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PLUS-MINUS PROPERTY AS A GENERALIZATION OF THE DAUGAVET PROPERTY

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Abstract. It was shown in [2] that the most natural equalities valid for every rank-one operator T in real Banach spaces lead either to the Daugavet equation $\|I+T\| = 1+\|T\|$ or to the equation $\|I-T\| = \|I+T\|$. We study if the spaces where the latter condition is satisfied for every finite-rank operator inherit the properties of Daugavet spaces.

1. Introduction. A Banach space X has the Daugavet property if the Daugavet equation $\|I+T\| = 1+\|T\|$ holds for every rank-one operator $T : X \to X$. It was shown in [4], [5] that for such spaces the Daugavet equation automatically extends to wider classes of operators, e.g., operators that do not fix copies of l_1, and strong Radon-Nikodým operators (meaning the operators that map the unit ball into a set with the Radon-Nikodým property). Classical examples of Banach spaces having the Daugavet property are $C(K)$ for every

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perfect compact Hausdorff topological space K, and $L_1(\mu)$ for every non atomic measure μ.

There are several geometrical characterizations of the Daugavet property which helped to prove a number of results of isomorphic nature. For instance, a Banach space with the Daugavet property contains l_1, it does not have the Radon-Nikodým property (moreover, every slice of the unit ball of such a space has diameter 2), and it does not have an unconditional basis.

In [2] spaces with other norm equalities for certain classes of bounded linear operators were studied. It was shown there that the most natural equalities for rank-one operators lead either to the Daugavet property or to the strictly weaker property that for some norm-one scalar ω every bounded rank-one operator T satisfies the condition $\|I+\omega T\| = \|I+T\|$. So, in particular, in the real case we get only two different properties of the kind – the Daugavet property and the property which is derived from the operator equality $\|I-T\| = \|I+T\|$. Since the first one is already well-studied it is quite natural to investigate the other one.

Let X be a Banach space, $\mathcal{G} \subset L(X)$ be a class of operators. We say that a Banach space X satisfies the *plus-minus property with respect to* $\mathcal{G}$ and write $X \in D_{\pm}(\mathcal{G})$, if every $T \in \mathcal{G}$ satisfies the equation $\|I+T\| = \|I-T\|$. In the most valuable for us case when $\mathcal{G}$ is the class of finite-rank operators we call this property just *plus-minus property* and denote it just $D_{\pm}$. In [2] some results on spaces with plus-minus property with respect to rank-one operators ($D_{\pm}(rank\,1)$) were obtained. For example, it was shown that as well as in spaces with the Daugavet property, every slice of the unit ball of a space with the $D_{\pm}(rank\,1)$ has diameter 2, and so these spaces can not have the Radon-Nikodým property (see Proposition 5.2). It was also mentioned there that if $X \in D_{\pm}(rank\,1)$ then this space is a "space with bad projections", which means that every one-codimensional projection in $L(X)$ is at least of norm 2.

In this paper we obtain some new results on spaces with the plus-minus property and also introduce a geometrical approach to such spaces.

As we mentioned above, spaces with the Daugavet property do not possess an unconditional basis. Since the plus-minus property is a generalization of the Daugavet property, it is natural to find out whether a Banach space with the plus-minus property can have an unconditional basis. In Section 2 we give a sufficient condition for $D_{\pm}$-spaces not to have an unconditional basis. Also we show that if the algebraic Conjecture 2.4. from [1] was true, then we could prove that no space with the plus-minus property can have an unconditional basis. However, we present a counterexample to this conjecture, and so the question whether a

$D_\pm$-space can have unconditional basis remains open.

It was more convenient to work with geometrical characterizations of the Daugavet property than with the original definition. Since we could not find such characterizations for the plus-minus property, we introduce in Section 3 the notion of the strong plus-minus property ($SD_\pm$), which implies the plus-minus property and has a geometrical definition. We prove that in such spaces the equation $\|I-T\| = \|I+T\|$ is satisfied even for strong Radon-Nikodým operators. Also we show that the Daugavet property implies the strong plus-minus property and is strictly stronger. Actually, in section 4 we present a class of examples of spaces with the strong plus-minus property but without the Daugavet property. These examples also serve as examples of $D_\pm$-spaces without the Daugavet property. In [2] an example from this class was given, namely $C[0,1] \oplus_2 C[0,1]$, but the proof was using heavily the specifics of the space $C[0,1]$, whereas now we get the whole class as a simple corollary of the Proposition 4.4. We conclude Section 3 by proving that $SD_\pm$ is inherited by subspaces of finite co-dimension.

In Section 4 we study questions concerning the stability of the strong plus-minus property. Namely, we prove that if $X \in SD_\pm$ and E is an arbitrary Banach function space (see Definition 4.1 below) then $E(X) \in SD_\pm$. Then we show that a 1-unconditional sum of two spaces with the strong plus-minus property also has this property – this is the Proposition 4.4, from which we get the examples mentioned above. And finally, we prove the converse to the statement of Proposition 4.4.

We end the paper by listing several open questions in Section 5.

2. Unconditional basis in spaces with the plus-minus property and one algebraical counterexample.

We start this section by giving a sufficient condition for an $SD_\pm$ space not to have unconditional basis. The proof is quite similar to those of Theorem 2.1 in [3] but we still present it for the sake of completeness.

