finite family $\gamma''_1 = \{U_\alpha : \alpha \in A''_1\}$ of γ'' and an open subset U_1 of X such that $F \subseteq U_1 \subseteq cl_X U_1 \subseteq \cup\{U_\alpha : \alpha \in A''_1\}$ and $F \cap U_\alpha \neq \emptyset$ for any $\alpha \in A''_1$. For every $\alpha \in A''_1$ fix $V_\alpha = V_\beta$ for some $\beta \in B_1$ such that $clU_\alpha \subseteq \theta^{-1}(V_\alpha)$. Let $Y_1 = X \setminus U_1$. Since $k(Y_1) = \tau'_1 < \tau$, there exists an open subfamily $\gamma'_1 = \{U_\alpha : \alpha \in A'_1\}$ of γ' such that $|A'_1| \leq \tau_1$, $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A'_1\}$ and $cl_X U_1 \cap (\cup\{cl_X U_\alpha : \alpha \in A'_1\} = \emptyset$. For any $\alpha \in A'_1$ fix $V_\alpha \in \eta'$ such that $cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$. Let $A_1 = A'_1 \cup A''_1$, $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$ and $\eta_1 = \{V_\alpha : \alpha \in A_1\}$.

Consider that the objects $\{\gamma_i, \xi_i, \pi_{i-1}, U_i, \tau_i, : i \leq n\}$ are constructed.

We put $A_{im} = \{\alpha \in A_i : U_\alpha \cap U_m \neq \emptyset\}$ for all $i, m \leq n$.

Fix $\alpha \in A_n$.

Let $\eta_\alpha = \{V : V$ *is open in* Y, $cl_Y V \subseteq V_\alpha$ *and* $\operatorname{diam}(V) < 2^{-n-1}\}$ and $\gamma'_\alpha = \{W : W$ *is open in* X *and* $cl_X W \subseteq \theta^{-1}(V)$ *for some* $V \in \eta_\alpha\}$.

Assume that $\alpha \in A''_n$.

Since $F_\alpha = F \cap cl_X U_\alpha$ is a compact subset of X there exists a finite subfamily $\gamma_{0\alpha} = \{W_\beta : \beta \in A''_{0\alpha}\}$ of γ'_α such that $F_\alpha \subseteq \cup\{W_\beta : \beta \in A''_{0\alpha}\}$, $F_\alpha \cap W_\beta \neq \emptyset$ for any $\beta \in A''_{0\alpha}$ and for any $\beta \in A''_{0\alpha}$ there exists $V_\beta \in \eta_\alpha$ such that $V_\beta \subseteq H_{n+1}$ and $cl_X W_\beta \subseteq \theta^{-1}(V_\beta)$. Now we put $U_\beta = W_\beta \cap U_\alpha$.

Let $A''_{n+1} = \cup\{A_{0\alpha} : \beta \in A''_n\}$, $\gamma''_{n+1} = \{U_\beta : \beta \in A''_{n+1}\}$ and $\eta''_{n+1} = \{V_\beta : \beta \in A''_{n+1}\}$.

Let $\Phi_\alpha = cl_X U_\alpha \setminus \cup\{U_\beta : \beta \in A''_{0\alpha}\}$ and $U'_n = U_n \setminus \cup\{\Phi_\alpha : \alpha \in A''_n\}$. Then U'_n is an open subset of X and $F \subseteq U'_n \cap \cup(\{U_\beta : \beta \in A''_{0\alpha}\})$.

There exists an open subset U_{n+1} of X such that $U_{n+1} \subseteq cl_X U_{n+1} \subseteq U'_n \cap U_n \cap (\cup\{U_\beta : \beta \in A''_{0\alpha}\})$.

Let $X_i = X \setminus U_i$ for any $i \leq n+1$. Then $\tau_i = k(X_i)$ for any $i \leq n+1$.

For any $\alpha \in A_n$ there exist the subfamilies $\gamma'_{i\alpha} = \{W_\beta : \beta \in A'_{in\alpha}\}$, $i \leq n+1$, of γ'_α and the subfamilies $\eta'_{i\alpha} = \{V_\beta : \beta \in A'_{in\alpha}\}$, $i \leq n+1$, of γ'_α such that:

— $|A'_{in\alpha}| < \tau_i$ for any $i \leq n+1$;

— $X_i \cap cl_X U_\alpha \subseteq \cup\{W_\beta : \beta \in \cup\{A_{jn\alpha} : j \leq i\}\}$ for any $i \leq n+1$;

— $X_i \cap (\cup\{W_\beta : \beta \in \cup\{A_{jn\alpha} : i < j \leq n+1\}\}) = \emptyset$ for any $i < n+1$.

Now we put $A_{n\alpha} = \cup\{A_{in\alpha} : 0 \leq i \leq n+1\}$, $A_{n+1} = \cup\{A_{n\alpha} : \alpha \in A_n\}$, $U_\beta = W_\beta \cap U_\alpha$, $\gamma_{n+1} = \{U_\beta : \beta \in A_{n+1}\}$, $\eta_{n+1} = \{V_\beta : \beta \in A_{n+1}\}$ and $\pi_{n+1}^{-1}(\alpha) = A_{n\alpha}$.

The objects $\{\gamma_n, \xi_n, \pi_n, U_n, \tau_n : n \in \mathbb{N}\}$ are constructed.

Let $x \in X$. Denote by $A(x)$ the set of all sequences $\alpha = (\alpha_n : n \in \mathbb{N})$ for which $\alpha_n \in A_n$ and $x \in U_{\alpha_n}$ for any $n \in \mathbb{N}$. For any $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$ there exists a unique point $y(\alpha) \in Y$ such that $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$. It is obvious that $y(\alpha) \in \theta(x)$. Let $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$. Then ϕ is a selection of θ. By construction:

— $U_\alpha \subseteq \phi^{-1}(V_\alpha)$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$;

— the mapping ϕ is lower semi-continuous;

— if $Z = \phi(X)$, then $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup\{A_n : n \in \mathbb{N}\}\}$ is an open base of the subspace Z.

