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THE DIRECT AND INVERSE SPECTRAL PROBLEMS FOR SOME BANDED MATRICES

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Abstract. In this paper we introduced a notion of the generalized spectral function for a matrix $J = (g_{k,l})_{k,l=0}^{\infty}$, $g_{k,l} \in \mathbb{C}$, such that $g_{k,l} = 0$, if $|k-l| > N$; $g_{k,k+N} = 1$, and $g_{k,k-N} \neq 0$. Here N is a fixed positive integer. The direct and inverse spectral problems for such matrices are stated and solved. An integral representation for the generalized spectral function is obtained.

1. Introduction. The main object of our present investigation is a matrix $J = (g_{k,l})_{k,l=0}^{\infty}$, $g_{k,l} \in \mathbb{C}$, such that

$$g_{k,l} = 0, \qquad k,l \in \mathbb{Z}_+ : \ |k-l| > N, \tag{1}$$

$$g_{k,k+N} = 1, \qquad k \in \mathbb{Z}_+, \tag{2}$$

and

$$g_{k,k-N} \neq 0, \qquad k = N, N+1, \ldots. \tag{3}$$

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Here N is a fixed positive integer.
Thus, the matrix J has the following form:
(4)
$$J = \begin{pmatrix} g_{0,0} & g_{0,1} & g_{0,2} & \cdots & g_{0,N-1} & 1 & 0 & 0 & \ldots \\ g_{1,0} & g_{1,1} & g_{1,2} & \cdots & g_{1,N-1} & g_{1,N} & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ldots & \\ g_{N,0} & g_{N,1} & g_{N,2} & \cdots & g_{N,N-1} & g_{N,N} & g_{N,N+1} & g_{N,N+2} & \cdots \\ 0 & g_{N+1,1} & g_{N+1,2} & \cdots & g_{N+1,N-1} & g_{N+1,N} & g_{N+1,N+1} & g_{N+1,N+2} & \cdots \\ 0 & 0 & g_{N+2,2} & \cdots & g_{N+1,N-1} & g_{N+1,N} & g_{N+1,N+1} & g_{N+1,N+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The direct and inverse spectral problems for Jacobi matrices (with matrix elements) are described, e.g., in [1], [2]. For Jacobi fields these problems were studied in [3], [4].

As far as we know, for the first time the direct and inverse spectral problems for non-selfadjoint Jacobi matrices were investigated by Guseinov in [5]. In [6], by using a different method we extended Guseinov's result to the case of $(2N + 1)$-diagonal complex symmetric matrices.

The direct and inverse spectral problems for the block Jacobi type unitary matrices and for the block Jacobi type bounded normal matrices were solved in [7], [8].

Spectral problems for generalized Jacobi matrices connected with the indefinite product inner spaces were studied in [9].

For the case of $(2N + 1)$-diagonal complex skew-symmetric matrices, the direct and inverse spectral problems were investigated in [10]. In [11] we obtained an integral representation for the spectral function of some three-diagonal complex symmetric matrices. We remark that complex symmetric and complex skew-symmetric banded matrices are closely related to J-symmetric and J-skew-symmetric operators (see [12] and References therein). Notice that the direct and inverse spectral problems for finite nonself-adjoint Jacobi matrices were recently investigated by Guseinov in [13].

Our aim here is to introduce a notion of the generalized spectral function for the matrix J. Then we state and solve the direct and inverse spectral problems for such banded matrices.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\mathbb{C}_{n\times m}$ we denote the set of all $n \times m$ matrices with complex elements, $n, m \in \mathbb{N}$. If $V \in \mathbb{C}_{n\times m}$, then V^T stands for its transpose. Denote

$E_M = (\delta_{k,l})_{k,l=0}^M$, $M \in \mathbb{Z}_+$. By $\mathbb{P}$ we denote the set of all polynomials with complex coefficients.

