Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **37** (2011), 113–142

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DUNKL-SCHRÖDINGER EQUATIONS WITH AND WITHOUT QUADRATIC POTENTIALS

Hatem Mejjaoli

Communicated by T. Gramchev

Dedicated to Khalifa Trimèche for his 65 birthday

Abstract. The purpose of this paper is to study the dispersive properties of the solutions of the Dunkl-Schrödinger equation and their perturbations with potential. Furthermore, we consider a few applications of these results to the corresponding nonlinear Cauchy problems.

1. Introduction. Dunkl operators T_j $(j = 1, \dots, d)$ introduced by Dunkl in [7] are parameterized differential-difference operators on $\mathbb{R}^d$ that are related to finite reflection groups. Over the last years, much attention has been paid to these operators in various mathematical (and even physical) directions. In this prospect, Dunkl operators are naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many-body systems ([1, 6, 14]). Moreover, Dunkl operators allow generalizations of several analytic structures,

2010 *Mathematics Subject Classification:* Primary 42A38. Secondary 42B10.
Key words: Nonlinear Dunkl-Schrödinger equation, Strichartz estimate, local and global well-posedness, blow up, scattering, wave operators.

such as Laplace operator, Fourier transform, heat semigroup, wave equations, and Schrödinger equations ([2, 8, 9, 19, 20, 21, 24]).

In the present paper, we intend to continue our study of Dunkl-Schrödinger equations started in [20, 21]. Indeed, in [20] we provided a general theory for the equation

$$\begin{cases} i\partial_t u + \Delta_k u &= f(t,x), \quad (t,x) \in I \times \mathbb{R}^d \\ u_{|t=0} &= u_0, \end{cases}$$

where $\Delta_k := \sum_{j=1}^{d} T_j^2$ is the Dunkl Laplace operator. Furthermore, we studied the dispersive phenomena and we proved the Strichartz estimate for this equation. Next in [21], we have studied a class of nonlinear Dunkl-Schrödinger equations:

$$i\partial_t u + \Delta_k u = F(u), \tag{1.1}$$

where F is a nonlinear complex valued function. We have studied a general theory for the above equation focusing on the following problems:

(1) Existence and uniqueness of solutions of the Cauchy problem, and

(2) asymptotic behavior in time of the solutions, and in particular scattering theory for the pair of equations consisting of nonlinear equations and of homogeneous equation.

The main subject of this paper is the study of the dispersive properties of the Dunkl-Schrödinger equation

$$i\partial_t u + \Delta_k u = 0,$$

and their perturbation with a potential:

$$i\partial_t u + \Delta_k u + V(t,x)u = 0.$$

We shall also consider a few applications of these results to the corresponding nonlinear Cauchy problems.

A first main question studied in the present paper is: what part of the dispersive properties is preserved if we perturb the equation with a potential term of the form $V(t,x)u$ or simply $V(x)u$? The importance of this question is clear both form the point view of the applications, and as a first step to the general case of equations with variables coefficients. Notice that classical Schrödinger equation perturbed with a potential has been studied by many authors, as references see [3].

The paper is organized as follows. In Section 2, we recall main results in harmonic analysis associated with the Dunkl operators. In Section 3, we consider

Dunkl-Schrödinger equation with and without quadratic potential. Instead of the stronger dispersive estimate, our goal here is to prove the Strichartz estimates. We give two quite general results of this type. In the first one, we consider the model case: when $V(x) = \|x\|^2$, we study and we give some properties of the semigroup generated by the generalized harmonic oscillator $\mathcal{H}_k := -\Delta_k + \|x\|^2$, noted by $e^{-z\mathcal{H}_k}$ with $z \in \mathbb{C}$, $\Re z \geq 0$. Next we establish the Strichartz estimate for the inohomegeneous equation. In our second result, we consider a perturbation of the equation of the form

$$\begin{cases} i\partial_t u - \mathcal{H}_k u & = & V(t,x)u, \quad (t,x) \in I \times \mathbb{R}^d \\ u_{|t=0} & = & u_0 \in L^2_k(\mathbb{R}^d), \end{cases}$$

we prove this problem is well posedness under the assumption on V. In Section 4 we use a change of variables that turns the critical nonlinear Dunkl-Schrödinger equation into the critical nonlinear Dunkl-Schrödinger equation with isotropic harmonic potential, in any space dimension. This change of variables is isometric on $L^2_k(\mathbb{R}^d)$, and bijective on some time intervals. Using the known results for the critical nonlinear Dunkl-Schrödinger equation (cf. [21]), this provides the information for the critical nonlinear Dunkl-Schrödinger equation with isotropic harmonic potential. In Section 5 we consider the perturbation of the Dunkl-Schrödinger equation with quadratic potential. We prove that no finite time blow up can occur for the nonlinear Dunkl-Schrödinger equation with quadratic potentials, provided that the potential has a strong repulsive component.

