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ON DIFFERENTIAL INCLUSIONS WITH UNBOUNDED RIGHT-HAND SIDE

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Abstract. The classical Filippov's Theorem on existence of a local trajectory of the differential inclusion $\dot{x}(t) \in \Phi(t, x(t))$ requires the right-hand side $\Phi(\cdot, \cdot)$ to be Lipschitzian with respect to the Hausdorff distance and then to be bounded-valued. We give an extension of the quoted result under a weaker assumption, used by Ioffe in [6], allowing unbounded right-hand side.

1. Introduction and notation. The well-known Filippov's Theorem on existence of a local trajectory of the differential inclusion $\dot{x}(t) \in \Phi(t, x(t))$ requires the right-hand side $\Phi(\cdot, \cdot)$ to be Lipschitzian with respect to the Hausdorff distance, and to be bounded-valued. When dealing with a multifunction taking unbounded values, Ioffe introduced in [6] a weakening of the Lipschitzian

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assumption in terms of pseudo-Lipschitzness: namely, there exists $\beta \geq 0$ such that for every $(t, x_1), (t, x_2)$,

$$y_1 \in \Phi(t, x_1) \Rightarrow d(y_1, \Phi(t, x_2)) \leq (k(t) + \beta\|y_1\|)\|x_1 - x_2\|,$$

where $k(\cdot)$ is summable and nonnegative. In this note, we give an existence result for a local solution of the differential inclusion under such condition. The structure of the proof is very different from that in [6], it is based on a fixed point theorem given in Section 2, and which is the principal result of the paper.

Let us give some notation. We let X be a metric space endowed with the metric d. The open (resp. closed) ball with center x and radius r will be denoted by $B_r(x)$ (resp. $B_r[x]$). Given a subset C of a metric space X, we denote by $d(x, C)$ the distance from x to C, that is, $d(x, C) = \inf_{z \in C} d(x, z)$, and we denote by $e(C, D)$ the Hausdorff–Pompeiu excess of C into D, defined by $e(C, D) = \sup_{x \in C} d(x, D)$, with the conventions $e(\emptyset, D) = 0$, and $e(C, \emptyset) = +\infty$ whenever $C \neq \emptyset$. We denote by ι_S the indicator function of the subset $S \subset X$ defined by $\iota_S(x) = 0$ if $x \in S$, $\iota(x) = +\infty$ otherwise.

A multifunction from a set X to a set Y is a subset of the cartesian product $X \times Y$. For $x \in X$, we set $F(x) = \{y \in Y : (x, y) \in F\}$. In section 2 we give a fixed point result generalizing [5, 3] and we apply it in section 3 in order to obtain a local existence result for a differential equation with an unbounded right-hand side.

2. A fixed point result.

Our main result is the following one, on fixed points of generalized contractions, for which we give a proof based on the Ekeland's principle.

In the line of [2], let us recall some basic facts about it. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, a point $x \in X$ is said to be a d-*point* of f if

$$f(x) < f(z) + d(z, x) \qquad \text{for all } z \in X,\ z \neq x.$$

We also define, for $x \in X$, the set $M_f(x) := \{z \in X : f(z) + d(z, x) \leq f(x)\}$. It is important to observe (it is an immediate consequence of the triangle inequality) that z is a d-point of f whenever it is a d-point of the restriction of f to some subset $M_f(x)$, $x \in X$. Ekeland's variational principle ([4]) under its simpler form, given in [7], says that any bounded from below and lower semicontinuous function defined on a complete metric space admits a d-point.

Theorem 2.1. *Let* (X, d) *be a complete metric space, and let* $G \subset X \times X$ *be a closed multifunction. Let* $x_0 \in X$, $\alpha > 0$, *and* $\kappa \in [0, 1)$. *Assume that:*

(a) *For all* $x \in B_\alpha(x_0)$ *and for all* $y \in G(x) \cap B_\alpha(x_0)$ *we have* $d(y, G(y)) \leq \kappa d(x, y)$;

(b) $d(x_0, G(x_0)) < \alpha(1 - \kappa)$.