2. The direct spectral problem. Consider the banded matrix (4). Define a set of *monic* polynomials $\{p_n(\lambda)\}_{n=0}^\infty$, $\deg p_n = n$, by the following relations:

$$p_k(\lambda) = \lambda^k, \qquad k = 0, 1, \dots, N-1; \tag{5}$$

$$p_{n+N}(\lambda) + \sum_{j=n-N}^{n+N-1} g_{n,j} p_j(\lambda) = \lambda^N p_n(\lambda), \qquad n = 0, 1, 2, \dots. \tag{6}$$

Here we set $p_{-1} = \cdots = p_{-N} = 0$, and $g_{n,j}$ with negative indices are zeros.

Relation (5) may be written in the matrix form:

$$J\vec{p}(\lambda) = \lambda^N \vec{p}(\lambda), \tag{7}$$

where $\vec{p}(\lambda) = (p_0(\lambda), p_1(\lambda), p_2(\lambda), \dots)^T$.

Let $\sigma = \sigma(u, v)$, $u, v \in \mathbb{P}$, be an arbitrary sesquilinear functional (i.e. linear in the first argument, antilinear in the second argument, but *not necessarily* $\sigma(u, v) = \overline{\sigma(v, u)}$). The functional σ is uniquely determined by complex numbers

$$f_{n,j} := \sigma(p_n(\lambda), \lambda^j), \qquad n, j \in \mathbb{Z}_+. \tag{8}$$

We shall construct a special functional σ. We set

$$f_{n,j} = \delta_{n,j}, \qquad 0 \le n, j \le N-1; \tag{9}$$

$$f_{n,j} = 0, \qquad 0 \le j \le N-1; \quad n \ge N. \tag{10}$$

Consider the following matrices:

$$F = F_\sigma = (f_{n,j})_{n,j=0}^\infty = \begin{pmatrix} f_{0,0} & f_{0,1} & \cdots \\ f_{1,0} & f_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \ F_M = F_{\sigma,M} = (f_{n,j})_{n,j=0}^M, \tag{11}$$

where $M \in \mathbb{Z}_+$.

Observe that the first N columns of the matrix F_σ are determined. We sequently define columns $N+1$, $N+2$, $N+3$, $\ldots$; by the following relation:

$$f_{n,j+N} = f_{n+N,j} + \sum_{l=n-N}^{n+N-1} g_{n,l} f_{l,j}, \qquad n,j \in \mathbb{Z}_+. \tag{12}$$

(With $j=0$ and $n \in \mathbb{Z}_+$ we define the $(N+1)$-th column, and so on).

In this way, we construct the matrix F_σ which satisfies relations (9), (10) and (12). Moreover, there exists a unique sesquilinear functional $\sigma(u,v)$, $u,v \in \mathbb{P}$, with the matrix F_σ which satisfies (9),(10) and (12).

Observe that relations (9), (10) and (12) are equivalent to the relations

$$\sigma(p_n(\lambda), p_j(\lambda)) = \delta_{n,j}, \qquad 0 \leq n,j \leq N-1; \tag{13}$$

$$\sigma(p_n(\lambda), p_j(\lambda)) = 0, \qquad 0 \leq j \leq N-1; \quad n \geq N; \tag{14}$$

and

$$\sigma(\lambda^N p_n(\lambda), \lambda^j) = \sigma(p_n(\lambda), \lambda^{j+N}), \qquad n,j \in \mathbb{Z}_+, \tag{15}$$

respectively. (In (15) one may use relation (6) to get (12)).

If in relation (12) we choose $n > j+N$ then in the right-hand side of this equation we get elements $f_{r,s}$ with $r > s$. By induction we obtain that

$$f_{n,j} = 0, \qquad n,j \in \mathbb{Z}_+ : \ n > j. \tag{16}$$

Thus, F_σ is an upper triangular matrix. Condition (16) is equivalent to the following condition:

$$\sigma(p_n(\lambda), p_j(\lambda)) = 0, \qquad n,j \in \mathbb{Z}_+ : \ n > j. \tag{17}$$

By linearity and antilinearity, from (15) it follows that

$$\sigma(\lambda^N u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda^N v(\lambda)), \qquad u,v \in \mathbb{P}. \tag{18}$$

Definition 1. *Let $J = (g_{k,l})_{k,l=0}^\infty$ be a complex numerical matrix satisfying* (1)–(3). *Let $\{p_n(\lambda)\}_{n=0}^\infty$ be a set of polynomials determined by relations* (5), (6). *A sesquilinear functional $\sigma = \sigma(u,v)$, $u,v \in \mathbb{P}$,* (*linear in the first argument, antilinear in the second argument, but* not necessarily $\sigma(u,v) = \overline{\sigma(v,u)}$) *which satisfies relations* (13), (17) *and* (18) *is said to be* ***the (generalized) spectral function of the matrix*** J.