Proposition 2.1. *Let X be a Banach space, $X \in D_\pm$ and let $\max\limits_{\pm} \|I \pm P\|^2 \geq 1 + \|P\|^2$ for every finite-rank projection P. Then there is no unconditional basis in X.*

P r o o f. Suppose for the contrary that X has an unconditional basis $\{e_k\}_{k=1}^\infty$. Let D be the set of all finite subsets of $\mathbb{N}$. For every $A \in D$ denote by P_A the finite-rank projection from X onto $lin\{e_k\}_{k\in A}$. Then since $\{e_k\}_{k=1}^\infty$ is an unconditional basis we have that $\sup\limits_{A\in D} \|P_A\|$ is finite and we denote it by M. Pick

$A \in D$ such that $M^2 < \|P_A\|^2 + 1$ and consider the operator $I - P_A$. Obviously, this operator is a projection on $lin\{e_k\}_{k\in\mathbb{N}\setminus A}$ and therefore

$$\|I - P_A\| = \sup_{B\in D,\ B\subset\mathbb{N}\setminus A} \|P_B\| \leq \sup_{B\in D} \|P_B\| = M.$$

So we got that $\|I - P_A\| \leq M$. But on the other hand since $X \in D_\pm$ and $\max\limits_{\pm} \|I \pm P\|^2 \geq 1 + \|P\|^2$ for every finite-rank projection P, we have that

$$\|I - P_A\| = \|I + P_A\| = \max_{\pm} \|I \pm P_A\| \geq \sqrt{1 + \|P_A\|^2} > M$$

by the choice of A. The contradiction obtained proves that X can not have an unconditional basis. □

In [1] there was a conjecture (Conjecture 2.4) that the inequality $\max\limits_{\pm} \|1 \pm x\|^2 \geq 1 + \|x\|^2$ holds for every element x of any unital Banach algebra. If it was so, we could prove the following theorem:

No Banach space with the plus-minus property can have an unconditional basis.

Indeed, for $X \in D_\pm$ we could consider the unital Banach algebra $L(X)$ of all bounded linear operators on X and obtain that $\max\limits_{\pm} \|I \pm T\|^2 \geq 1 + \|T\|^2$ for every $T \in L(X)$. In particular, $\max\limits_{\pm} \|I \pm P\|^2 \geq 1 + \|P\|^2$ for every finite-rank projection P on X and then the result of the theorem would follow immediately from the Proposition 2.1.

However it appears that there is a counterexample to the conjecture mentioned above and so the question whether a Banach space with the plus-minus property can have an unconditional basis remains open. We are now going to present an algebra B that is a counterexample to Conjecture 2.4 of [1].

Let B be a two-dimensional real Banach algebra spanned on a unit element I and an idempotent element A with $\|A\| = p \geq 3$. Every element of such algebra has the form $\alpha A + \beta I$ ($\alpha, \beta \in \mathbb{R}$), and we define the norm on this algebra as follows:

$$\|\alpha A + \beta I\| = \max\{|\alpha|(p-1) + |\beta|, |\alpha|p\}$$

(Function $f(\alpha, \beta) = \max\{|\alpha|(p-1) + |\beta|, |\alpha|p\}$ defines a norm on B for $p > 1$: maximum of two seminorms is a seminorm and the condition $(f(\alpha, \beta) = 0) \Rightarrow ((\alpha, \beta) = 0)$ is evident.)

Note that if we define the norm as above, then we will have that

$$\max\{\|I+A\|,\|I-A\|\} = p = \|A\| < \sqrt{1+\|A\|^2},$$

and so this indeed will be a required counterexample.

The only thing we need to check is that the norm defined is an algebra norm, i.e. that

$$\|(\alpha A+\beta I)(\gamma A+\delta I)\| \le \|\alpha A+\beta I\|\|\gamma A+\delta I\|$$

for every $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. Since the norm is positive homogeneous and continuous, it is enough to consider only the case when $\beta=\delta=1$. So, we just need to prove that

$$\|(\alpha A+I)(\gamma A+I)\| \le \|\alpha A+I\|\|\gamma A+I\|. \tag{1}$$

By the assumption $A^2=A$ (A is an idempotent), so using our definition of the norm we can rewrite (1) as follows:

$$\begin{aligned}\max\{|\alpha+\gamma+\alpha\gamma|(p-1)+1, |\alpha+\gamma+\alpha\gamma|p\}\\ \le \max\{|\alpha|(p-1)+1,|\alpha|p\}\cdot\max\{|\gamma|(p-1)+1,|\gamma|p\}.\end{aligned}$$

To prove this inequality it is enough to prove that

$$|\alpha+\gamma+\alpha\gamma|(p-1)+1 \le (|\alpha|(p-1)+1)(|\gamma|(p-1)+1), \tag{2}$$

and

$$\begin{aligned}|\alpha+\gamma+\alpha\gamma|p &\le (|\alpha|+|\gamma|+|\alpha||\gamma|)p\\ &\le \max\{|\alpha|(p-1)+1,|\alpha|p\}\cdot\max\{|\gamma|(p-1)+1,|\gamma|p\}.\end{aligned} \tag{3}$$

The inequality (2) is almost evident since $p\ge 3$ and so $(p-1)^2\ge(p-1)$:

$$|\alpha+\gamma+\alpha\gamma|(p-1)+1 \le (|\alpha|+|\gamma|+|\alpha||\gamma|)(p-1)+1 \le (|\alpha|(p-1)+1)(|\gamma|(p-1)+1).$$

Now we will prove (3). Consider two cases: $(|\alpha|-1)(|\gamma|-1)\ge 0$ and $(|\alpha|-1)(|\gamma|-1)<0$. For the first case we will prove that

$$(|\alpha|+|\gamma|+|\alpha||\gamma|)p \le (|\alpha|(p-1)+1)(|\gamma|(p-1)+1).$$

Indeed, this inequality is equivalent to the following:

$$0 \le ((p-1)^2-p)|\alpha||\gamma| - (|\alpha|+|\gamma|)+1.$$

We have that $(p-1)^2 - p = p^2 - 3p + 1 \geq 1$, since $p \geq 3$, thus

$$((p-1)^2 - p)|\alpha||\gamma| - (|\alpha| + |\gamma|) + 1 \geq |\alpha||\gamma| - (|\alpha| + |\gamma|) + 1 = (|\alpha| - 1)(|\gamma| - 1) \geq 0,$$

and this case is proved.