We affirm that $k(cl_Y Z) \leq \tau$.

Subcase 3.1. τ is not a sequential cardinal.

In this subcase $m = sup\{|A_n| : n \in \mathbb{N}\} < \tau$ and $w(Z) \leq |A| \leq m < \tau$. In this subcase we are proved the implication $(K1 \to K11)$.

Subcase 3.2. τ is a sequential cardinal.

Let $Z_n = \phi(X_n)$ and $A_{nk} = \{\alpha \in A_n : X_k \cap U_\alpha \neq \emptyset\}$. Then $|Ank| < \tau_n$ for all $n, k \in \mathbb{N}$. Thus $w(Z_n) < \tau_n$.

Since $\phi(X) \setminus Z_n \subseteq H_n$, we have $k(cl_Y \phi(X) \leq \tau$.

In this subcase we have proved the implication $(K1 \to K2)$.

Let $H = \cap\{U_n : n \in \mathbb{N}\}$, $\mu = \{\mu_n = \{H \cap U_\alpha : \alpha \in A_n''\}$ and $q = \{q_n = \pi_n | A_{n+1}'' : A_{n+1}'' \to A_n'' : n \in \mathbb{N}\}$. By construction, we have $\cup\{W_\beta ; \beta \in q_n^{-1}(\alpha)\} = W_\alpha \subseteq cl_X W_\alpha \subseteq \theta^{-1}(V_\alpha)$ for any $\alpha \in A_n''$ and $n \in \mathbb{N}$. Let $x \in H$. Denote by $B(x)$ the set of all sequences $\alpha = (\alpha_n : n \in \mathbb{N})$ for which $\alpha_n \in A_n''$ and $x \in cl_X W_{\alpha_n}$ for any $n \in \mathbb{N}$. For any $\alpha = (\alpha_n : n \in \mathbb{N}) \in B(x)$ there exists a unique point $y(\alpha) \in Y$ such that $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$. It is obvious that $y(\alpha) \in \Phi \cap \theta(x)$. Let $\mu_1(x) = \{y(\alpha) : \alpha \in B(x)\}$. The mapping $\mu_1 : H \to \Phi$ is compact-valued and upper semi-continuous. Let $\mu(x) = \phi(x)$ for $x \in X \setminus H$ and $\mu(x) = \mu_1(x)$ for $x \in H$. Then μ is a selection of θ. Fix a closed subset $Z \subseteq Y \setminus \Phi$ of the space Y. Then $Z \cap \mu(X) \subseteq \phi(Y_n)$ for some $n \in \mathbb{N}\}$. Thus $w(Z \cap \mu(X) < \tau$. In this subcase we have proved the implication $(K1 \to K12)$, too.

Lemma 2.1 completes the proof of the theorem. □

The last theorem and Lemma 1.3 imply

Corollary 2.6. *Let X be a μ-complete space and τ be a sequential cardinal number. Then assertions $K1 - K8$ are equivalent.*

Theorem 2.5 is signicative for a sequential cardinal τ. Every compact subset of X is paracompact in X. In fact we have

Theorem 2.7. *Let X be a regular space, F be a paracompact subspace of X, τ be an infinite cardinal number, $k(F) \leq \tau$, $k(X') < \tau$ for any closed subset $X' \subseteq X \setminus F$ of X. Then assertions $K1 - K8$ are equivalent. Moreover, if the cardinal number τ is not sequential, then Properties $K1 - K8$ and $K11$ are equivalent.*

Proof. It is obvious that for any open in X set $U \supseteq F$ there exists an open subset V of X such that $F \subseteq U \subseteq cl_X U \subseteq V$

Case 1. τ is a regular cardinal number.

In this case Theorem 2.3 completes the proof.

Case 2. τ is a sequential cardinal number.

In this case Theorem 2.5 and Lemma 1.4 complete the proof.

Case 3. τ is a limit non-sequential cardinal.

Let $\tau^* = cf(\tau) < \tau$. Obviously, τ^* is a regular cardinal and $\tau^* < \tau$.

There exist a sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open covers of the space X, a sequence $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open families of the space Y, a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, a sequence $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\}$ of mappings and a sequence $\{\tau_n : n \in \mathbb{N}\}$ of cardinal numbers such that:

— $\cup\{U_\beta; \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ for every $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $\cup\{cl_Y V_\beta; \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ and $\mathrm{diam}(V_\alpha) < 2^{-n}$ for every $\alpha \in A_n$ and $n \in \mathbb{N}$;

— $|A_n| < \tau_n \leq \tau_{n+1} < \tau$ for every $n \in \mathbb{N}$;

— if $A'_n = \{\alpha \in A_n : F \cap cl_X U_\alpha = \emptyset\}$ and $A''_n = A_n \setminus A'_n$, then $| A''_n |< \tau^*$ and $F \subseteq U_n \subseteq cl_X U_n \subseteq \cup\{U_\alpha : \alpha \in A''_n\}$;

— the family $\gamma''_n = \{U_\alpha : \alpha \in A''_n\}$ is locally finite in X for every $n \in \mathbb{N}$;

— $cl_X U_n \cap cl_X U_\alpha = \emptyset$ for every $n \in \mathbb{N}$ and $\alpha \in A'_n$.