The direct spectral problem for the banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying (1)–(3) consists in searching for answers on the following questions:

1) Does the spectral function exist?

2) If the spectral function exists, is it unique?

3) If the spectral function exists, how to find it (or them)?

The answer on the first two questions is surely affirmative as it follows from the preceding considerations.

The procedure of the construction of the spectral function was presented above. Thus, the direct spectral problem for the banded matrix J is solved in full.

3. The inverse spectral problem. The inverse spectral problem for the banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying (1)–(3) consists in searching for answers on the following questions:

1) Is it possible to reconstruct the matrix J using its spectral function? If it is possible, what is the procedure of the reconstruction?

2) What are the necessary and sufficient conditions for a sesquilinear functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, to be the spectral function of a complex banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying (1)–(3)?

An answer on the second question provides the following theorem.

Theorem 1. *A sesquilinear functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, is the spectral function of a complex banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying* (1)–(3) *if and only if:*

1) $\sigma(\lambda^N u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda^N v(\lambda)), \qquad u, v \in \mathbb{P};$

2) $\sigma(\lambda^k, \lambda^l) = \delta_{k,l}, \qquad k, l = 0, 1, \ldots, N-1;$

3) $\det \Gamma_M \neq 0$, $M \in \mathbb{Z}_+$,

where $\Gamma_M = (\gamma_{k,l})_{k,l=0}^{M}$, $\gamma_{k,l} = \sigma(\lambda^k, \lambda^l)$.

P r o o f. *Necessity.* Conditions 1) and 2) follows directly from the definition of the spectral function. Consider the following matrices:

$$\Gamma = (\gamma_{k,l})_{k,l=0}^{\infty}, \quad \Gamma_M = (\gamma_{k,l})_{k,l=0}^{M}, \quad \gamma_{k,l} = \sigma(\lambda^k, \lambda^l), \quad M \in \mathbb{Z}_+. \tag{19}$$

We may write

$$\lambda^k = p_k(\lambda) + \sum_{r=0}^{k-1} b_{k,r} p_r(\lambda), \qquad b_{k,r} \in \mathbb{C},\ k \in \mathbb{Z}_+. \tag{20}$$

Therefore

$$\gamma_{k,l} = \sigma(\lambda^k, \lambda^l) = \sigma(p_k(\lambda), \lambda^l) + \sum_{r=0}^{k-1} b_{k,r}\sigma(p_r(\lambda), \lambda^l)$$
$$= f_{k,l} + \sum_{r=0}^{k-1} b_{k,r} f_{r,l}, \qquad k, l \in \mathbb{Z}_+.$$

Then

$$\begin{pmatrix} \gamma_{0,l} \\ \gamma_{1,l} \\ \vdots \\ \gamma_{k,l} \\ \vdots \\ \gamma_{M,l} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_{1,0} & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{k,0} & \dots & b_{k,k-1} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{M,0} & \dots & \dots & \dots & \dots & b_{M,M-1} & 1 \end{pmatrix} \begin{pmatrix} f_{0,l} \\ f_{1,l} \\ \vdots \\ f_{k,l} \\ \vdots \\ f_{M,l} \end{pmatrix}, \tag{21}$$

for $l \in \mathbb{Z}_+$ and $M \in \mathbb{Z}_+$.