If $(|\alpha|-1)(|\gamma|-1) < 0$ then without loss of generality we can assume that $|\alpha| \leq 1 \leq |\gamma|$. Then we will prove that

$$(|\alpha| + |\gamma| + |\alpha||\gamma|)p \leq (|\alpha|(p-1) + 1)|\gamma|p.$$

In this last inequality we have that both parts of it are linear in α and since we have that $0 \leq |\alpha| \leq 1$, we just need to check that for $|\alpha| = 0$ and $|\alpha| = 1$ the inequality is satisfied. The case of $|\alpha| = 0$ is evident and for $|\alpha| = 1$ we need to check that $(p-2)|\gamma| \geq 1$, but this is true because $p \geq 3$ and $|\gamma| \geq 1$ by our assumptions. So, we proved (3) and this concludes the proof that our norm is well-defined on the Banach algebra B.

3. Strong plus-minus property.

Definition 3.1. *We say that a Banach space X has a* strong plus-minus *property and write $X \in SD_{\pm}$ if for every relatively weakly open set U in B_X and every element $y \in X$:*

$$\sup_{x \in U} \|x + y\| = \sup_{x \in U} \|x - y\|.$$

Remark 3.2. A finite-dimensional space can not have the strong plus-minus property because in such a space there are weakly open sets of arbitrarily small diameter.

We will now give a characterization of the strong plus-minus property which is more appropriate to work with:

Lemma 3.3. *For a Banach space X the following conditions are equivalent:*

(*i*) *X has the strong plus-minus property;*

(*ii*) *for every $x \in S_X$, every $y \in X$, every U – weak neighborhood of x in B_X, and every $\varepsilon > 0$ there exists $z \in U$ such that $\|y - z\| \geq \|x + y\| - \varepsilon$.*

Proof. First let X be a space with the strong plus-minus property. Consider arbitrary $x \in S_X$, $y \in X$, U – weak neighborhood of x in B_X, and $\varepsilon > 0$. Since $X \in SD_{\pm}$, we have that $\|x+y\| \leq \sup_{u \in U} \|u+y\| = \sup_{u \in U} \|u-y\|$.

So there exists $z \in U$ such that $\|z - y\| \geq \sup_{u \in U} \|u - y\| - \varepsilon \geq \|x + y\| - \varepsilon$ and we are done.

Conversely, let X satisfy (ii). Take arbitrary relatively weakly open U in B_X, $y \in X$ and $\varepsilon > 0$. We first show that we can choose $x \in U \cap S_X$ such that $\|x+y\| > \sup_{u \in U} \|u+y\| - \varepsilon$. Let $x_0 \in U$ be such that $\|x_0+y\| > \sup_{u \in U} \|u+y\| - \varepsilon$. If $x_0 \in S_X$ then we are done, so suppose that $\|x_0\| < 1$. U is a weak neighborhood of x_0 in B_X and so $U = V \cap B_X$ for a weakly open set V in X. Now from the Remark 3.2 it follows that X is infinite-dimensional and thus there exists a straight line l in X such that $x_0 \in l$ and $l \subset V$. Since $\|x_0\| < 1$ and $x_0 \in l$, we have that l has two points of intersection with S_X – say x_1 and x_2. Then $x_1, x_2 \in U$ and $x_0 \in [x_1, x_2]$. Now because the function $f(u) = \|u + y\|$ is convex, we get that $\max_{i=1,2} \|x_i + y\| \geq \|x_0 + y\| > \sup_{u \in U} \|u + y\| - \varepsilon$. This means that taking $x = x_i$ for the appropriate i we will obtain $x \in U \cap S_X$ such that $\|x + y\| > \sup_{u \in U} \|u + y\| - \varepsilon$. Now we can use (ii) to find $z \in U$ such that $\|y - z\| \geq \|x + y\| - \varepsilon$. Then we will get that

$$\sup_{u \in U} \|u - y\| \geq \|y - z\| \geq \|x + y\| - \varepsilon > \sup_{u \in U} \|u + y\| - 2\varepsilon,$$

and since ε was arbitrary this proves that $\sup\limits_{u \in U} \|u - y\| \geq \sup_{u \in U} \|u + y\|$. The reversed inequality can be obtained by taking $(-y)$ instead of y. □

Using this characterization we can prove that the strong plus-minus property implies the plus-minus property.

Theorem 3.4. *Let X be a Banach space and $X \in SD_{\pm}$. Then every strong Radon-Nikodým operator $T : X \to X$ satisfies the equality $\|I + T\| = \|I - T\|$. In particular, every finite-rank operator satisfies this equality and so $X \in SD_{\pm}$ implies $X \in D_{\pm}$.*

Proof. We first prove that $\|I - T\| \geq \|I + T\|$. Denote $a = \|I + T\|$ and fix some $\varepsilon > 0$. Consider $A = \{x \in B_X : \|(I + T)x\| > a - \varepsilon\}$. Fix some $x_0 \in A$ and choose $x^* \in S_{X^*}$ such that $x^*((I + T)x_0) = \|(I + T)x_0\|$, i.e. x^* is a support functional for the element $(I + T)x_0$. Then the set $V = \{x \in B_X : x^*(I + T)x > a - \varepsilon\}$ is a slice of B_X and $V \subset A$. Let us consider $T(V)$. Since T is a strong Radon-Nikodým operator we can find a slice W of $T(V)$ with diam $W < \varepsilon$. Take $U = V \cap T^{-1}(W)$. We have that U is an intersection of two slices and is therefore a relatively weakly open subset of B_X. Choose an arbitrary element $x \in U \cap S_X$. Then U is a weak-neighborhood of x in B_X and so since $X \in SD_{\pm}$ we can apply Lemma 3.3 for x, U and $y = Tx$ to get $z \in U$ such that $\|y - z\| \geq \|x + y\| - \varepsilon$. Since $x, z \in U = V \cap T^{-1}(W)$, we have that $Tx, Tz \in W$. But diam $W < \varepsilon$

and thus $\|Tz - y\| = \|Tz - Tx\| < \varepsilon$. Also since $x \in U \subset V \subset A$ we have that $\|x + Tx\| > a - \varepsilon = \|I + T\| - \varepsilon$. Now we can make the following estimates:

$$\|I - T\| \geq \|z - Tz\| = \|(z - y) - (Tz - y)\| \geq \|z - y\| - \|Tz - y\| > \|z - y\| - \varepsilon$$

$$\geq \|x + y\| - 2\varepsilon = \|x + Tx\| - 2\varepsilon \geq \|I + T\| - 3\varepsilon.$$

So we finally proved that $\|I - T\| > \|I + T\| - 3\varepsilon$ and because of the arbitrariness of ε, we get that $\|I - T\| \geq \|I + T\|$.