Throughout this paper, C stands for a positive constant not necessarily the same in each occurrence.

2. Preliminaries. This section gives an introduction to the theory of Dunkl operators, Dunkl kernel and Dunkl transform. Main references are [7, 8, 9].

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle\cdot,\cdot\rangle$ and $\|x\|=\sqrt{\langle x,x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. We fix a positive root system $R_+ = \{\alpha \in R \,/ \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in R} H_\alpha$. We will assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2} \omega_k(x)\, dx \right)^{-1}. \tag{2.4}$$

In the following we denote by

$C(\mathbb{R}^d)$	the space of continuous functions on $\mathbb{R}^d$.
$C^p(\mathbb{R}^d)$	the space of functions of class C^p on $\mathbb{R}^d$.
$\mathcal{E}(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$.
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
$D(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.
$\mathcal{S}'(\mathbb{R}^d)$	the space of temperate distributions on $\mathbb{R}^d$.

Dunkl operators T_j, $j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d). \tag{2.5}$$

In the case $k = 0$, the T_j, $j = 1, \ldots, d$, reduce to the corresponding partial derivatives.

Throughout this paper, we will assume that the multiplicity is non-negative, that is $k(\alpha) \geq 0$ for all $\alpha \in R$. We write $k \geq 0$ for short.

We define the Dunkl-Laplace operator on $\mathbb{R}^d$ by

$$\begin{aligned}\Delta_k f(x) &= \sum_{j=1}^{d} T_j^2 f(x) \\ &= \Delta f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\left[\frac{\langle \nabla f(x),\alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x)-f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\right].\end{aligned} \tag{2.6}$$

For $y \in \mathbb{R}^d$, the system

$$\left\{\begin{array}{lcl} T_j u(x,y) &=& y_j u(x,y), \quad j=1,\dots,d, \\ u(0,y) &=& 1, \end{array}\right. \tag{2.7}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The Dunkl kernel possesses the following properties.

i) For $z, t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z); K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(z,x)| \le \|x\|^{|\nu|} \exp(\|x\|\,\|\Re z\|), \tag{2.8}$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\cdots \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

Notation. We denote by $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned}\|f\|_{L_k^p(\mathbb{R}^d)} &:= \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)\,dx\right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad 1 \le p < +\infty, \\ \|f\|_{L_k^\infty(\mathbb{R}^d)} &:= \operatorname{ess}\sup_{x\in\mathbb{R}^d} |f(x)| < +\infty.\end{aligned}$$

The Dunkl transform of a function f in $L_k^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x) K(-iy,x)\omega_k(x)dx, \quad \text{for all } y \in \mathbb{R}^d. \tag{2.9}$$

Next, we give some properties of this transform (cf. [8, 9]).

i) For f in $L_k^1(\mathbb{R}^d)$ we have

$$\|\mathcal{F}_D(f)\|_{L_k^\infty(\mathbb{R}^d)} \le \|f\|_{L_k^1(\mathbb{R}^d)}. \tag{2.10}$$

ii) For f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\mathcal{F}_D(T_jf)(y) = iy_j\mathcal{F}_D(f)(y), \quad \text{for all} \quad j = 1, \dots, d \quad \text{and} \quad y \in \mathbb{R}^d. \tag{2.11}$$

Proposition 1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathcal{S}(\mathbb{R}^d)$*

$$\overline{\mathcal{F}_D}(f)(y) = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\mathcal{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.12}$$

we have

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 2. i) **Plancherel formula for $\mathcal{F}_D$.**
For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)\, dx = \frac{c_k^2}{4^{\gamma+\frac{d}{2}}}\int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)\, d\xi. \tag{2.13}$$

ii) **Plancherel theorem for $\mathcal{F}_D$.**
The renormalized Dunkl transform $f \to 2^{-(\gamma+\frac{d}{2})}c_k\mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.