Denote the square matrix in the right-hand side of the last equality by B_M, $M \in \mathbb{Z}_+$. Observe that $\det B_M = 1$. We may write

$$\Gamma_M = B_M F_M. \tag{22}$$

By relations (12) and (16) we get

$$f_{n,n} = f_{n+N,n-N} + \sum_{l=n-N}^{n+N-1} g_{n,l} f_{l,n-N} = g_{n,n-N} f_{n-N,n-N}, \qquad n \geq N. \tag{23}$$

Using relations (3) and (9), by induction we obtain that

$$f_{n,n} \neq 0, \qquad n \in \mathbb{Z}_+. \tag{24}$$

Therefore $\det F_M \neq 0$ and

$$\det \Gamma_M = \det B_M \det F_M \neq 0. \tag{25}$$

Sufficiency. Let a sesquilinear functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, satisfying conditions 1), 2), 3) be given. Define a sequence of polynomials $p_n(\lambda)$ in the following way:

$$p_k(\lambda) = \lambda^k, \qquad 0 \leq k \leq N-1; \tag{26}$$

and for $n \geq N$ we set

$$p_n(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_{n,i}\lambda^i, \tag{27}$$

where the coefficients $a_{n,i}$ are uniquely determined by the following linear system of equations:

$$\begin{cases} \gamma_{0,0}a_{n,0} + \gamma_{1,0}a_{n,1} + \cdots + \gamma_{n-1,0}a_{n,n-1} = -\gamma_{n,0} \\ \gamma_{0,1}a_{n,0} + \gamma_{1,1}a_{n,1} + \cdots + \gamma_{n-1,1}a_{n,n-1} = -\gamma_{n,1} \\ \dots \\ \gamma_{0,n-1}a_{n,0} + \gamma_{1,n-1}a_{n,1} + \cdots + \gamma_{n-1,n-1}a_{n,n-1} = -\gamma_{n,n-1} \end{cases} \tag{28}$$

Notice that the matrix of this linear system is Γ_{n-1}^T. By (26) and condition 2) we get

$$\sigma(p_k(\lambda), p_l(\lambda)) = \delta_{k,l}, \qquad 0 \leq k, l \leq N-1; \tag{29}$$

and by (27) and (28) we obtain

$$\sigma(p_n(\lambda), \lambda^j) = \sigma(\lambda^n, \lambda^j) + \sum_{i=0}^{n-1} a_{n,i}\sigma(\lambda^i, \lambda^j)$$

$$= \gamma_{n,j} + \sum_{i=0}^{n-1} a_{n,i}\gamma_{i,j} = 0, \tag{30}$$

where $j = 0, 1, \dots, n-1$; $n \geq N$. Therefore

$$\sigma(p_n(\lambda), p_j(\lambda)) = 0, \qquad n, j \in \mathbb{Z}_+ : \ n > j. \tag{31}$$

By (28) we may write

$$\begin{cases} \gamma_{0,0}a_{n,0} + \gamma_{1,0}a_{n,1} + \cdots + \gamma_{n-1,0}a_{n,n-1} + \gamma_{n,0} \cdot 1 = 0 \\ \gamma_{0,1}a_{n,0} + \gamma_{1,1}a_{n,1} + \cdots + \gamma_{n-1,1}a_{n,n-1} + \gamma_{n,1} \cdot 1 = 0 \\ \dots \\ \gamma_{0,n-1}a_{n,0} + \gamma_{1,n-1}a_{n,1} + \cdots + \gamma_{n-1,n-1}a_{n,n-1} + \gamma_{n,n-1} \cdot 1 = 0 \\ \gamma_{0,n}a_{n,0} + \gamma_{1,n}a_{n,1} + \cdots + \gamma_{n-1,n}a_{n,n-1} + \gamma_{n,n} \cdot 1 = t_n \end{cases}, \tag{32}$$

where t_n is a complex number. Suppose that $t_n = 0$. The matrix of the linear system (32) is Γ_n^T. Therefore it should possess a unique solution. However relation (32) shows that there exist a non-trivial and trivial solutions. We obtained a contradiction. Consequently, we conclude that $t_n \neq 0$. Therefore

$$\sigma(p_n(\lambda), \lambda^n) = \sigma(\lambda^n, \lambda^n) + \sum_{i=0}^{n-1} a_{n,i}\sigma(\lambda^i, \lambda^n) = \gamma_{n,n} + \sum_{i=0}^{n-1} a_{n,i}\gamma_{i,n}$$

$$= t_n \neq 0, \qquad n \geq N;$$

$$\sigma(p_n(\lambda), \lambda^n) \neq 0, \qquad n \in \mathbb{Z}_+. \tag{33}$$

We may write

$$\lambda^N p_n(\lambda) = p_{n+N}(\lambda) + \sum_{j=0}^{n+N-1} \xi_{n,j} p_j(\lambda), \qquad n \in \mathbb{Z}_+, \tag{34}$$

with some complex coefficients $\xi_{n,j}$.