The reversed inequality is just the same inequality for the strong Radon-Nikodým operator $(-T)$. $\square$

Remark 3.5. In this theorem we used the characterization of the strong plus-minus property and so the strong plus-minus property itself only for those relatively weakly open sets that can be obtained by an intersection of two slices. We do not know if it is enough to use just slices here.

The next proposition will give us a lot of examples of spaces with the strong plus-minus property.

Proposition 3.6. *The Daugavet property implies $SD_{\pm}$.*

Proof. If a space X possesses the Daugavet property, then from the characterization of this property from [5, Lemma 2.2] it easily follows that for every $x \in S_X$, every $y \in X$, every U – weak neighborhood of x, and every $\varepsilon > 0$ there exists $z \in U$ such that $\|z - y\| = \|z + (-y)\| \geq 1 + \|y\| - \varepsilon \geq \|x + y\| - \varepsilon$ and so $X \in SD_{\pm}$ by Lemma 3.3. $\square$

There are Banach spaces with the strong plus-minus property that do not have the Daugavet property. We will present examples of such spaces in the next section.

We conclude this part by proving a result that gives us more examples of Banach spaces with the strong plus-minus property.

Proposition 3.7. *The strong plus-minus property is inherited by finite-codimensional subspaces.*

Proof. Let X be a Banach space, $X \in SD_{\pm}$, and Y be a subspace of X of finite co-dimension. We want to prove that then $Y \in SD_{\pm}$. First we note that an ε-neighborhood Y_ε of Y ($Y_\varepsilon = \{x \in X : \operatorname{dist}(x, Y) < \varepsilon\}$) is weakly-open in X for every $\varepsilon > 0$. Indeed, Y_ε is a pre-image of the ball B_ε of radius ε under the quotient map $q : X \to X/Y$, and B_ε is weakly open in X/Y since this space is finite-dimensional. Now, let $x \in S_Y$, U be a weak neighborhood of x in B_Y, $y \in Y$ and $\varepsilon > 0$. According to the Lemma 3.3 it is enough to find $z \in U$ such that $\|z - y\| \geq \|x + y\| - \varepsilon$. Without loss of generality we can assume that $U = \{u \in B_Y : |f_i(u - x)| < \delta, \ i \in \overline{1,n}\}$ for some $\{f_i\}_{i=1}^n \in S_{Y^*}$ and $\delta > 0$. Let $\hat{f}_i \in S_{X^*}$ be an extension of f_i for $i \in \overline{1,n}$. Then $V = \{u \in B_X : |\hat{f}_i(u - x)| < \delta/2, \ i \in \overline{1,n}\}$ is a relatively weakly open set in X. Take $\varepsilon_0 = \min\{\varepsilon, \delta\}/4$ and consider $U_0 = V \cap Y_{\varepsilon_0}$. Since Y_{ε_0} is weakly open U_0 is a weak neighborhood of $x \in S_Y \subset S_X$ in B_X. So we can use that $X \in SD_\pm$ to find $z_0 \in U_0$ such that $\|z_0 - y\| \geq \|x + y\| - \varepsilon/2$. Since $z_0 \in U_0 \subset Y_{\varepsilon_0}$ there is $\hat{z} \in Y$ such that $\|\hat{z} - z_0\| < \varepsilon_0$. The problem now is that this $\hat{z}$ does not necessarily belong to B_Y. So there are two possibilities: $\|\hat{z}\| \leq 1$ and $\|\hat{z}\| > 1$. In the first case we take $z = \hat{z}$ and so we will automatically have that $\|z - z_0\| < \varepsilon_0$. In the second case we take $z = \hat{z}/\|\hat{z}\|$ and prove that $\|z - z_0\| < 2\varepsilon_0$. Since $\|z - z_0\| \leq \|z - \hat{z}\| + \|\hat{z} - z_0\| < \|z - \hat{z}\| + \varepsilon_0$ it is enough to prove that $\|z - \hat{z}\| < \varepsilon_0$. We have:

$$\|z - \hat{z}\| = \left\| \frac{\hat{z}}{\|\hat{z}\|} - \hat{z} \right\| = |1 - \|\hat{z}\|| = \|\hat{z}\| - 1 \leq \|\hat{z} - z_0\| + \|z_0\| - 1 < \varepsilon_0,$$

because $z_0 \in U_0 \subset B_X$ and so $\|z_0\| \leq 1$. So, in both cases we got $z \in B_Y$ such that $\|z - z_0\| < 2\varepsilon_0$. We will prove that this z meets all our requirements, i.e. that $z \in U$ and $\|z - y\| \geq \|x + y\| - \varepsilon$. To prove that $z \in U$ it is enough to show that $|f_i(z - x)| < \delta$ for all $i \in \overline{1,n}$ because we already know that $z \in B_Y$. Since $z_0 \in U_0$ we have that $|\hat{f}_i(z_0 - x)| < \delta/2$ and so

$$\begin{aligned} |f_i(z - x)| = |\hat{f}_i(z - x)| &\leq |\hat{f}_i(z - z_0)| + |\hat{f}_i(z_0 - x)| \\ &\leq \|\hat{f}_i\| \cdot \|z - z_0\| + \delta/2 < 2\varepsilon_0 + \delta/2 \leq \delta \end{aligned}$$