Then the fixed-point set $\mathcal{F}_G = \{x \in X : \ x \in G(x)\}$ *of* G *is nonempty, and*

$$d(x_0, \mathcal{F}_G) \leq (1 - \kappa)^{-1} d(x_0, G(x_0)).$$

Proof. Let $\tilde{\kappa} \in (\kappa, 1)$ be such that $d(x_0, G(x_0)) < \alpha(1 - \tilde{\kappa})$. Let us endow $X \times X$ with the distance $d((x_1, v_1), (x_2, v_2)) = \max\{d(x_1, x_2), \tilde{\kappa}^{-1} d(v_1, v_2)\}$, and let $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ be the lower semicontinuous function defined by

$$f(x, v) = (1 - \tilde{\kappa})^{-1} d(x, v) + \iota_G(x, v).$$

Let $v_0 \in G(x_0)$ be such that $d(x_0, v_0) < \alpha(1 - \tilde{\kappa})$, so that

$$M_f(x_0, v_0) \subset B_{f(x_0, v_0)}(x_0, v_0) \subset B_\alpha(x_0, v_0).$$

From Ekeland's variational principle, f has a d-point $(\bar{x}, \bar{v})$ belonging to the closed set $M_f(x_0, v_0)$. Now, given $(x, v) \in B_\alpha(x_0, v_0) \cap G$ with $f(x, v) > 0$, and taking into account that $B_{\kappa\alpha}(v_0) \subset B_\alpha(x_0)$, we can find $w \in G(v)$ such that $d(v, w) < \tilde{\kappa} d(x, v)$, so that $(x, v) \neq (v, w)$ and

$$f(x, v) - f(v, w) = (1 - \tilde{\kappa})^{-1}(d(x, v) - d(v, w)) \geq d((x, v), (v, w)),$$

which shows that (x, v) is not a d-point of f . It follows that $f(\bar{x}, \bar{v}) = 0$, which means that $\bar{v} = \bar{x} \in G(\bar{x})$, that is, $\bar{x} \in \mathcal{F}_G$. Since $(\bar{x}, \bar{x}) \in M_f(x_0, v_0)$ we have $d(x_0, \bar{x}) \leq f(x_0, v_0) = (1 - \tilde{\kappa})^{-1} d(x_0, v_0)$, yielding the conclusion of the lemma since $\tilde{\kappa}$ can be chosen arbitrarily close to κ , and v_0 can be chosen arbitrarily in $G(x_0)$. □

Remark 2.1. The previous result widely extends the one given in [5, 3]. Indeed, in the quoted results, the existence of a fixed point is obtained under the stronger assumption that for all x_1 , $x_2 \in B_\alpha(x_0)$ we have $e(G(x_1) \cap B_\alpha(x_0), G(x_2)) \leq \kappa d(x_1, x_2)$.

3. Differential inclusion. Given $t_0 \in \mathbb{R}$, $a > 0$, and $\alpha > 0$, we consider a base function $x_0 \in W^{1,1}([t_0, t_0 + a]; \mathbb{R}^n)$, and a multifunction $\Phi \subset Q \times \mathbb{R}^n$ with closed values, where

$$Q = \{(t, \xi) : t \in [t_0, t_0 + a],\ \|\xi - x_0(t)\| \leq 2\alpha\}.$$

We make the following assumptions on the multifunction Φ:

(H1) For all $\xi \in \mathbb{R}^n$, the multifunction $\Phi(\cdot, \xi)$ is $\mathcal{L}$-measurable;

(H2) There exists $\beta \geq 0$ such that for every $(t, x_1), (t, x_2) \in Q$,

$$y_1 \in \Phi(t, x_1) \Rightarrow d(y_1, \Phi(t, x_2)) \leq (k(t) + \beta\|y_1\|)\|x_1 - x_2\|,$$

where k is summable and nonnegative.

Assumption (H1) means that for every $\xi \in \mathbb{R}^n$ and for every open set $O \subset \mathbb{R}^n$, the set $\{t \in \mathbb{R} : \Phi(t, \xi) \cap O \neq \emptyset\}$ is (Lebesgue-)measurable. From our assumptions (H1) and (H2) and from the results of [1, Chapter 8], it follows that, for any measurable $y : [t_0, t_0 + a] \to \mathbb{R}^n$ with graph in Q, the multifunction $\Phi(\cdot, y(\cdot))$ is measurable, and that, for every measurable $v : [t_0, t_0 + a] \to \mathbb{R}^n$, there exists a measurable $u : [t_0, t_0 + a] \to \mathbb{R}^n$ such that $u(t) \in \Phi(t, y(t))$ and $\|v(t) - u(t)\| = d(v(t), \Phi(t, y(t)))$ for all $t \in [t_0, t_0 + a]$. Moreover, u is summable if so is v.

Let us set:

$$D(a) = \int_{t_0}^{t_0+a} e^{-K(t-t_0)} d(\dot{x}_0(t), \Phi(t, x_0(t)))\, dt,\ V(a) = \int_{t_0}^{t_0+a} e^{-K(t-t_0)} \|\dot{x}_0(t)\|\, dt,$$

where $K(t) = \int_{t_0}^{t_0+t} k(s)\, ds$ for $t \in [0, a]$.