Let us check that

$$\lambda^N p_n(\lambda) = p_{n+N}(\lambda) + \sum_{j=n-N}^{n+N-1} \xi_{n,j} p_j(\lambda), \qquad n \in \mathbb{Z}_+, \tag{35}$$

where we set $p_{-1} = \cdots = p_{-N} = 0$, and $\xi_{n,j}$ with negative indices are zeros.

If $0 \leq n \leq N$ this is already obtained.

Let $n = N + r$, $r \geq 1$. By (34), (31) we get

$$\sigma(\lambda^N p_n(\lambda), p_0(\lambda)) = \sigma(p_{n+N}(\lambda), p_0(\lambda)) + \sum_{j=0}^{n+N-1} \xi_{n,j}\sigma(p_j(\lambda), p_0(\lambda))$$

$$= \xi_{n,0}\sigma(p_0(\lambda), p_0(\lambda)); \tag{36}$$

On the other hand, by condition 1) and (31) we obtain

$$\sigma(\lambda^N p_n(\lambda), p_0(\lambda)) = \sigma(p_n(\lambda), \lambda^N p_0(\lambda)) = 0. \tag{37}$$

By (36),(37) and (33) we conclude that

$$\xi_{n,0} = 0. \tag{38}$$

As a consequence, if $r = 1$, then relation (35) is proven.

Consider the case $r \geq 2$. Suppose that

$$\xi_{n,0} = \cdots = \xi_{n,k} = 0, \tag{39}$$

where $0 \leq k \leq r-2$. Let us check that

$$\xi_{n,k+1} = 0. \tag{40}$$

In fact, we may write

$$\sigma(\lambda^N p_n(\lambda), p_{k+1}(\lambda)) = \sigma(p_{n+N}(\lambda), p_{k+1}(\lambda)) + \sum_{j=k+1}^{n+N-1} \xi_{n,j}\sigma(p_j(\lambda), p_{k+1}(\lambda))$$

$$= \xi_{n,k+1}\sigma(p_{k+1}(\lambda), p_{k+1}(\lambda)). \tag{41}$$

By condition 1) and (31) we obtain

$$\sigma(\lambda^N p_n(\lambda), p_{k+1}(\lambda)) = \sigma(p_n(\lambda), \lambda^N p_{k+1}(\lambda)) = 0. \tag{42}$$

By (41), (42) we obtain that relation (40) is true.

By induction we conclude that relation (35) holds. Notice that a similar idea of the proof of a recurrence relation was applied in [14, pp. 364–365].

Observe that for an arbitrary $n \geq N$ we may write

$$\sigma(\lambda^N p_n(\lambda), p_{n-N}(\lambda)) = \sigma(p_{n+N}(\lambda), p_{n-N}(\lambda)) + \sum_{j=n-N}^{n+N-1} \xi_{n,j}\sigma(p_j(\lambda), p_{n-N}(\lambda))$$

$$= \xi_{n,n-N}\sigma(p_{n-N}(\lambda), p_{n-N}(\lambda)). \tag{43}$$

By (33) and (31) we obtain that

$$\sigma(\lambda^N p_n(\lambda), p_{n-N}(\lambda)) = \sigma(p_n(\lambda), \lambda^N p_{n-N}(\lambda)) \neq 0. \tag{44}$$

Therefore

$$\xi_{n,n-N} \neq 0, \qquad n \geq N. \tag{45}$$

Set

$$\xi_{n,n+N} = 1, \quad \xi_{n,j} = 0, \qquad n \in \mathbb{Z}_+,\ j > n+N; \tag{46}$$

and

$$J_\sigma := (\xi_{n,j})_{n,j=0}^{\infty}. \tag{47}$$

The matrix J_σ satisfies the conditions (1)–(3). The polynomials from (5), (6) coincide with the defined above polynomials $p_n(\lambda)$ since they satisfy the same recurrence relation. By (29), (31), condition 1) and Definition 1 we conclude that σ is the spectral function of the matrix J_σ. $\square$

Let us investigate the first question in the inverse spectral problem. Consider an arbitrary complex banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying (1)–(3). Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be the polynomials defined by (5), (6) and σ be the spectral function of J.