Let $\eta = \{V : V \textit{ is open in } Y \textit{ and } \mathrm{diam}(V) < 2^{-1}\}$ and $\gamma' = \{U : U \textit{ is open in } X \textit{ and } cl_X U \subseteq \theta^{-1}(V)) \textit{ for some } V \in \eta\}$. There exist a locally finite subfamily $\gamma''_1 = \{U_\alpha : \alpha \in A''_1\}$ of γ' such that $| A''_1 |< \tau^* < k(F)$ and an open subset U_1 of the space X such that $F \subseteq U_1 \subseteq cl_X U_1 \subseteq \cup\{U_\alpha : \alpha \in A''_1\}$ and $F \cap U_\alpha \neq \emptyset$ for every $\alpha \in A''_1$. For every $\alpha \in A''_1$ fix $V_\alpha \in \eta$ such that $clU_\alpha \subseteq \theta^{-1}(V_\alpha)$. Let $X_1 = X \setminus U_1$ and $\tau_1 = k(F) + \tau^*$. Since $k(X_1) \leq \tau_1 < \tau$, there exists an open subfamily $\gamma_1' = \{U_\alpha : \alpha \in A_1'\}$ of γ' such that $|A_1'| \leq \tau_1$, $X_1 \subseteq \cup\{U_\alpha : \alpha \in A_1'\}$ and $cl_X U_1 \cap (\cup\{cl_X U_\alpha : \alpha \in A_1'\}) = \emptyset$. For every $\alpha \in A_1'$ fix $V_\alpha \in \eta'$ such that $cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$. Let $A_1 = A_1' \cup A_1''$, $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$ and $\eta_1 = \{V_\alpha : \alpha \in A_1\}$.

The objects $\{\gamma_1, \xi_1, U, \tau_1\}$ are constructed.

Consider that the objects $\{\gamma_i, \xi_i, \pi_{i-1}, U_i, \tau_i, : i \leq n\}$ are constructed. Fix $\alpha \in A_n$.

Let $\eta_\alpha = \{V : V$ *is open in* Y, $cl_Y V \subseteq V_\alpha$ *and* $\mathrm{diam}(V) < 2^{-n-1}\}$ and $\gamma_\alpha^* = \{W : W$ *is open in* X *and* $cl_X W \subseteq \theta^{-1}(V)$ *for some* $V \in \eta_\alpha\}$.

Assume that $\alpha \in A_n''$.

Since $F_\alpha = F \cap cl_X U_\alpha$ is a closed subset of X, then there exists a locally finite subfamily $\gamma_\alpha'' = \{W_\beta : \beta \in A_\alpha''\}$ of γ_α^*, where $\mid A_\alpha'' \mid < \tau^*$ such that $F_\alpha \subseteq \cup\{W_\beta : \beta \in A_\alpha''\}$, $F_\alpha \cap W_\beta \neq \emptyset$ for every $\beta \in A_\alpha''\}$ and for every $\beta \in A_\alpha''$ there exists $V_\beta \in \eta_\alpha$ such that $cl_X W_\beta \subseteq \theta^{-1}(V_\beta)$. We put $U_\beta = U_\alpha \cap W_\beta$ for every $\beta \in A_\alpha''$.

Let $A_{n+1}'' = \cup\{A_\alpha'' : \alpha \in A_n''\}$, $\gamma_{n+1}'' = \{U_\alpha : \alpha \in A_{n+1}''\}$ and $\eta_{n+1}'' = \{V_\alpha : \alpha \in A_{n+1}''\}$.

The family γ_{n+1}'' is locally finite.

Let $\Phi_\alpha = cl_X U_\alpha \backslash \cup\{U_\beta : \beta \in A_\alpha''$ and $U_n' = U_n \backslash \cup\{\Phi_\alpha : \alpha \in A_n''\}$. Since the family γ_n'' is locally finite, the set U_n' is open in X and $F \subseteq U_n' \subseteq \cup\{U_\beta : \beta \in A_\alpha''\}$.

There exists an open subset U_{n+1} of X such that $U_{n+1} \subseteq cl_X U_{n+1} \subseteq \cup\{U_\beta : \beta \in A_\alpha''\}$.

Let $X_{n+1} = X \setminus U_{n+1}$ and $\tau_{n+1} = k(X_{n+1}) + \tau_n$.

For every $\alpha \in A_n$ there exist the subfamily $\gamma_\alpha' = \{W_\beta : \beta \in A_\alpha'\}$ of γ_α^* and the subfamily $\eta_{i\alpha}' = \{V_\beta : \beta \in A_\alpha'\}$ of γ_α' such that:

— $|A_\alpha'| < \tau_{n+1}$;

— $cl_X U_\alpha \setminus U_n \subseteq \cup\{W_\beta : \beta \in A_\alpha'\}$;

— $cl_X W\beta \cap cl_X U_{n+1} = \emptyset$ for any $\beta \in A_\alpha'$.

Now we put $A_\alpha = A_\alpha' \cup A_\alpha''$, $A_{n+1} = \cup\{A_\alpha : \alpha \in A_n\}$, $U_\beta = U_\alpha \cap U_\beta$ for any $\beta \in A_\alpha$, $\gamma_{n+1} = \{U_\alpha : \alpha \in A_{n+1}\}$, $\eta_{n+1} = \{V_\alpha : \alpha \in A_{n+1}\}$ and $\pi_{n+1}^{-1}(\alpha) = A_{n\alpha}$.

The objects $\{\gamma_n, \xi_n, \pi_n, U_n, \tau_n : n \in \mathbb{N}\}$ are constructed.

Since τ is not sequential, we have $m = sup\{\tau_n : n \in \mathbb{N}\} < \tau$.

Let $x \in X$. Denote by $A(x)$ the set of all sequences $\alpha = (\alpha_n : n \in \mathbb{N})$ for which $\alpha_n \in A_n$ and $x \in U_{\alpha_n}$ for every $n \in \mathbb{N}$. For every $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$ there exists a unique point $y(\alpha) \in Y$ such that $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$. It is obvious that $y(\alpha) \in \theta(x)$. Let $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$. Then ϕ is a selection of θ. By construction:

— $U_\alpha \subseteq \phi^{-1}(V_\alpha)$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$;

— the mapping ϕ is lower semi-continuous;

— if $Z = \phi(X)$, then $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup\{A_n : n \in \mathbb{N}\}\}$ is an open base of the subspace Z and $w(Z) \leq m$.