Theorem 3.1. *Under assumptions* (H1) *and* (H2), *let us assume that* $a > 0$ *and* $\alpha > \delta > 0$ *are such that:*

$$e^{K(a)}\beta(\alpha + V(a)) < 1 \tag{1}$$

and

$$e^{2K(a)}\Big(D(a) + \delta(1 - e^{-K(a)} + \beta(D(a) + V(a)))\Big) < \alpha(1 - e^{K(a)}\beta(\alpha + V(a)))\,. \tag{2}$$

Then, for all $s_0 \in [t_0, t_0 + a]$ *and for all* $\xi_0 \in B_\delta(x_0(s_0))$, *there exists a solution* $x \in W^{1,1}([t_0, t_0 + a]; \mathbb{R}^n)$ *of*

$$\begin{cases} \dot{x}(t) \in \Phi(t, x(t)) \quad \text{for a.e. } t \in [t_0, t_0 + a] \\ \\ x(s_0) = \xi_0 \end{cases} . \tag{3}$$

P r o o f. Let $X = \{x \in W^{1,1}([t_0, t_0 + a]; \mathbb{R}^n) : x(s_0) = \xi_0\}$ be endowed with the norm

$$\|x\|_X = \int_{t_0}^{t_0+a} e^{-K(t-t_0)} \|\dot{x}(t)\| \, dt \,,$$

and let $\hat{\alpha} = e^{-K(a)}\alpha$. Let us define a multifunction $G_a \subset X \times X$ by

$$(x, y) \in G_a \Leftrightarrow \dot{y}(t) \in \Phi(t, x(t)) \text{ for a.e. } t \in [t_0, t_0 + a].$$

Observe that Φ is closed, due to assumption (H2) and to the fact that Φ is closed-valued. It then follows that the multifunction G_a is closed. Let $z_0 = x_0 + \xi_0 - x_0(s_0)$, so that $z_0 \in X$, let $x \in B_{\hat{\alpha}}(z_0)$, and let $y \in G_a(x) \cap B_{\hat{\alpha}}(z_0)$. Using the fact that, for any $x \in B_{\hat{\alpha}}(z_0)$, we have $(t, x(t)) \in Q$ for all $t \in [t_0, t_0 + a]$, and relying on assumption (H2), we find $v \in G_a(y)$ such that

$$\|\dot{y}(t) - \dot{v}(t)\| \le (k(t) + \beta\|\dot{y}(t)\|)\|x(t) - y(t)\|$$

for a.e. $t \in [t_0, t_0 + a]$. Setting $\chi(t) = k(t) + \beta\|\dot{y}(t)\|$, we get

$$\begin{aligned} &\int_{t_0}^{t_0+a} e^{-K(t-t_0)} \|\dot{y}(t) - \dot{v}(t)\| dt \\ &\qquad \le \int_{t_0}^{t_0+a} e^{-K(t-t_0)} \chi(t) \left(\int_{s_0}^{t} \|\dot{x}(s) - \dot{y}(s)\| \, ds \right) dt \\ &\qquad \le \int_{t_0}^{t_0+a} \left(\int_{s}^{t_0+a} e^{-K(t-t_0)} \chi(t) dt \right) \|\dot{x}(s) - \dot{y}(s)\| ds. \end{aligned}$$

As

$$\int_{s}^{t_0+a} e^{-K(t-t_0)} k(t) \, dt = e^{-K(s-t_0)} - e^{-K(a)} \le e^{-K(s-t_0)} \left(1 - e^{-K(a)} \right),$$

and

$$\beta \int_{s}^{t_0+a} e^{-K(t-t_0)} \|\dot{y}(t)\| \, dt \leq \beta \|y\|_X \leq \beta(\hat{\alpha} + \|z_0\|_X) = \beta(\alpha + V(a)) \, ,$$

we derive that

$$\|v - y\|_X \leq \left(1 - e^{-K(a)} + \beta(\alpha + V(a))\right) \|y - x\|_X ,$$

leading to

$$d(y, G_a(y)) \leq \kappa(a) \|x - y\|_X ,$$

for all $x \in B_{\hat{\alpha}}(z_0)$ and all $y \in G_a(x) \cap B_{\hat{\alpha}}(z_0)$, where $\kappa(a) = 1 - e^{-K(a)}(1 - e^{K(a)}\beta(\alpha + V(a)))$ belongs to $[0, 1)$ thanks to assumption (1).