Denote by $J_\sigma = (\xi_{k,l})_{k,l=0}^{\infty}$ the banded matrix which is constructed for the functional σ by the procedure in the Sufficiency of the proof of the last theorem. To omit the confusion, we denote the polynomials $p_n(\lambda)$ which are constructed in the Sufficiency by $r_n(\lambda)$. Let us check that $J = J_\sigma$. Polynomials $p_0, p_1, \dots, p_{N-1}$ and $r_0, r_1, \dots, r_{N-1}$ coincide by their definitions.

Polynomials $r_n(\lambda)$ are defined by relations (27), (28) which are equivalent to the following condition:

$$\sigma(r_n(\lambda), \lambda^j) = 0, \qquad j = 0, 1, \dots, n-1;\ n \geq N. \tag{48}$$

By (17) we see that polynomials $p_n(\lambda)$ satisfy this relation, as well. Since the solution of the linear system (28) is unique we conclude that $r_n = p_n$, $n \in \mathbb{Z}_+$.

Since $\deg p_n = n$, $n \in \mathbb{Z}_+$, the polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ form a linear basis in $\mathbb{P}$ and an arbitrary polynomial $u(\lambda) \in \mathbb{P}$ has a unique representation as the linear combination of polynomials p_n. Therefore $\xi_{k,l} = g_{k,l}$, $k, l \in \mathbb{Z}_+$, and $J = J_\sigma$.

Thus, the procedure of the construction of the matrix J_σ provides a tool to reconstruct a complex banded matrix satisfying (1)–(3) by its generalized spectral function.

Theorem 1 can be reformulated in another form. We shall use the following definition (cf. [15, Definition 3.2, p. 16]).

Definition 2. *Let $\sigma = \sigma(u, v)$, $u, v \in \mathbb{P}$, be a sesquilinear functional. The functional σ is said to be* **quasi-definite** *if*

$$\det \Gamma_M \neq 0, \qquad M \in \mathbb{Z}_+, \tag{49}$$

where $\Gamma_M = (\gamma_{k,l})_{k,l=0}^{M}$, $\gamma_{k,l} = \sigma(\lambda^k, \lambda^l)$.

Theorem 2. *A sesquilinear functional $\sigma(u,v)$, $u,v \in \mathbb{P}$, is the spectral function of a complex banded matrix $J = (g_{k,l})_{k,l=0}^{\infty}$ satisfying (1)–(3) if and only if σ is quasi-definite and the matrix*

$$\Gamma = (\gamma_{i,j})_{i,j=0}^{\infty}, \quad \gamma_{i,j} = \sigma(\lambda^i, \lambda^j), \tag{50}$$

is a $(N \times N)$ block Hankel matrix:

$$\Gamma = (G_{k+l})_{k,l=0}^{\infty}, \quad G_j \in \mathbb{C}_{N\times N}, \tag{51}$$

such that $G_0 = E_N$.

P r o o f. Consider an arbitrary sesquilinear functional $\sigma(u,v)$, $u,v \in \mathbb{P}$ and define the matrix Γ by (50). Split the matrix Γ into $(N \times N)$ blocks:

$$\Gamma = (G_{k,l})_{k,l=0}^{\infty}, \quad G_{k,l} \in \mathbb{C}_{N\times N}. \tag{52}$$

Observe that

$$G_{k,l} = \begin{pmatrix} \gamma_{kN,lN} & \gamma_{kN,lN+1} & \cdots & \gamma_{kN,lN+N-1} \\ \gamma_{kN+1,lN} & \gamma_{kN+1,lN+1} & \cdots & \gamma_{kN+1,lN+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{kN+N-1,lN} & \gamma_{kN+N-1,lN+1} & \cdots & \gamma_{kN+N-1,lN+N-1} \end{pmatrix}, \quad k,l \in \mathbb{Z}_+. \tag{53}$$