Thus we have proved the implication $(K1 \to K11)$.

Lemma 2.1 completes the proof of the theorem. □

Remark 2.8. Let X be a paracompact space and $Y \subseteq X$. Then $l(cl_X Y) \leq l(Y)$ and $k(cl_X Y) \leq k(Y)$.

Theorem 2.7, Corollary 2.6 and Lemma 1.4 yield

Corollary 2.9. *Let X be a paracompact and τ be an infinite cardinal. Then Properties $K1 - K10$ are equivalent.*

One can observe that the Corollary 2.9 follows from Proposition 2.2, Lemma 2.1 and Theorem 0.1, too.

Corollary 2.10. *Let X be a space and τ be an uncountable not sequential cardinal number. Then the following assertions are equivalent:*

1. *X is a paracompact space and $k(X) \leq \tau$.*
2. *X is a paracompact space and for every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a lower semi-continuous selection $\phi : X \to Y$ of θ such that $w(\phi(X)) < \tau$.*
3. *X is a paracompact space and for every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a single-valued selection $g : X \to Y$ such that $w(g(X)) < \tau$.*
4. *X is a paracompact space and for every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists a single-valued selection $g : X \to Y$ such that $\mid g(X) \mid < \tau$.*
5. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exist a compact-valued lower semi-continuous mapping $\varphi : X \to Y$ and a compact-valued upper semi-continuous mapping $\psi : X \to Y$ such that $w(\psi(X)) < \tau$ and $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in X$.*
6. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists an upper semi-continuous selection $\phi : X \to Y$ of θ such that $w(\phi(X)) < \tau$.*

Example 2.11. Let τ be an uncountable limit cardinal number and $m = cf(\tau)$. Fix a well ordered set A and a family of regular cardinal numbers $\{\tau_\alpha : \alpha \in A\}$ such that $\sup\{\tau_\alpha : \alpha \in A\} = \tau$ and $\tau_\alpha < \tau_\beta < \tau$ for all $\alpha, \beta \in A$ and $\alpha < \beta$. For every $\alpha \in A$ fix a zero-dimensional complete metric space X_α

such that $w(X_\alpha) = \tau_\alpha$. Let X' be the discrete sum of the spaces $\{X_\alpha : \alpha \in A\}$. Then X' is a complete metrizable space and $w(X') = \tau$. Thus $l(X') = \tau$ and $k(X') = \tau^+$. Fix a point $b \notin X'$. Put $X = \{b\} \cup X'$ with the topology generated by the open base $\{U \subseteq X' : U \textit{ is open in } X'\} \bigcup \{X \setminus \bigcup\{X_\beta : \beta \leq \alpha\} : \alpha \in A\}$. Then X is a zero-dimensional paracompact space and $\chi(X) = \chi(b, X) = cf(\tau)$. If $cf(\tau) = \aleph_0$, then X is a complete metrizable space. If $Y \subseteq X'$ is a closed subspace of X, then there exists $\alpha \in A$ such that $Y \subseteq \cup\{X_\beta : \beta < \alpha\}$, $w(Y) < \tau_\alpha$ and $k(Y) < \tau$. Therefore $k(X) = \tau$.

Let $Z = X \times [0,1]$. Then $k(Z) = \tau$ and $k(Z, \tau) = \{b\} \times [0,1]$.

Suppose that τ is not a sequential cardinal number, $\mathbb{N}$ is a discrete space and $S = X \times \mathbb{N}$. Then $k(S) = \tau$ and $k(S, \tau) = \{b\} \times \mathbb{N}$.

Moreover, if $m = cf(\tau)$ is uncountable, X_τ is a complete metrizable space, $w(X_\tau) < m$ and $Z_\tau = X \times X_\tau$, then $k(Z_\tau) = \tau$ and $k(Z_\tau, \tau) = \{b\} \times X_\tau$.

3. On the geometry of paracompact spaces. Our aim is to prove that the classes $\Pi(\tau)$ may be characterized in terms of selections. The main results of the section are the following two theorems.

Theorem 3.1. *Let X be a space and τ be an uncountable non-sequential cardinal number. Then the following assertions are equivalent:*

1. *$X \in \Pi(\tau)$, i.e. X is paracompact and $k(c\omega(X)) \leq \tau$.*
2. *X is a paracompact space and for every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a lower semi-continuous selection $\phi : X \to Y$ of θ such that $w(\phi(c\omega(X))) < \tau$.*
3. *X is a paracompact space and for every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exists a single-valued selection $g : X \to Y$ such that $w(g(c\omega(X))) < \tau$.*
4. *X is a paracompact space and for every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists a single-valued selection $g : X \to Y$ such that $\mid g(c\omega(X)) \mid < \tau$.*
5. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metrizable space Y there exist a compact-valued lower semi-continuous mapping $\varphi : X \to Y$ and a compact-valued upper semi-continuous mapping $\psi : X \to Y$ such that $w(\psi(X)) < \tau$ and $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in c\omega(X)$.*
6. *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exist a closed G_δ-set H of X and an upper semi-continuous compact-valued selection $\psi : X \to Y$ such that:*

i) $c\omega(X) \subseteq H$ *and* $w(\psi(H)) < \tau$*;*

ii) $\psi(x)$ *is a one-point set of* Y *for every* $x \in X \setminus H$*;*

iii) $cl_Y\psi(H) = cl_Y\psi(c\omega(X))$.

7. *For every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metric space* Y *there exists an upper semi-continuous compact-valued selection* $\psi : X \to Y$ *such that* $k(\psi^\infty(x)) < \tau$ *for every* $x \in X$.
8. *For every lower semi-continuous mapping* $\theta : X \to Y$ *into a discrete space* Y *there exists an upper semi-continuous selection* $\psi : X \to Y$ *such that* $| \psi^\infty(x) | < \tau$ *for every* $x \in X$.