3. Dunkl-Schrödinger equations with and without harmonic potential

In this section we will discuss Strichartz type estimates for the Dunkl-Schrödinger equations with and without harmonic potential.

3.1. Dunkl-Schrödinger semigroup. This subsection deals with the Dunkl-type analogues of the classical Schrödinger semigroup on several Banach spaces. These semigroups are generated by the Dunkl Laplacian, and they are governed by a generalized Schrödinger kernel which was introduced and studied in [20]. Firstly we collect some notations and results which we need in the sequel.

Notations. We denote by $e^{it\Delta_k}$ the Dunkl-Schrödinger semigroup on $L_k^2(\mathbb{R}^d)$ defined by

$$e^{it\Delta_k}v := \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}}e^{-i(d+2\gamma)\frac{\pi}{4}\operatorname{sgn} t}e^{i\frac{\|.\|^2}{4t}}\left[\mathcal{F}_D\left(e^{i\frac{\|.\|^2}{4t}}v\right)\right]\left(\frac{.}{2t}\right). \tag{3.1}$$

For any interval I of $\mathbb{R}$ (bounded or unbounded), we define the mixed space-time $L^p(I; L_k^q(\mathbb{R}^d))$ Banach space of (classes of) measurable functions u :

$I \to L^q_k(\mathbb{R}^d)$ such that $\|u\|_{L^p(I;L^q_k(\mathbb{R}^d))} < \infty$, with

$$\begin{aligned} \|u\|_{L^p(I;L^q_k(\mathbb{R}^d))} &= \left(\int_I \|u(t,.)\|^p_{L^q_k(\mathbb{R}^d)} dt\right)^{\frac{1}{p}}, \quad \text{if} \quad 1 \leq p, q < \infty, \\ \|u\|_{L^\infty(I;L^q_k(\mathbb{R}^d))} &= \operatorname*{ess\,sup}_{t\in I} \|u(t,.)\|_{L^q_k(\mathbb{R}^d)}, \qquad \text{if} \quad 1 \leq q < \infty. \end{aligned}$$

For any interval I of $\mathbb{R}$ (bounded or unbounded) and a Banach space X, we define the mixed space-time $C(\overline{I};X)$ space of continuous functions $\overline{I} \to X$. When I is bounded, $C(\overline{I};X)$ is a Banach space with the norm of $L^\infty(I,X)$.

For $1 \leq p < \infty$, we denote by $WL^p(I)$ the weak $L^p(I)$ Lebesgue space defined as the set of locally integrable functions f on I with the finite norm

$$\|f\|_{WL^p(I)} = \sup_{r>0} r\mu\Big\{x \in I:\ |f(x)| > r\Big\}^{\frac{1}{p}},$$

where μ designates the Lebesgue measure.

Definition 1. *We say that the exponent pair* (q,r) *is* $\frac{d+2\gamma}{2}$*-admissible if* $q, r \geq 2$, $\left(q, r, \frac{d+2\gamma}{2}\right) \neq (2,\infty,1)$ *and*

$$\frac{1}{q} + \frac{d+2\gamma}{2r} \leq \frac{d+2\gamma}{4}. \tag{3.2}$$

If equality holds in (3.2) *we say that* (q,r) *is sharp* $\frac{d+2\gamma}{2}$*-admissible, otherwise we say that* (q,r) *is nonsharp* $\frac{d+2\gamma}{2}$*-admissible. Note in particular that when* $d+2\gamma > 2$ *the endpoint*

$$P = (2, \frac{2d+4\gamma}{d+2\gamma-2})$$

is sharp $\frac{d+2\gamma}{2}$*-admissible.*

The following Proposition generalize, in the setting of Dunkl's theory, the Strichartz estimates studied by Strichartz, Ginibre, Velo, Keel and Tao and others (cf. [13, 18, 25]).