If Condition 1) of Theorem 1 holds then

$$\gamma_{kN+r,lN+s} = \gamma_{(k+l)N+r,s}, \qquad 0 \le r,s \le N-1, \quad k,l \in \mathbb{Z}_+, \tag{54}$$

and hence

$$G_{k,l} = G_{k+l}, \qquad k,l \in \mathbb{Z}_+.$$

On the other hand, if G is a $(N \times N)$ block Hankel matrix, then relation (54) holds. Then

$$\begin{aligned} \sigma(\lambda^{aN+r}, \lambda^N \lambda^{cN+s}) &= \sigma(\lambda^{aN+r}, \lambda^{(c+1)N+s}) = \sigma(\lambda^{(a+c+1)N+r}, \lambda^s) \\ &= \sigma(\lambda^{(a+1)N+r}, \lambda^{cN+s}) = \sigma(\lambda^N \lambda^{aN+r}, \lambda^{cN+s}), \end{aligned}$$

where $0 \le r,s \le N-1$, $a,c \in \mathbb{Z}_+$. By linearity we conclude that Condition 1 holds.

Thus, Condition 1 of Theorem 1 is equivalent to the condition (51).

If conditions of Theorem 1 hold, then $G_0 = E_N$ and relation (49) holds.

Conversely, if conditions of Theorem 2 hold, then Condition 2) and Condition 3) hold. $\square$

4. An integral representation for the spectral function. Recall that the matrix Hamburger moment problem consists of finding a non-decreasing $\mathbb{C}_{n\times n}$-valued function $M(x)$ such that

$$S_k = \int_{\mathbb{R}} x^k dM(x), \quad k \in \mathbb{Z}_+, \tag{55}$$

where $\{S_k\}_{k=0}^\infty$ is a prescribed set of complex Hermitian $N \times N$ matrices, $N \in \mathbb{N}$. This problem has a solution if and only if ([16])

$$(S_{k+l})_{k,l=0}^M \geq 0, \qquad M \in \mathbb{Z}_+. \tag{56}$$

On the other hand, by applying [15, Theorem 6.3, p. 74] we may state that for an *arbitrary* sequence $\{S_k\}_{k=0}^\infty$, $S_k \in \mathbb{C}_{N\times N}$, there exists a $\mathbb{C}_{n\times n}$-valued function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ with functions $m_{k,l}(x)$ of bounded variation such that relation (55) holds.

Consider an arbitrary complex banded matrix $J = (g_{k,l})_{k,l=0}^\infty$ satisfying (1)–(3). Let σ be the generalized spectral function of J. We shall obtain an integral representation for the functional σ. Define the matrix Γ by (50). By Theorem 2 the matrix Γ has the form (51). By the above mentioned result, there exists a $\mathbb{C}_{n\times n}$-valued function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ with functions $m_{k,l}(x)$ of bounded variation such that

$$G_k = \int_{\mathbb{R}} x^k dM(x), \quad k \in \mathbb{Z}_+. \tag{57}$$

We shall use the following operator [17]:

$$R_{N,m}(p) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} t^n, \qquad p \in \mathbb{P},\ 0 \leq m \leq N-1. \tag{58}$$

Observe that

$$R_{N,m}(\lambda^{kN+r}) = \begin{cases} x^k, & \text{if } m = r \\ 0, & \text{if } m \neq r \end{cases}, \qquad 0 \leq m, r \leq N-1;\ k \in \mathbb{Z}_+. \tag{59}$$

By (53) we may write

$$\gamma_{kN+r,lN+s} = \vec{e}_r G_{k+l} \vec{e}_s^{\,*}, \qquad k, l \in \mathbb{Z}_+;\ 0 \leq r, s \leq N-1, \tag{60}$$

where $\vec{e}_r = (\delta_{r,0}, \delta_{r,1}, \ldots, \delta_{r,N-1})$.