because by the definition $\varepsilon_0 \leq \delta/4$. Thus we have that indeed $z \in U$. Finally,

$$\|z - y\| \geq \|z_0 - y\| - \|z_0 - z\| > \|x + y\| - \varepsilon/2 - 2\varepsilon_0 \geq \|x + y\| - \varepsilon$$

since $\varepsilon_0 \leq \varepsilon/4$, which completes the proof. □

4. Banach function spaces and sums of spaces with the strong plus-minus property.

Definition 4.1. *Let (Ω, Σ, μ) be a measure space. A Banach space E of equivalence classes of scalar measurable functions on Ω we will call in the sequel a Banach function space if for every $f \in E$ and for every measurable g on Ω a condition $|g| \leq |f|$ almost everywhere implies that $g \in E$ and $\|g\| \leq \|f\|$.*

Now let E be a Banach function space on a measure space (Ω, Σ, μ) and X be a Banach space. Then we define a space $E(X)$ as a space of strongly measurable functions $f : \Omega \to X$ for which $\hat{f}(t) = \|f(t)\|_X$ is an element of E. The norm on $E(X)$ is defined as follows: $\|f\|_{E(X)} := \|\hat{f}\|_E$. With this norm $E(X)$ becomes a Banach space. We will prove that if X has the strong plus-minus property then so does $E(X)$. For this we need the following technical lemma.

Lemma 4.2. *Let E be a Banach function space on a measure space (Ω, Σ, μ) and X be a Banach space. Then for every $f \in E(X)$ and every $\varepsilon > 0$ there exists a function $f_\varepsilon \in E(X)$ of the form $f_\varepsilon = \sum\limits_{k=1}^{\infty} x_k \cdot \chi_{A_k}$, $x_k \in X$ $(k \in \mathbb{N})$, with $\{A_k\}_{k=1}^{n}$ pairwise disjoint, such that $\|f - f_\varepsilon\|_{E(X)} < \varepsilon$.*

Proof. Since our function f is strongly measurable there exists a set $U_0 \subset \Omega$ such that $\mu(U_0) = 0$ and $f(\Omega \setminus U_0)$ is separable. Also the norm of any function from $E(X)$ does not depend on the values of this function on a set of zero measure, so we can assume that $f(\Omega)$ is separable and hence that X is separable. For every $n \in \mathbb{Z}$ we define $B^n = \{x \in X : 2^n \leq \|x\| < 2^{n+1}\}$. Then the sets B^n are disjoint and $\bigcup\limits_{n=-\infty}^{\infty} B^n = X \setminus \{0\}$. We now fix some $m \in \mathbb{N}$. Since X is separable, all the B^n are also separable and so for every $n \in \mathbb{Z}$ we can find a sequence $\{y_k^n\}_{k=1}^{\infty} \subset B^n$ which is dense in B^n. Then it follows that $B^n \subset \bigcup\limits_{k=1}^{\infty} B(y_k^n, 2^{n-1}/m)$, where $B(y_k^n, 2^{n-1}/m)$ is a ball with center y_k^n and radius $2^{n-1}/m$. If we denote $B_k^{n,m} = B(y_k^n, 2^{n-1}/m) \cap B^n$, then we will get that $B^n = \bigcup\limits_{k=1}^{\infty} B_k^{n,m}$ and $\operatorname{diam}(B_k^{n,m}) < 2^n/m$. Now we take $C_1^{n,m} = B_1^{n,m}$ and $C_k^{n,m} = B_k^{n,m} \setminus \bigcup\limits_{j=1}^{k-1} B_j^{n,m}$ for $k \geq 2$, to obtain for a fixed m a set $\{C_k^{n,m}\}_{n \in \mathbb{Z},\, k \in \mathbb{N}}$ of pairwise disjoint Borel sets such that $\operatorname{diam}(C_k^{n,m}) < 2^n/m$ and $B^n = \bigcup\limits_{k=1}^{\infty} C_k^{n,m}$. Without loss of generality we can assume that all $C_k^{n,m}$ are non-empty. Let $x_k^{n,m}$ be an arbitrary element of $C_k^{n,m}$ and $A_k^{n,m} = f^{-1}(C_k^{n,m})$. Then since f is strongly measurable we get that all $A_k^{n,m}$ are measurable and also these sets are pairwise disjoint for a fixed m. Define $f_m : \Omega \to X$ as follows: $f_m = \sum\limits_{n=-\infty}^{\infty} \sum\limits_{k=1}^{\infty} x_k^{n,m} \cdot \chi_{A_k^{n,m}}$. We will now prove that $f_m \in E(X)$ for every natural m and $\|f - f_m\|_{E(X)} \to 0$ as $m \to \infty$.

Fix some natural m. Since $\bigcup\limits_{n=-\infty}^{\infty} B^n = X \setminus \{0\}$ and $B^n = \bigcup\limits_{k=1}^{\infty} C_k^{n,m}$, we get that $\bigcup\limits_{n=-\infty}^{\infty} \bigcup\limits_{k=1}^{\infty} C_k^{n,m} = X \setminus \{0\}$. This implies that for every $t \in \Omega$ we have two possibilities:

1) $(f(t) \in C_k^{n,m}) \Longleftrightarrow (t \in A_k^{n,m})$ for some $n \in \mathbb{Z}$, $k \in \mathbb{N}$, or 2) $f(t) = 0$.