Proposition 3. *Let* $(U(t))_{t\in I}$ *be a bounded family of continuous opera-*

tors on $L^2_k(\mathbb{R}^d)$ *such that, we have*

$$\|U(t)U^*(t')f\|_{L^\infty_k(\mathbb{R}^d)} \leq C\Big(\Theta(t-t')\Big)^{\frac{d}{2}+\gamma}\|f\|_{L^1_k(\mathbb{R}^d)}, \tag{3.3}$$
$$\text{with} \quad \Theta \geq 0 \quad \text{and} \quad \Theta \in WL^1(I).$$

i) *If* I *is an unbounded interval. Then, the estimates*

$$\|U(t)u_0\|_{L^q(I;L^r_k(\mathbb{R}^d))} \leq C\|u_0\|_{L^2_k(\mathbb{R}^d)} \tag{3.4}$$

$$\left\|\int_{\mathbb{R}} U^*(t)f(t,\cdot)dt\right\|_{L^2_k(\mathbb{R}^d)} \leq C\|f\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))} \tag{3.5}$$

holds for any sharp $\dfrac{d+2\gamma}{2}$*-admissible exponent* $(q,r)\neq P$*, where* q'*,* r' *denotes, as in all that follows, the conjugate exponent of* q *and* r *and* U^* *the adjoint operator of* U*.*

ii) *If* I *is a bounded interval. Then, the estimates* (3.4) *and* (3.5) *are valid for any sharp* $\dfrac{d+2\gamma}{2}$*-admissible exponent* $(q,r)\neq P$*, and for pairs* (q,r) *such that* $1\leq q\leq 2$*,* $2\leq r<\dfrac{2(d+2\gamma)}{d+2\gamma-2}$*.*

P r o o f. i) For the convenience of the reader, we prove the result, with $I=\mathbb{R}$. We have

$$\begin{aligned}\|U(t)u_0\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} &= \sup_{\varphi\in B^{q,r}_k}\int_{\mathbb{R}^{d+1}} U(t)u_0(x)\overline{\varphi(t,x)}dt\omega_k(x)dx\\ &= \sup_{\varphi\in B^{q,r}_k}\left\langle u_0, \int_{\mathbb{R}} U^*(t)\varphi(t,.)dt\right\rangle_{L^2_k(\mathbb{R}^d)},\end{aligned}$$

where $B^{q,r}_k$ denotes the set of elements of $D(\mathbb{R}^{d+1},\mathbb{C})$ such that the norm $\|\cdot\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))}$ is less or equal to 1, and U^* the adjoint operator of U.

Thus, using Cauchy-Schwarz inequality, we deduce that

$$\|U(t)u_0\|_{L^q(\mathbb{R};L^r_k(\mathbb{R}^d))} \leq \|u_0\|_{L^2_k(\mathbb{R}^d)}\sup_{\varphi\in B^{q,r}_k}\left\|\int_{\mathbb{R}} U^*(t)\varphi(t,\cdot)dt\right\|_{L^2_k(\mathbb{R}^d)}.$$

This duality argument simply says that inequality (3.5) implies (3.4). In order to prove (3.5), let us write

$$\begin{aligned}\left\|\int_{\mathbb{R}} U^*(t)\varphi(t,\cdot)dt\right\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^2} \langle U^*(t)\varphi(t,\cdot), U^*(t')\varphi(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)} dtdt' \\ &= \int_{\mathbb{R}^2} \langle U(t')U^*(t)\varphi(t,\cdot), \varphi(t',\cdot)\rangle_{L^2_k(\mathbb{R}^d)} dtdt'.\end{aligned}$$

As $(U(t))_{t\in\mathbb{R}}$ is a bounded family of operators on $L^2_k(\mathbb{R}^d)$ and using the dispersive estimate (3.3), we get, thanks to the interpolation theorem, for all $r \in [2,\infty]$,

$$\|U(t)U^*(t')\varphi(t,.)\|_{L^r_k(\mathbb{R}^d)} \leq C\big(\Theta(t-t')\big)^{\beta(r)+1} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)}, \tag{3.6}$$

where $\beta(r) = \left(\frac{d}{2}+\gamma\right)\left(1-\frac{2}{r}\right) - 1.$