Let us now estimate $d(z_0, G_a(z_0))$. To this end, let $u_0 \in \mathcal{L}^1([t_0, t_0 + a]; \mathbb{R}^n)$ be such that $u_0(t) \in \Phi(t, x_0(t))$ and

$$\|\dot{x}_0(t) - u_0(t)\| = d(\dot{x}_0(t), \Phi(t, x_0(t))) \tag{4}$$

for a.e. $t \in [t_0, t_0 + a]$. From assumption (H2), we have

$$d(u_0(t), \Phi(t, z_0(t))) \leq (k(t) + \beta \|u_0(t)\|) \|z_0(t) - x_0(t)\| < (k(t) + \beta \|u_0(t)\|) \hat{\delta}$$

for some $\hat{\delta} \in (0, \delta)$. Now, let $v_0 \in \mathcal{L}^1([t_0, t_0 + a]; \mathbb{R}^n)$ be such that $v_0(t) \in \Phi(t, z_0(t))$ and

$$\|u_0(t) - v_0(t)\| = d(u_0(t), \Phi(t, z_0(t))) \tag{5}$$

for a.e. $t \in [t_0, t_0 + a]$, and let $w_0(t) = \xi_0 + \int_{s_0}^{t} v_0(s) \, ds$ for $t \in [t_0, t_0 + a]$, so that $w_0 \in G_a(z_0)$. From (4) and (5) we then get, for a.e. $t \in [t_0, t_0 + a]$:

$$\begin{aligned} \|\dot{x}_0(t) - v_0(t)\| &\leq \|\dot{x}_0(t) - u_0(t)\| + \|u_0(t) - v_0(t)\| \\ &< d(\dot{x}_0(t), \Phi(t, x_0(t))) + (k(t) + \beta \|u_0(t)\|) \hat{\delta} \end{aligned}$$

and

$$\begin{aligned} \|z_0 - w_0\|_X &= \int_{t_0}^{t_0+a} e^{-K(t-t_0)} \|\dot{x}_0(t) - v_0(t)\| \, dt \\ &\leq \int_{t_0}^{t_0+a} e^{-K(t-t_0)} \left(d(\dot{x}_0(t), \Phi(t, x_0(t))) + (k(t) + \beta \|u_0(t)\|) \hat{\delta} \right) dt \end{aligned}$$

$$< \quad D(a) + \delta(1 - e^{-K(a)} + \beta(D(a) + V(a)))\,.$$

Assumption (2) tells us that

$$D(a) + \delta\big(1 - e^{-K(a)} + \beta(D(a) + V(a))\big) < \hat{\alpha}(1 - \kappa(a)),$$

yielding $d(z_0, G_a(z_0)) < \hat{\alpha}(1 - \kappa(a))$. Thus, we can apply Theorem 2.1 to find $x \in \mathcal{F}_{G_a}$ such that

$$\|x - z_0\|_X < \frac{e^{K(a)}(D(a) + \delta(1 - e^{-K(a)} + \beta(D(a) + V(a)))}{1 - e^{K(a)}\beta(\alpha + V(a))}.$$

As any fixed point $x \in \mathcal{F}_{G_a}$ is a solution of (3), we are led to the conclusion of the theorem. □

Remark 3.1. Assuming that $\alpha\beta < 1$, there clearly exists $a > 0$ satisfying conditions (1) and (2).

Remark 3.2. In the case when $\beta = 0$, conditions (1) and (2) reduce to

$$e^{2K(a)}\Big(D(a) + \delta(1 - e^{-K(a)})\Big) < \alpha\,,$$

a condition close to the one in the classical Filippov's Theorem.

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REFERENCES

[1] J.-P. Aubin, H. Frankowska. Set-Valued Analysis. Birkhäuser, Boston, 1990.

[2] D. Azé, J.-N. Corvellec. A variational method in fixed point results with inwardness conditions. *Proc. Amer. Math. Soc.* **134** (2006), 3577–3583.

[3] A. L. Dontchev, W. M. Hager. An inverse mapping theorem for set-valued maps. *Proc. Amer. Math. Soc.* **121** (1994), 481–489.

[4] I. Ekeland. On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.

[5] A. D. Ioffe, V. M. Tihomirov. Theory of Extremal Problems. Studies in Mathematics and its applications, vol. **6**, North Holland, 1979.

[6] A. D. Ioffe. Existence and relaxation theorems for unbounded differential inclusions. *J. Convex Anal.* **13** (2006), 353–362.

[7] J.-P. Penot. The drop theorem, the petal theorem and Ekeland's variational principle. *Nonlinear Anal.* **10** (1986), 813–822.

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