If $t \in A_k^{n,m}$ then $f(t) \in B^n$ and so $\|f(t)\| \geq 2^n$. On the other hand if $t \in A_k^{n,m}$ then $f_m(t) = x_k^{n,m} \in B^n$ and so $\|f_m(t)\| < 2^{n+1}$. From the last two inequalities we obtain that $\|f_m(t)\| \leq 2\|f(t)\|$ in the first case. Now, if $f(t) = 0$ and so $t \notin A_k^{n,m}$ for any n and k then we will have that $f_m(t) = 0 = f(t)$. This means that in the second case we also have that $\|f_m(t)\| \leq 2\|f(t)\|$. So we conclude that for every $t \in \Omega$:

$$(*) \qquad \|f_m(t)\| \leq 2\|f(t)\|.$$

Since $f \in E(X)$, we have that the function $\hat{f}$ defined by $\hat{f}(t) = \|f(t)\|$ belongs to E. Now since E is a Banach function space $(*)$ implies that $\hat{f}_m \in E$ and thus $f_m \in E(X)$. We will now estimate $\|f - f_m\|_{E(X)}$.

For every $t \in \Omega$ we again have two possibilities: either $t \in A_k^{n,m}$ for some n and k or $f_m(t) = f(t) = 0$. If $t \in A_k^{n,m}$ then $f(t) \in C_k^{n,m}$, $\|f(t)\| \geq 2^n$ and $f_m(t) = x_k^{n,m} \in C_k^{n,m}$. So in this case both $f(t)$ and $f_m(t)$ are in $C_k^{n,m}$ and since the diameter of this set is less than $2^n/m$ we obtain that $\|f(t) - f_m(t)\| < 2^n/m \leq \|f(t)\|/m$. Also the inequality $\|f(t) - f_m(t)\| \leq \|f(t)\|/m$ evidently holds if $f_m(t) = f(t) = 0$ and so it holds for all $t \in \Omega$. Then it follows that $\|f - f_m\|_{E(X)} \leq \|f\|_{E(X)}/m$. This estimate implies that $\|f - f_m\|_{E(X)} \to 0$ as $m \to \infty$ and since all the f_m are of the desired form this concludes the proof. □

We are now ready to prove the following theorem:

Theorem 4.3. *Let X be a Banach space with the strong plus-minus property. Then for every Banach function space E on a measure space (Ω, Σ, μ) we have that $E(X) \in SD_{\pm}$.*

Proof. Take some $f \in S_{E(X)}$, U – weak neighborhood of f in $B_{E(X)}$, $g \in E(X)$ and $\varepsilon > 0$. To prove the proposition we need to find $h \in U$ such that $\|g - h\|_{E(X)} \geq \|f + g\|_{E(X)} - \varepsilon$. Of course it is enough to do this for $U = \{\tilde{f} \in B_{E(X)} : \ |f_j^*(f - \tilde{f})| < \varepsilon, \ j = 1, 2, \ldots, n\}$, where $f_j^* \in (E(X))^*$. Lemma 4.2 allows us to assume without loss of generality that $f = \sum\limits_{k=1}^{\infty} x_k \cdot \chi_{A_k}$, $g = \sum\limits_{k=1}^{\infty} y_k \cdot \chi_{A_k}$, where A_k are measurable disjoint subsets of Ω and $x_k, \ y_k \in X$.

We define functionals $x^*_{j,k}$ on X for $j \in \overline{1,n}$, $k \in \mathbb{N}$:

$$x^*_{j,k}(x) = f^*_j(x \cdot \chi_{A_k}).$$

All $\{x^*_{j,k}\}$ are, evidently, linear and we will prove that they are also continuous. Indeed, for $x \in B_X$ we have:

$$|x^*_{j,k}(x)| = |f^*_j(x\cdot\chi_{A_k})| \le \|f^*_j\|\cdot\|x\cdot\chi_{A_k}\|_{E(X)} = \|f^*_j\|\cdot\|x\|\cdot\|\chi_{A_k}\|_E \le \|f^*_j\|\cdot\|\chi_{A_k}\|_E,$$

and so $\|x^*_{j,k}\| \le \|f^*_j\| \cdot \|\chi_{A_k}\|_E < \infty$ which means that $x^*_{j,k}$ are continuous. Now for $x_k \ne 0$ we use that $X \in SD_\pm$ to find $z_k \in X$ such that

$$\|z_k\| \le \|x_k\|,\ |x^*_{j,k}(x_k - z_k)| < \frac{\varepsilon}{2^k}\ (j \in \overline{1,n})\ \text{ and }\ \|y_k - z_k\| \ge \|x_k + y_k\| - \frac{\varepsilon}{2^k\,\|\chi_{A_k}\|_E}.$$

If $x_k = 0$ we put $z_k = 0$. We use these z_k to define our h: $h = \sum\limits_{k=1}^{\infty} z_k \cdot \chi_{A_k}$. First we check that $h \in E(X)$. Indeed, we have that $\|z_k\| \le \|x_k\|$ for every $k \in \mathbb{N}$ and so for every $t \in \Omega$ we have that $\|h(t)\| \le \|f(t)\|$. This implies that $h \in E(X)$ and $\|h\| \le \|f\| = 1$, so $h \in B_{E(X)}$. Now we will show that $h \in U$, i.e. that for every $j = 1, 2, \dots, n$ we have that $|f^*_j(f - h)| < \varepsilon$:

$$|f^*_j(f-h)| = \left| f^*_j \left(\sum_{k=1}^{\infty} (x_k - z_k)\chi_{A_k} \right) \right| = \left| \sum_{k=1}^{\infty} x^*_{j,k}(x_k - z_k) \right|$$

$$\le \sum_{k=1}^{\infty} |x^*_{j,k}(x_k - z_k)| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

The only thing left to prove now is that $\|g - h\| \ge \|f + g\| - \varepsilon$. Define $h_-(t) = \|g(t) - h(t)\|$ and $h_+(t) = \|f(t) + g(t)\|$. Then $h_-,\ h_+ \in E$ and we need to show that $\|h_-\|_E \ge \|h_+\|_E - \varepsilon$ or, equivalently, that $\|h_+\|_E \le \|h_-\|_E + \varepsilon$. What we know from our conditions is that for $t \in A_k$:

$$h_-(t) = \|g(t) - h(t)\| = \|y_k - z_k\| \ge \|x_k + y_k\| - \frac{\varepsilon}{2^k\,\|\chi_{A_k}\|_E} = h_+(t) - \frac{\varepsilon}{2^k\,\|\chi_{A_k}\|_E}.$$