In the sharp $\gamma + \frac{d}{2}$-admissible case we have

$$\frac{1}{q'} - \frac{1}{q} = -\beta(r).$$

The relation (3.6) and Hölder's inequality gives

$$\begin{aligned}&\left\|\int_{\mathbb{R}} U^*(t)\varphi(t,\cdot)dt\right\|^2_{L^2_k(\mathbb{R}^d)} \\ &\qquad \leq \int_{\mathbb{R}^2} C(\Theta(t-t'))^{\beta(r)+1} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt'.\end{aligned}$$

We put

$$k(t) = \int_{\mathbb{R}} (\Theta(t-t'))^{\beta(r)+1} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dt'.$$

Hence

$$\int_{\mathbb{R}^2} (\Theta(t-t'))^{\beta(r)+1} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt' = \int_{\mathbb{R}} k(t) \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dt.$$

Hölder's inequality implies that

$$\begin{aligned}&\int_{\mathbb{R}^2} (\Theta(t-t'))^{\beta(r)+1} \|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \|\varphi(t',\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} dtdt' \\ &\qquad\qquad \leq \|\varphi\|_{L^{q'}(\mathbb{R};L^{r'}_k(\mathbb{R}^d))} \left(\int_{\mathbb{R}} |k(t)|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

On the other hand, since (q,r) is sharp $\gamma+\frac{d}{2}$-admissible (and it not an endpoint), then we can apply weak Young inequality, obtaining

$$\left\|\int_{\mathbb{R}} U^*(t)\varphi(t,\cdot)dt\right\|^2_{L^2_k(\mathbb{R}^d)} \leq C\|\Theta\|^{\frac{q}{2}}_{WL^1(I)}\|\varphi\|^2_{L^{q'}(I;L^{r'}_k(\mathbb{R}^d))}.$$

As Θ is in $WL^1(I)$, (3.5) follows.

ii) We proceed as above we prove the result for any sharp $\frac{d+2\gamma}{2}$-admissible exponent $(q,r)\neq P$. Moreover it is easy to see that the weak spaces $WL^\rho(I)$ are in L^1_{loc} for $\rho>1$. Thus as $\Theta^{\beta(r)+1}\in WL^{\frac{1}{\beta(r)+1}}$, we deduce that $\Theta^{\beta(r)+1}\in L^1(I)$ for $2\leq r<\frac{2(d+2\gamma)}{d+2\gamma-2}$. Hence, by Minkowski inequality for integrals,

$$\|k(t)\|_{L^q(I)} \leq \|\Theta\|_{L^{\beta(r)+1}(I)}\|\varphi\|_{L^q(I;L^{r'}_k(\mathbb{R}^d))}.$$

Integrating this inequality for $q=\infty$ over I yields

$$\|k(t)\|_{L^1(I)} \leq C\|\Theta\|_{L^{\beta(r)+1}(I)}\|\varphi\|_{L^\infty(I;L^{r'}_k(\mathbb{R}^d))}.$$

Interpolation this with the above L^q estimate for $q=2$ we get

$$\|k(t)\|_{L^q(I)} \leq C\|\Theta\|_{L^{\beta(r)+1}(I)}\|\varphi\|_{L^{q'}(I;L^{r'}_k(\mathbb{R}^d))} \quad \text{for} \quad 1\leq q\leq 2.$$

Thus, using Hölder's inequality and the following estimate

$$\left\|\int_I U^*(t)\varphi(t,\cdot)dt\right\|^2_{L^2_k(\mathbb{R}^d)} \leq \int_I k(t)\|\varphi(t,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)}dt$$

we deduce the result. $\square$

Next, we recall the result proved in [20].

Proposition 4 (Strichartz-type Schrödinger estimate)**.** *Suppose that $d\geq 1$ and (q,r) and (q_1,r_1) are $\frac{d+2\gamma}{2}$-admissible pairs. If u is a solution to the problem*

$$\begin{cases} i\partial_t u(t,x)+\Delta_k u(t,x) = f(t,x), \ (t,x)\in I\times\mathbb{R}^d \\ u_{|t=0} = u_0 \end{cases} \tag{3.7}$$

for some data, u_0, f and an interval I of $\mathbb{R}$ (bounded or not), then

$$\|u\|_{L^q(I;L^r_k(\mathbb{R}^d))}+\|u\|_{C(\overline{I};L^2_k(\mathbb{R}^d))} \leq C\left(\|u_0\|_{L^2_k(\mathbb{R}^d)}+\|f\|_{L^{q_1'}(I;L^{r_1'}_k(\mathbb{R}^d))}\right). \tag{3.8}$$

Remark 1. We can prove this Proposition otherwise for admissible pairs different from of extremal point P. Indeed we apply Proposition 3 i) with $U(t) := e^{it\Delta_k}$ the Dunkl-Schrödinger semigroup, which verifies the assumptions of Proposition 3.