By (57), (60) we get

$$\gamma_{kN+r,lN+s} = \vec{e}_r \int_{\mathbb{R}} x^{k+l} dM(x) \vec{e}_s^{\,*} = \int_{\mathbb{R}} x^k \vec{e}_r dM(x) x^l \vec{e}_s^{\,*}$$

$$= \int_{\mathbb{R}} (R_{N,0}(x^{kN+r}), R_{N,1}(x^{kN+r}), \ldots, R_{N,N-1}(x^{kN+r})) dM(x) \overline{\begin{pmatrix} R_{N,0}(x^{lN+s}) \\ R_{N,1}(x^{lN+s}) \\ \ldots \\ R_{N,N-1}(x^{lN+s}) \end{pmatrix}}.$$

Therefore

$$\gamma_{i,j} = \int_{\mathbb{R}} (R_{N,0}(x^i), R_{N,1}(x^i), \ldots, R_{N,N-1}(x^i)) dM(x) \overline{\begin{pmatrix} R_{N,0}(x^j) \\ R_{N,1}(x^j) \\ \ldots \\ R_{N,N-1}(x^j) \end{pmatrix}}, \tag{61}$$

$i, j \in \mathbb{Z}_+$.

By linearity we obtain that

$$\sigma(u,v) = \int_{\mathbb{R}} (R_{N,0}(u)(x), R_{N,1}(u)(x), \ldots, R_{N,N-1}(u)(x)) dM(x) \overline{\begin{pmatrix} R_{N,0}(v)(x) \\ R_{N,1}(v)(x) \\ \ldots \\ R_{N,N-1}(v)(x) \end{pmatrix}},$$

$$u, v \in \mathbb{P}. \tag{62}$$

We have proved the following theorem.

Theorem 3. *Let $J = (g_{k,l})_{k,l=0}^{\infty}$ be a complex banded matrix satisfying* (1)–(3) *and σ be its generalized spectral function. The spectral function σ admits an integral representation* (62) *where $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ is a $\mathbb{C}_{n\times n}$-valued function with functions $m_{k,l}(x)$ of bounded variation.*

Theorem 3 can be reformulated from another point of view. Namely, we can state it as a Favard-type theorem (see [15, Theorem 4.4, p. 21] and [15, Theorem 6.4, p. 75]). We shall make use of the following definition (cf. [15, Definition 3.2, p. 16]).

Definition 3. *Let $\sigma = \sigma(u,v)$, $u, v \in \mathbb{P}$, be a quasi-definite sesquilinear functional. A set of monic polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n$, such that*

$$\sigma(p_n(\lambda), \lambda^j) = 0, \qquad n, j \in \mathbb{Z}_+ : \; n > j, \tag{63}$$

is said to be ***the set of monic left-orthogonal polynomials with respect to*** σ.

Theorem 4. *Let* $\{p_n(\lambda)\}_{n=0}^{\infty}$, *be a set of polynomials satisfying relations* (5), (6), *where* $g_{k,j}$ *are arbitrary complex numbers satisfying* (3). *Then there exists a quasi-definite sesquilinear functional* σ *such that* $\{p_n(\lambda)\}_{n=0}^{\infty}$ *is the set of monic left-orthogonal polynomials with respect to* σ. *Moreover, the functional* σ *admits an integral representation* (62) *where* $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ *is a* $\mathbb{C}_{n\times n}$*-valued function with functions* $m_{k,l}(x)$ *of bounded variation.*

Remark. Condition (5) in the last theorem can be replaced by the following more general initial conditions:

$$p_k(\lambda) = \varphi_k(\lambda), \qquad k = 0, 1, \ldots, N-1, \tag{64}$$

where $\{\varphi_k(\lambda)\}_{k=0}^{N-1}$, $\deg \varphi_k = k$, is an arbitrary prescribed set of monic polynomials. In fact, we can determine numbers $f_{n,j} := \sigma(p_n(\lambda), \lambda^j)$, $n, j \in \mathbb{Z}_+$, by relations (9), (10) and (12). Then relation (16) holds. Relation (12) is equivalent to the relation (18). Therefore the matrix

$$\Gamma = (\gamma_{i,j})_{i,j=0}^{\infty}, \quad \gamma_{i,j} = \sigma(\lambda^i, \lambda^j), \tag{65}$$

is a $(N \times N)$ block Hankel matrix (see the Proof of Theorem 2). Repeating considerations after (19) we conclude that σ is quasi-definite. Then we repeat the construction after (57) to obtain an integral representation for σ.

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