Consider $h_\varepsilon = \sum\limits_{k=1}^{\infty} (\varepsilon \cdot \chi_{A_k})/(2^k\,\|\chi_{A_k}\|_E)$. Then the above inequality means that $h_- \ge h_+ - h_\varepsilon$ or, equivalently, $h_- + h_\varepsilon \ge h_+$. We will now show that $h_\varepsilon \in E$. For this we notice that $\|(\varepsilon \cdot \chi_{A_k})/(2^k\,\|\chi_{A_k}\|_E)\|_E = \varepsilon/2^k$ and so the series from the definition of h_ε is absolutely convergent in E. Therefore this series is also convergent in E and $\|h_\varepsilon\|_E \leq \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. Now we use the inequality $h_- + h_\varepsilon \geq h_+$. Since on both sides of this inequality we have non-negative functions from E and E is a Banach function space, we can write that $\|h_+\|_E \leq \|h_- + h_\varepsilon\|_E \leq \|h_-\|_E + \|h_\varepsilon\|_E \leq \|h_-\|_E + \varepsilon$, which was to be proved. □

In the sequel by a 1-unconditional sum $X_1 \oplus_E X_2$ of Banach spaces X_1 and X_2 we will mean a Banach space $X = \{(x_1, x_2)|\ x_i \in X_i,\ i = 1, 2\}$ with the norm $\|(x_1, x_2)\| = \|(\|x_1\|_{X_1}, \|x_2\|_{X_2})\|_E$ where E is a Banach space of pairs of real numbers satisfying $\|(a, b)\|_E = \|(|a|, |b|)\|_E$ for every $a, b \in \mathbb{R}$ (i.e. E is a two-dimensional space with 1-unconditional basis).

Theorem 4.3 implies in particular that for a Banach space $X \in SD_\pm$ we have that every 1-unconditional sum $X \oplus_E X$ also has the strong plus-minus property. However, a more general result holds.

Proposition 4.4. *The* 1*-unconditional sum of two spaces with the strong plus-minus property also has this property.*

Proof. Let $X_1,\ X_2 \in SD_\pm$ and $X = X_1 \oplus_E X_2$ – a 1-unconditional sum of X_1 and X_2. Take arbitrary $x = (x_1, x_2) \in S_X$, $y = (y_1, y_2) \in X$, U – weak-neighborhood of x, and $\varepsilon > 0$. Without loss of generality we can assume that

$$U = \{u = (u_1, u_2) \in B_X :\ |f_i^*(x - u)| < \varepsilon_i,\ i = 1, ..., n\},$$

where $f_i^* = ((f_i^*)_1, (f_i^*)_2) \in X^*$ and $\varepsilon_i > 0$, $i \in \overline{1, n}$. Consider then $U_1 = \{u_1 \in X_1 :\ |(f_i^*)_1(x_1 - u_1)| < \varepsilon_i/2,\ i = 1, ..., n\}$ – weak neighborhood of x_1 in X_1 and $U_2 = \{u_2 \in X_2 :\ |(f_i^*)_2(x_2 - u_2)| < \varepsilon_i/2,\ i = 1, ..., n\}$ – weak neighborhood of x_2 in X_2. Then we use that $X_k \in SD_\pm$ to find $z_k \in U_k \cap B_{\|x_k\|}$ such that $\|y_k - z_k\| \geq \|x_k + y_k\| - \varepsilon$, $k = 1, 2$. Now consider $z = (z_1, z_2)$. For this z we have:

1. $\|z\| = \|(z_1, z_2)\| = \|(\|z_1\|, \|z_2\|)\|_E \leq \|(\|x_1\|, \|x_2\|)\|_E = \|(x_1, x_2)\| = 1$ because $\|z_k\| \leq \|x_k\|$ and our sum is 1-unconditional,
2. $z \in U$ since $z_k \in U_k$, and
3. $$\begin{aligned}\|y - z\| &= \|(y_1 - z_1, y_2 - z_2)\| = \|(\|y_1 - z_1\|, \|y_2 - z_2\|)\| \\ &\geq \|(\|x_1 + y_1\| - \varepsilon, \|x_2 + y_2\| - \varepsilon)\| \geq \|(\|x_1 + y_1\|, \|x_2 + y_2\|)\| - \|(\varepsilon, \varepsilon)\| \\ &= \|(x_1 + y_1, x_2 + y_2)\| - \varepsilon\|(1, 1)\| = \|(x - y)\| - \varepsilon\|(1, 1)\|.\end{aligned}$$

So, since ε was arbitrary, z satisfies all the necessary properties to guarantee that $X \in SD_\pm$. □

We remark that this proposition can be easily extended to the 1-unconditional sum of a finite number of spaces by induction.

From Proposition 4.4 it follows in particular that $X = X_1 \oplus_E X_2$ satisfies the strong plus-minus property whenever X_1 and X_2 have the Daugavet property. And since not all unconditional sums preserve the Daugavet property, this will give us lots of examples of spaces with $SD_\pm$ but without the Daugavet property that we promised earlier. To be more precise, only l_1 and l_∞ 1-unconditional sums preserve the Daugavet property(see [4]), and so we get the following corollary:

Corollary 4.5. *Let X_1 and X_2 be two Banach spaces with the Daugavet property. Then a* 1*-unconditional sum $X_1 \oplus_E X_2$ will have the strong plus-minus property but will not possess the Daugavet property except for cases when $E = l_1$ or $E = l_\infty$.*

This result works also for the ordinary plus-minus property because $SD_\pm$ implies $D_\pm$.