The last of this subsection is motivated by a different kind of uncertainty principles written via the Dunkl-Schrödinger semigroup. Indeed, the following identity proved in [20]

$$u(t,x) = e^{it\Delta_k}u_0(x) = \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}} e^{-i(d+2\gamma)\frac{\pi}{4}\operatorname{sgn} t} e^{i\frac{\|\cdot\|^2}{4t}} \left[\mathcal{F}_D(e^{i\frac{\|\cdot\|^2}{4t}}u_0)\right]\left(\frac{x}{2t}\right), \tag{3.9}$$

tells us that this kind of results for the free solution of the Dunkl-Schrödinger equation with data u_0

$$\begin{cases} i\partial_t u(t,x) + \Delta_k u(t,x) &= 0, \ (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u_{|t=0} &= u_0, \end{cases} \tag{3.10}$$

is related to uncertainty principles. In this regards we use some uncertainty principles for the Dunkl transform proved in [5, 12, 26] for obtain the following.

Proposition 5. (i) *Let u_0 a mesurable function on $\mathbb{R}^d$ and $a, b > 0$ such that*

$$u_0(x) = O(e^{-a\|x\|^2}), \qquad e^{it\Delta_k}u_0(x) = O(e^{-b\|x\|^2}).$$

If $ab > \dfrac{1}{16t^2}$, then $u_0 \equiv 0$. Moreover, if $ab = \dfrac{1}{16t^2}$, then u is solution with initial data, $Ce^{-\left(a+\frac{i}{4t}\right)\|x\|^2}$.

(ii) *Let u_0 a mesurable function on $\mathbb{R}^d$ and $a, b > 0$ such that*

$$e^{a\|x\|^2}u_0(x) \in L^p_k(\mathbb{R}^d), \qquad e^{b\|x\|^2}e^{it\Delta_k}u_0(x) \in L^q_k(\mathbb{R}^d)$$

with $p, q \in [1,\infty]$, with at least one of them finite. If $ab \geq \dfrac{1}{16t^2}$, then $u_0 \equiv 0$.

(iii) *If $u_0 \in L^2_k(\mathbb{R}^d)$, $p \in (1,2)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $a, b > 0$ such that for some $t \neq 0$*

$$\int_{\mathbb{R}^d} |u_0(x)| e^{\frac{(2a)^p}{p}\|x\|^p}\omega_k(x)dx + \int_{\mathbb{R}^d} |e^{it\Delta_k}u_0(x)| e^{\frac{(2b)^q}{(2t)^q q}\|x\|^q}\omega_k(x)dx < +\infty.$$

If $ab \geq \dfrac{1}{4}$, then $u_0 \equiv 0$.

(iv) *If* $u_0 \in L^2_k(\mathbb{R}^d)$ *such that for some* $t \neq 0$

$$\int_{\mathbb{R}^{2d}} |u_0(x)||e^{it\Delta_k}u_0(y)|e^{\frac{\|x\|\|y\|}{2t}}\omega_k(x)dxdy < \infty,$$

then $u_0 \equiv 0$.

(v) *Let* u_0 *a mesurable function on* $\mathbb{R}^d$ *such that*

$$e^{a\|x\|^2}u_0 \in L^1_k(\mathbb{R}^d) + L^\infty_k(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} \log^+ \frac{|e^{it\Delta_k}u_0(\xi)e^{b\|\xi\|^2}|}{\lambda}d\xi < \infty,$$

for some constants $a > 0$, $b > 0$, $\lambda > 0$.

If $ab > \frac{1}{16t^2}$, *then* $u_0 = 0$ *almost everywhere.*

If $ab = \frac{1}{16t^2}$, *then* u *is solution with initial data,* $Ce^{-(a+\frac{i}{4t})\|x\|^2}$.

Proof. We only prove the estimate (i), the proofs of (ii)–(v) being similar.