Theorem 3.2. *Let* X *be a space and* τ *be an infinite cardinal number. Then the following assertions are equivalent:*

1. $X \in \Pi(\tau)$*, i.e.* X *is paracompact and* $k(c\omega(X)) \leq \tau$.
2. X *is a paracompact space and for every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metrizable space* Y *there exists a lower semi-continuous selection* $\phi : X \to Y$ *of* θ *such that* $k(cl_Y\phi(c\omega(X))) \leq \tau$.
3. X *is a paracompact space and for every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metrizable space* Y *there exists a single-valued selection* $g : X \to Y$ *such that* $k(cl_Y g(c\omega(X))) \leq \tau$.
4. X *is a paracompact space and for every lower semi-continuous mapping* $\theta : X \to Y$ *into a discrete space* Y *there exists a single-valued selection* $g : X \to Y$ *such that* $| g(c\omega(X)) | < \tau$.
5. *For every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metrizable space* Y *there exist a compact-valued lower semi-continuous mapping* $\varphi : X \to Y$ *and a compact-valued upper semi-continuous mapping* $\psi : X \to Y$ *such that* $k(cl_Y(\psi(c\omega(X))) \leq k(\psi(c\omega(X))) \leq \tau$ *and* $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ *for any* $x \in c\omega(X)$.
6. *For every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metric space* Y *there exist a closed* G_δ*-set* H *of* X *and an upper semi-continuous compact-valued selection* $\psi : X \to Y$ *such that:*

   i) $c\omega(X) \subseteq H$ *and* $k(\psi(H)) \leq \tau$*;*

   ii) $\psi(x)$ *is a one-point set of* Y *for every* $x \in X \setminus H$*;*

   iii) $cl_Y\psi(H) = cl_Y\psi(c\omega(X))$.
7. *For every lower semi-continuous closed-valued mapping* $\theta : X \to Y$ *into a complete metric space* Y *there exists an upper semi-continuous compact-valued selection* $\psi : X \to Y$ *such that* $k(\psi^n(x)) < \tau$ *for every* $x \in X$ *and*

any $n \in \mathbb{N}$.

8. *For every lower semi-continuous mapping* $\theta : X \to Y$ *into a discrete space* Y *there exists an upper semi-continuous selection* $\psi : X \to Y$ *such that* $|\psi^n(x)| < \tau$ *for every* $x \in X$ *and any* $n \in \mathbb{N}$.

P r o o f o f t h e T h e o r e m s: Let $X \in \Pi(\tau)$ and $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping into a complete metric space (Y, d). For every subset L of Y and every $n \in \mathbb{N}$ we put $O(L, n) = \{y \in Y : d(y, L) = inf\{d(x, z) : z \in L\} < 2^{-n}\}$. Obviously, $cl_Y L = \cap\{O(L, n) : n \in \mathbb{N}$ and $cl_Y O(L, n+1) \subseteq O(L, n)$ for any $n \in \mathbb{N}$.

By virtue of the Michael's Theorem 0.1, there exist a compact-valued lower semi-continuous mapping $\varphi : X \to Y$ and a compact-valued upper semi-continuous mapping $\psi : X \to Y$ such that $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$ for any $x \in c\omega(X)$.

From Proposition 2.2 it follows that $k(cl_Y(\psi(c\omega(X))) \leq k(\psi(c\omega(X))) \leq \tau$ and $k(cl_Y(\varphi(c\omega(X))) \leq k(cl_Y(\psi(c\omega(X)))) \leq \tau$. Moreover, if τ is a not sequential cardinal number, then $w(\varphi(c\omega(X)) \leq w(\psi(c\omega(X))) < \tau$.

Therefore, the assertions 2, 3, 4 and 5 of Theorems follow from the assertion 1.

It will be affirmed that there exist a sequence $\{\phi_n : X \to Y : n \in \mathbb{N}\}$ of lower semi-continuous compact-valued mappings, a sequence $\{\psi_n : X \to Y : n \in \mathbb{N}\}$ of upper semi-continuous compact-valued mappings, a sequense $\{V_n : n \in \mathbb{N}\}$ of open subsets of Y and a sequense $\{H_n : n \in \mathbb{N}\}$ of open-and-closed subsets of X such that:

1) $\psi_{n+1}(x) \subseteq \phi_n(x) \subseteq \psi_n(x) \subseteq \theta(x)$ for every $x \in X$ and every $n \in \mathbb{N}$;

2) $\phi_n(x) = \psi_n(x)$ is a one-point subset of Y for every $x \in X \setminus H_n$ and for every $n \in \mathbb{N}$;

3)$H_{n+1} \subseteq \{x \in X : \psi_n(x) \subseteq V_n\}$, $H_{n+1} \subseteq H_n$ and $V_{n+1} = O(\psi_n(c\omega(X))$ for every $n \in \mathbb{N}$;

Let $V_1 = O(\theta(c\omega(X))$ and $U_1 = \theta^{-1}V_1$. From Lemma 0.2 it follows that there exists an open-and-closed subset H_1 of X such that $c\omega(X) \subseteq H_1 \subseteq U_1$.

Since $\dim(X \setminus H_1) = 0$ there exists a single-valued continuous mapping $h_1 : X \setminus H_1 \to Y$ such that $h_1(x) \in \theta(x)$ for every $x \in X \setminus H_1$. Since H_1 is a paracompat space, V_1 is a complete metrizable space and $\theta_1 : H_1 \to V_1$, where $\theta_1(x) = V_1 \cap \theta(x)$, is a lower semicontinuous closed-valued in V_1 mapping, by virtue of Theorem 0.1, there exist a compact-valued lower semi-continuous mapping $\varphi_1 : H_1 \to V_1$ and a compact-valued upper semi-continuous mapping $\lambda_1 : H_1 \to V_1$ such that $\varphi_1(x) \subseteq \lambda_1(x) \subseteq \theta_1(x)$ for any $x \in H_1$.