The converse to the Proposition 4.4 is also true:

Proposition 4.6. *Let X_1 and X_2 be two Banach spaces such that a* 1*-unconditional sum $X_1 \oplus_E X_2 \in SD_\pm$. Then both X_1 and X_2 have the strong plus-minus property.*

P r o o f. It is enough to prove that $X_1 \in SD_\pm$. For this we need to prove that for arbitrary $x_1 \in S_{X_1}$, $f_1^1, \dots, f_n^1 \in X_1^*$, $y_1 \in X_1$ and $\varepsilon > 0$ there exists $z_1 \in B_{X_1}$ such that $|f_j^1(x_1 - z_1)| < \varepsilon$ $(j = 1, 2, \dots, n)$ and $\|y_1 - z_1\| \geq \|x_1 + y_1\| - \varepsilon$. Fix some $\delta > 0$ and find $x^* \in S_{X_1^*}$ such that $x^*(x_1) > 1 - \delta$. Using that $X = X_1 \oplus_E X_2 \in SD_\pm$ we find $(z_1, z_2) \in B_X$ such that: $|f_j^1(x_1 - z_1)| < \delta$ $(j = 1, 2, \dots, n)$, $|x^*(x_1 - z_1)| < \delta$ and $\|(y_1 - z_1, -z_2)\| \geq \|(x_1 + y_1, 0)\| - \delta$ (we use Lemma 3.3 for $x = (x_1, 0) \in S_X$, $U = \{z \in B_X : \ |(f_j, 0)(x - z)| < \delta, j \in \overline{1, n}, |(x^*, 0)(x - z)| < \delta\}$, $y = (y_1, 0) \in X$ and $\delta > 0$). From this point we need to distinguish two cases depending on whether B_E contains a vertical line segment passing through $(1, 0)$:

1. For every $\alpha > 0$ there exists $\beta > 0$ such that whenever $(t_1, t_2) \in B_E$ and $|t_1| > 1 - \beta$ we have that $|t_2| < \alpha$. In this case we will prove that for an appropriately small δ the z_1 that we found meets all the requirements. Since $\|x^*\| = 1$, $x^*(x) > 1 - \delta$ and $|x^*(x - z_1)| < \delta$ we obtain that $|x^*(z_1)| > 1 - 2\delta$ and thus $\|z_1\| > 1 - 2\delta$. So, if we take $\alpha = \varepsilon/2$, find the corresponding β and take $\delta = \min\{\varepsilon/2, \beta/2\}$ then we would have that $\|z_1\| > 1 - \beta$ and so since $(z_1, z_2) \in B_X$ this implies that $\|z_2\| < \alpha = \varepsilon/2$. Finally we have:

$$\|y_1 - z_1\|_{X_1} = \|(y_1 - z_1, 0)\|_X \geq \|(y_1 - z_1, -z_2)\| - \|(0, z_2)\|$$

$$\geq (\|(x_1 + y_1, 0)\| - \delta) - \|z_2\| \geq (\|x_1 + y_1\| - \varepsilon/2) - \varepsilon/2 = \|x_1 + y_1\| - \varepsilon,$$

which was to be proved.

2. Now, if the condition from the first case is not satisfied, we can find $\lambda > 0$ such that $(1, \lambda) \in B_E$ and λ is the biggest number with this property. This implies that for every $\alpha > 0$ there exists $\beta > 0$ such that whenever $(t_1, t_2) \in B_E$ and $|t_1| > 1 - \beta$ we have that $|t_2| < \lambda + \alpha$. Applying the same argument as in the previous case we get that $\|z_1\| > 1 - 2\delta$. Then we take $\alpha = \lambda\varepsilon/4$, find the corresponding β and define $\delta = \min\{\varepsilon/4, \beta/2\}$. Note that if z_1 that we get for this δ satisfies $\|y_1 - z_1\| \geq \|x_1 + y_1\| - \delta = \|x_1 + y_1\| - \varepsilon/4$ then this z_1 satisfies all our conditions and we are done. Now, if $\|y_1 - z_1\| < \|x_1 + y_1\| - \varepsilon/4$, then since $\|(y_1 - z_1, -z_2)\| \geq \|x_1 + y_1\| - \varepsilon/4$ we get that $\|z_2\| \geq (\|x_1 + y_1\| - \varepsilon/4)\lambda$. On the other hand, since $\|z_1\| > 1 - 2\delta > 1 - \beta$ and $(z_1, z_2) \in B_X$ we obtain that $\|z_2\| < \lambda + \alpha = (1 + \varepsilon/4)\lambda$. So, we have:

$$(\|x_1 + y_1\| - \varepsilon/4)\lambda \leq \|z_2\| < (1 + \varepsilon/4)\lambda,$$

which implies that $\|x_1 + y_1\| - \varepsilon/4 < 1 + \varepsilon/4$, i.e. that $\|x_1 + y_1\| < 1 + \varepsilon/2$, because $\lambda > 0$. Now we show that in this case x_1 instead of z_1 meets all our requirements. Obviously, $x_1 \in U$ and so we need only to check that $\|y_1 - x_1\| \geq \|x_1 + y_1\| - \varepsilon$. By the triangle inequality $\|y_1 - x_1\| \geq 2\|x_1\| - \|x_1 + y_1\| = 2 - \|x_1 + y_1\|$. Then we use that $\|x_1 + y_1\| < 1 + \varepsilon/2$ to obtain:

$$\|y_1 - x_1\| \geq 2 - \|x_1 + y_1\| > 2 - (1 + \varepsilon/2) = 1 - \varepsilon/2 = (1 + \varepsilon/2) - \varepsilon > \|x_1 + y_1\| - \varepsilon,$$

which was to be proved. □

5. Open questions.

Question 5.1. *Does the property $D_\pm(rank\, 1)$ imply $D_\pm$?*

Question 5.2. *Can a Banach space with the plus-minus property have an unconditional basis?*

Question 5.3. *Does the class of strong plus-minus Banach spaces coincide with the class of plus-minus spaces?*

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