Set $h(y) = e^{i\frac{\|y\|^2}{4t}}u_0(y)$. Then from (3.9) we get

$$u(t,x) = \frac{1}{c_k|t|^{\gamma+\frac{d}{2}}}e^{-i(d+2\gamma)\frac{\pi}{4}\operatorname{sgn} t}e^{i\frac{\|\cdot\|^2}{4t}}\left[\mathcal{F}_D(h)\right]\left(\frac{x}{2t}\right).$$

From the hypothesis on $u(t,x)$, we have

$$|\mathcal{F}_D(h)|\left(\frac{x}{2t}\right) \le Ce^{-b\|x\|^2}.$$

Thus

$$|\mathcal{F}_D(h)|(x) \le Ce^{-4bt^2\|x\|^2}.$$

Clearly $|h(y)| \le Ce^{-a\|y\|^2}$. Now we apply Hardy's uncertainty principle for the Dunkl transform (cf. [12]) for h, we obtain the result. □

3.2. Generalized Hermite semigroup. In this subsection we recall some facts about generalized Hermite semigroup related to the Dunkl operators. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [24].

In the setting of general Dunkl's theory Rösler [24] constructed systems of naturally associated multivariables generalized Hermite polynomials and Hermite functions. The system of generalized Hermite polynomials $\{H^k_\mu,\ \mu \in \mathbb{N}^d\}$ is orthogonal and complete in $L^2(\mathbb{R}^d, e^{-\|\cdot\|^2}\omega_k)$, while the system $\{h^k_\mu,\ \mu \in \mathbb{N}^d\}$ of generalized Hermite functions

$$h^k_\mu(x) = \frac{1}{\sqrt{2^{|\mu|}\mu!c_k}}e^{-\frac{\|x\|^2}{2}}H^k_\mu(x), \quad x \in \mathbb{R}^d,\ \mu \in \mathbb{N}^d,$$

is an orthonormal basis in $L_k^2(\mathbb{R}^d)$. Moreover, h_μ^k are eigenfunctions of the generalized harmonic oscillator:

$$\mathcal{H}_k := -\Delta_k + \|x\|^2. \tag{3.11}$$

More specifically, one has

$$\mathcal{H}_k h_\mu^k = (2|\mu| + d + 2\gamma) h_\mu^k,$$

where $|\mu| = \mu_1 + \cdots + \mu_d$. We not that the operator $\mathcal{H}_k$ is positive and symmetric in $L_k^2(\mathbb{R}^d)$ on the domain $D(\mathbb{R}^d)$.

Thus every $f \in L_k^2(\mathbb{R}^d)$ has the expansion

$$f = \sum_\mu \langle f, h_\mu^k \rangle_{L_k^2(\mathbb{R}^d)} h_\mu^k, \tag{3.12}$$

where $\langle \cdot, \cdot \rangle_{L_k^2(\mathbb{R}^d)}$ denote the canonical inner product in $L_k^2(\mathbb{R}^d)$.

The above expansion may be written as

$$f = \sum_n P_n^k f, \tag{3.13}$$

where

$$P_n^k f = \sum_{|\mu|=n} \langle f, h_\mu^k \rangle_{L_k^2(\mathbb{R}^d)} h_\mu^k, \tag{3.14}$$

is the spectral projection corresponding to the eigenvalue $2n + d + 2\gamma$.

By orthonormality of the generalized Hermite functions

$$\|f\|_{L_k^2(\mathbb{R}^d)}^2 = \sum_n \|P_n^k f\|_{L_k^2(\mathbb{R}^d)}^2. \tag{3.15}$$

The semigroup $\left\{e^{-t\mathcal{H}_k}, t > 0\right\}$ generated by $\mathcal{H}_k$ (cf. [24]), can be defined also with a complex parameter z instead of t, for $\Re z > 0$. We remark that the semigroup complex $e^{-z\mathcal{H}_k}$ have the following integral representation:

$$e^{-z\mathcal{H}_k} f(x) = \int_{\mathbb{R}^d} H_z^k(x, y) f(y) \omega_k(y) dy, \quad z = r + it, \quad \Re z > 0. \tag{3.16}$$

with

$$H_z^k(x, y) = \sum_{\mu \in \mathbb{N}_0^d} e^{-(2|\mu|+d+2\gamma)z} h_\mu^k(x) h_\mu^k(y). \tag{3.17}$$

The sum is the weil-known Mehler kernel, which can be found for instance in [24].