Put $\psi_1(x) = \phi_1(x) = h_1(x)$ for $x \in X \setminus H_1$ and $\psi_1(x) = \lambda_1(x)$, $\phi_1(x) = \varphi_1(x)$ for $x \in H_1$.

The objects ϕ_1 and ψ_1 are constructed.

Suppose that $n > 1$ and the objects ϕ_{n-1},ψ_{n-1}, H_{n-1} and V_{n-1} had been constructed.

We put $F_n = cl_Y \psi_{n-1}(c\omega(X))$, $V_n = O(F_n, n)$ and $U_n = \{x \in H_{n-1} : \psi_{n-1}(x) \subseteq V_n\}$. From Lemma 0.2 it follows that there exists an open-and-closed subset H_n of X such that $c\omega(X) \subseteq H_n \subseteq U_n$.

Since $\dim(X \setminus H_n) = 0$ there exists a single-valued continuous mapping $h_n : X \setminus H_n \to Y$ such that $h_n(x) \in \phi_{n-1}(x)$ for every $x \in X \setminus H_n$. By construction, we have $\phi_{n-1} \subseteq \psi(x) \subseteq V_n$ for any $x \in H_n$. Since H_n is a paracompat space, V_n is a complete metrizable space and $\theta_n : H_n \to V_n$, where $\theta_n(x) = V_n \cap \phi_{n-1}(x)$, is a lower semicontinuous closed-valued in V_n mapping, by virtue of Theorem 0.1, there exist a compact-valued lower semi-continuous mapping $\varphi_n : H_n \to V_n$ and a compact-valued upper semi-continuous mapping $\lambda_n : H_n \to V_n$ such that $\varphi_n(x) \subseteq \lambda_n(x) \subseteq \theta_n(x)$ for any $x \in H_n$.

Put $\psi_n(x) = \phi_n(x) = h_n(x)$ for $x \in X \setminus H_n$ and $\psi_n(x) = \lambda_n(x)$, $\phi_n(x) = \varphi_n(x)$ for $x \in H_n$. The objects ϕ_n and ψ_n are constructed.

Now we put $\lambda(x) = \cap\{\psi_n(x) : n \in \mathbb{N}\}$ for any $x \in X$ and $H = \cap\{H_n : n \in \mathbb{N}\}$.

Sinse $\lambda^{-1}(\Phi) = \cap\{\psi_n^{-1}(\Phi) : n \in \mathbb{N}\}$ for any closed subset Φ of Y, the mapping λ is compact-valued and upper semi-continuous. By construction,

i) $c\omega(X) \subseteq H$ and $k(\lambda(H)) \leq \tau$;

ii) $\lambda(x)$ is a one-point set of Y for every $x \in X \setminus H$;

iii) $cl_Y \lambda(H) = cl_Y \lambda(c\omega(X))$;

iv) $\lambda(\lambda^{-1}(A)) \subseteq A \cup \lambda(H)$ for every subset A of Y.

Therefore, the assertions 6, 7 and 8 of Theorems follow from the assertion 1.

($8 \Rightarrow 1$) Let $\gamma = \{U_\alpha : \alpha \in A\}$ be an open cover of X. On A introduce the discrete topology and put $\theta(x) = \{\alpha \in A : x \in U_\alpha\}$ for $x \in X$. Since $\theta^{-1}(H) = \bigcup\{U_\alpha : \alpha \in H\}$ for every subset H of A, the mapping $\theta : X \to A$ is lower semi-continuous. Let $\psi : X \to A$ be an upper semi-continuous selection of θ with $| \psi^2(x) | < \tau$ for every $x \in X$. Then $\xi = \{\Psi_\alpha = \psi^{-1}(\alpha) : \alpha \in A\}$ is a closed closure-preserving τ^--star shrinking of the cover ξ. By virtue of Proposition 1.5, the assertion 1 follows from the assertion 8. $\square$

Corollary 3.3. *For a topological space X the following assertions are equivalent:*

1) *X is paracompact and $c\omega(X)$ is compact.*

2) *X is strongly paracompact and $c\omega(X)$ is compact.*
3) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exist an upper semi-continuous compact-valued selection $\psi : X \to Y$ and a closed G_δ-subset H of X such that $c\omega(X) \subseteq H$, $cl_Y(\psi(H))$ is compact and $\psi(x)$ is an one-point set for every $x \in X \setminus H$.*
4) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exists an upper semi-continuous selection $\psi : X \to Y$ such that $cl_Y \psi^\infty(x)$ is compact for every $x \in X$.*
5) *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists an upper semi-continuous selection $\psi : X \to Y$ such that the set $\psi^\infty(x)$ is finite for every $x \in X$.*
6) *For every open cover of X there exists an open star-finite shrinking.*

Proof. For the implication $(1 \Rightarrow 2)$ see Proposition 4, [4].
For the implications $(1 \Leftrightarrow 6)$ see Proposition 5, [4]. □

Corollary 3.4. *For a space and an infinite cardinal number τ the following assertions are equivalent:*

1) *X is paracompact and $l(c\omega(X)) \leq \tau$.*
2) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exist an upper semi-continuous compact-valued selection $\psi : X \to Y$ and a closed G_δ-subset H of X such that $c\omega(X) \subseteq H$ and $w(\psi(H)) \leq \tau$; $\psi(x)$ is an one-point set for every $x \in X \setminus H$.*
3) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exists an upper semi-continuous compact-valued selection $\psi : X \to Y$ such that $w(\psi^\infty(x)) \leq \tau$ for every $x \in X$.*
4) *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists an upper semi-continuous selection $\psi : X \to Y$ such that $| \psi^\infty(x) | \leq \tau$ for every $x \in X$.*

Corollary 3.5. *For a topological space X the following assertions are equivalent:*

1) *X is paracompact and $c\omega(X)$ is Lindelöf.*
2) *X is strongly paracompact and $c\omega(X)$ is Lindelöf.*
3) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exist an upper semi-continuous compact-valued selection $\psi : X \to Y$ and a closed G_δ-subset H of X such that*

$c\omega(X) \subseteq H$, $\psi(H)$ is separable and $\psi(x)$ is a one-point set for every $x \in X \setminus H$.

4) *For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space Y there exists an upper semi-continuous compact-valued selection $\psi : X \to Y$ such that $\psi^\infty(x)$ is separable for every $x \in X$.*
5) *For every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space Y there exists an upper semi-continuous selection $\psi : X \to Y$ such that the set $\psi^\infty(x)$ is countable for every $x \in X$.*
6) *For every open cover of X there exists an open star-countable shrinking.*

Example 3.6. Let A be an uncountable set and X_α be a non-empty compact space for every $\alpha \in A$. Let $X = \bigoplus\{X_\alpha : \alpha \in A\}$ be the discrete sum of the space $\{X_\alpha : \alpha \in A\}$. Let $B = \{\alpha \in A : \dim X_\alpha \neq 0\}$. Then $c\omega(X)$ is compact if and only if the set B is finite. If the set B is infinite then $l(c\omega(X)) =| B |$ and $k(c\omega(X)) =| B |^+$.

Example 3.7. Let τ be an uncountable non-sequential cardinal number. Fix an infinite set A_m for every cardinal number $m < \tau$ assuming that $A_m \cap A_n = \emptyset$ for $m \neq n$. Put $A = \bigcup\{A_m : m < \tau\}$. Let $\{X_\alpha : \alpha \in A\}$ be a family of non-empty compact spaces assuming that $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$. Put $B_m = \{\alpha \in A_m : \dim X_\alpha \neq 0\}$ and $1 \leq| B_m |\leq m$ for every $m < \tau$. Fix a point $b \notin \bigcup\{X_\alpha : \alpha \in A\}$. Let $X = \{b\} \cup (\bigcup\{X_\alpha : \alpha \in A\})$. Suppose that X_α is an open subset of X and $\{H_m = \{b\} \cup (\bigcup\{X_\alpha : \alpha \in A_n, n \leq m\}) : m < \tau\}$ is a base of X at b. If $Z = \{b\} \cup (\bigcup\{X_\alpha : \alpha \in B_m, m < \tau\})$, then $c\omega(X) \subseteq Z$ and $k(c\omega(X)) \leq k(Z) = l(Z) = \tau$.

Example 3.8. Let τ be a regular uncountable cardinal number, A be an infinite set, $\tau <| A |$, $\{X_\alpha : \alpha \in A\}$ be a family of non-empty compact spaces, $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$, $B = \{\alpha \in A : \dim X_\alpha \neq 0\}$, $\tau =| B |$ and $b \notin \bigcup\{X_\alpha : \alpha \in A\}$. Let $X = \{b\} \cup (\bigcup\{X_\alpha : \alpha \in A\})$. Suppose that X_α is an open subset of X and $\{U_H = X \setminus \bigcup\{X_\alpha : \alpha \in H\} : H \subseteq A, | H |< \tau\}$ is a base of X at b. If $Z = \{b\} \cup (\bigcup\{X_\alpha : \alpha \in B\})$, then $c\omega(X) \subseteq Z$ and $k(c\omega(X)) \leq k(Z) = l(Z) = \tau$.

Example 3.9. Let τ be a regular uncountable limit cardinal number and $2^m < \tau$ for any $m < \tau$. Let $\{m_\alpha : \alpha \in A\}$ be a family of infinite cardinal numbers such that $|A| = \tau$, the set A is well ordered and $m_\alpha < m_\beta$, $|\{\mu \in A : \mu \leq \alpha\}| < \tau$ provided $\alpha, \beta \in A$ and $\alpha < \beta$. For any $\alpha \in A$ fix a discrete space of the cardinality m_α. Let $X = \Pi\{X_\alpha : \alpha \in A\}$. If $x = (x_\alpha : \alpha \in A) \in X$ and $\beta \in A$, then $O(\beta, x) = \{y = (y_\alpha : \alpha \in A) \in X : y_\alpha = x_\alpha \textit{ for any } \alpha \leq \beta\}$. The family $\{O(\beta, x) : \beta \in A, x \in X\}$ form the open base of the space X. The space X is paracompact and $w(X) = l(X) = \tau$. It is obvious that $c(X,\tau) = X$, we have $k(X) = \tau^+$. If $\alpha \in A$, then $\gamma_\alpha = \{O(\alpha, x) : x \in X\}$ is an open discrete cover of X and $|\gamma_\alpha| = 2^{m_\alpha} < \tau$.

4. On the class $\Pi(0)$ of spaces. In the present section the class of all paracompact spaces X such that $\dim X = 0$ is studied.

Definition 4.1 *A set-valued mapping $\psi : X \longrightarrow Y$ is called virtual single-valued if $\psi^\infty(x) = \psi(x)$ for every $x \in X$.*

Remark 4.2 It is obvious that for a set-valued mapping $\theta : X \longrightarrow Y$ the following conditions are equivalent:

1. ψ is a virtual single-valued mapping;
2. $\psi^2(x) = \psi(x)$ for every $x \in X$;
3. $\psi^n(x) = \psi(x)$ for every $x \in X$ and some $n \geq 2$;
4. $\psi(x) = \psi(y)$ provided $x, y \in X$ and $\psi(x) \cap \psi(y) \neq \emptyset$.
5. $\psi^{-1}(y) = \psi^{-1}(z)$ provided $y, z \in Y$ and $\psi^{-1}(y) \cap \psi^{-1}(z) \neq \emptyset$.

Note that, if $f : X \longrightarrow Y$ is a single-valued mapping onto a space Y, then f^{-1} and f are virtual single-valued mappings.

Denote with $D = \{0, 1\}$ the two-point discrete space.

Theorem 4.3. *For a space X, the following assertions are equivalent:*

1. *X is normal and $\dim X = 0$;*
2. *For every lower semi-continuous mapping $\theta : X \longrightarrow D$ there exists a virtual single-valued lower semi-continuous selection;*
3. *For every lower semi-continuous mapping $\theta : X \longrightarrow D$ there exists a virtual single-valued upper semi-continuous selection;*
4. *For every lower semi-continuous mapping $\theta : X \longrightarrow D$ there exists a single-valued continuous selection.*

Proof. Implications $(1 \Leftrightarrow 4)$ is a well known fact. Implications $(4 \Rightarrow 2)$ and $(4 \Rightarrow 3)$ are obvious as every single-valued continuous selection is virtual single-valued.

$(2 \Rightarrow 1)$ and $(3 \Rightarrow 1)$ Let F_1 and F_2 be two disjoint closed subsets of X. Put $\theta(x) = \{0\}$ for $x \in F_1$, $\theta(x) = \{1\}$ for $x \in F_2$ and $\theta(x) = \{0, 1\}$ for $x \in X \setminus (F_1 \cup F_2)$. The mapping $\theta : X \longrightarrow D$ is lower semi-continuous. Suppose that $\lambda : X \longrightarrow D$ is a virtual single-valued selection of θ. Put $H_1 = \lambda^{-1}(0)$ and $H_2 = \lambda^{-1}(2)$. Then $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$., $X = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$. If

λ is lower semi-continuous (or upper semi-continuous)) the sets H_1, H_2 are open (closed). $\square$

Let τ be an infinite cardinal number. A topological space X is called *τ-paracompact* if X is normal and every open cover of X of the cardinality $\leq \tau$ has a locally finite open refinement.

Theorem 4.4. *For a space X and an infinite cardinal number τ the following assertions are equivalent:*

1. *X is a τ-paracompact space and* $\dim X = 0$.
2. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y of the weight $\leq \tau$ there exists a virtual single-valued lower semi-continuous selection;*
3. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y of the weight $\leq \tau$ there exists a virtual single-valued upper semi-continuous selection;*
4. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y of the weight $\leq \tau$ there exists a single-valued continuous selection;*
5. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a discrete space Y of the cardinality $\leq \tau$ there exists a single-valued continuous selection.*

Proof. Let $\gamma = \{U_\alpha : \alpha \in A\}$ be an open cover of X and $|A| \leq \tau$. Consider that A is a wellordered discrete space and $\theta(x) = \{\alpha \in A : x \in U_\alpha\}$ for any $x \in X$. Then θ is a lower semi-continuous mapping. Suppose that $\psi : X \to Y$ is a a virtual single-valued lower or upper semi-continuous selection of θ. For any $x \in X$ we denote by $f(x)$ the first element of the set $\psi(x)$. Then $f : X \to Y$ is a single-valued continuous selection of the mappings θ and ψ. Therefore $\{H_\alpha = f^{-1}(\alpha) : \alpha \in A\}$ is a discrete refinement of γ. The implications $(2 \Rightarrow 1)$, $(2 \Rightarrow 4)$, $(3 \Rightarrow 1)$, $(3 \Rightarrow 4)$ and $(5 \Rightarrow 1)$ are proved. The implications $(4 \Rightarrow 5)$, $(4 \Rightarrow 2)$ and $(4 \Rightarrow 3)$ are obvious. The implication $(1 \Rightarrow 4)$ is wellknown (see [1, 2]). $\square$

Corollary 4.5. *For a space X the following assertions are equivalent:*

1. *X is a paracompact space and* $\dim X = 0$.
2. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y there exists a virtual single-valued lower semi-continuous selection;*
3. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y there exists a virtual single-valued upper semi-continuous selection;*

4. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a complete metrizable space Y there exists a single-valued continuous selection;*
5. *For every lower semi-continuous mapping $\theta : X \longrightarrow Y$ into a discrete space Y there exists a single-valued continuous selection.*

Remark 4.6. Let Y be a topological space. Then:

1. If the space Y is discrete, then every lower semi-continuous virtual single-valued mapping or every upper semi-continuous virtual single-valued mapping $\theta : X \longrightarrow Y$ into the space Y is continuous.
2. If the space Y is not discrete, then there exist a paracompact space X and a virtual single-valued mapping $\theta : X \longrightarrow Y$ such that:
   — θ is upper semi-continuous and not continuous;
   — X has a unique not isolated point.
3. If Y has an open non-discrete subspace U and $|U| \leq |Y \setminus U|$, then there exist a paracompact space X and a virtual single-valued mapping $\theta : X \longrightarrow Y$ such that:
   — θ is lower semi-continuous and not continuous;
   — X has a unique not isolated point.

Remark 4.7 Let $\gamma = \{Hy : y \in Y\}$ be a cover of a space X, Y be a discrete space and $\theta_\gamma(x) = \{y \in Y : x \in Hy\}$. Then:

— the mapping θ_γ is lower semi-continuous if and only if γ is an open cover;

— the mapping θ_γ is upper semi-continuous if and only if γ is a closed and conservative cover;

— the mapping $\varphi : X \to Y$ is a selection of the mapping θ_γ if and only if $\{Vy = \varphi^{-1}(y) : y \in Y\}$ is a shrinking of γ.

Therefore, the study of the problem of the selections for the mappings into discrete spaces is an essential case of this problem.

Addendum. The main results of the present paper were announced in [5, 6, 7]. The main results from section 3 were announced in 2007 in [7]. At the time when this manuscript was in process, the authors were informed that results similar to those in section 3 were announced by V. Gutev and T. Yamauchi. These results of V. Gutev and T. Yamauchi are now published in [9].

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