For $\Re z > 0$ one has

$$(3.18)\quad H_z^k(x,y)=\frac{1}{(2\sinh 2z)^{\frac{d+2\gamma}{2}}}\exp\left(\frac{1}{2}(-\coth 2z(\|x\|^2+\|y\|^2)\right)K\left(\frac{x}{\sinh 2z},y\right),$$

where K designate the Dunkl kernel given by the relation (2.7). This expression is well defined also for z on the imaginary axis, except a multiples of $i\frac{\pi}{2}$. Indeed, for $t\in\mathbb{R}\backslash\frac{\pi}{2}\mathbb{Z}$ we get

$$(3.19)\qquad H_{it}^k(x,y)=\frac{e^{-i\pi\frac{d+2\gamma}{4}}}{(2\sin 2t)^{\frac{d+2\gamma}{2}}}\exp\left(\frac{i}{2}(\cot 2t(\|x\|^2+\|y\|^2))\right)K\left(\frac{-x}{\sin 2t},y\right).$$

From (3.17) it follows that $H_{\bar{z}}^k(x,y)=\overline{H_z^k(x,y)}$, and also that

$$H^k_{z+i\frac{\pi}{2}}(x,y)=e^{-i\pi\frac{d+2\gamma}{2}}H_z^k(-x,y).$$

Here $\Re z>0$, but if $t\in\mathbb{R}$ is not a multiple of $\frac{\pi}{2}$ also we conclude that

$$H^k_{-it}(x,y)=\overline{H^k_{it}}(x,y)\quad\text{ and }\quad H^k_{i(t+\frac{\pi}{2})}(x,y)=e^{-i\pi(\frac{d+2\gamma}{2})}H^k_{-it}(x,y).$$

For initial data u_0 belongs to the space of finite linear combinations of generalized Hermite functions, the solution of the following problem

$$(3.20)\qquad \begin{cases} i\dfrac{\partial u}{\partial t} &= \mathcal{H}_k u\\ u(\cdot,0) &= u_0\end{cases}$$

can be written

$$(3.21)\qquad e^{-it\mathcal{H}_k}u_0=\sum_{n=0}^{\infty}e^{-it(2n+d+2\gamma)}P_n^k(u_0).$$

Comparing (3.15) with (3.21) we see that

$$(3.22)\qquad \|e^{-it\mathcal{H}_k}u_0\|_{L_k^2(\mathbb{R}^d)}=\|u_0\|_{L_k^2(\mathbb{R}^d)},\quad t\in\mathbb{R}.$$

The solution given by (3.20) can also be formally expressed as an integral operator with kernel (3.19), which converges in the distribution sense. For real functions u_0, it follows that the $L_k^p(\mathbb{R}^d)$ norm of $e^{-it\mathcal{H}_k}u_0$ is even and $\frac{\pi}{2}$-periodic as a function of t, and thus determined by its values for $0<t<\frac{\pi}{4}$. The following kernel estimate is crucial in our analysis.

Proposition 6. *Let $H_z^k(x,y)$ be the kernel given by the Eq.* (3.17). *Then for $\Re z > 0$, H_z^k satisfies the uniform estimate*

$$|H_z^k(x,y)| \leq \frac{e^{-(2\gamma+d)r}}{|\sin(2t)|^{\frac{d+2\gamma}{2}}}, \quad z = r + it,\ 0 < |t| \leq \pi,\ x, y \in \mathbb{R}^d. \tag{3.23}$$

Proof. In view of (3.17), we can write (3.18) in the form

$$H_z^k(x,y) = \left(\frac{v}{1-v^2}\right)^{\frac{d+2\gamma}{2}} \exp\left[\frac{-(1+v^2)}{2(1-v^2)}(\|x\|^2+\|y\|^2)\right] K\left(\frac{2v}{1-v^2}x, y\right), \tag{3.24}$$

where $v = e^{-2(r+it)}$. As

$$1 - v^2 = 2e^{-2(r+it)} \sinh[2(r+it)],$$

and

$$|\sinh[2(r+it)]| = |\cos(2t)\sinh(2r) + i\sin(2t)\cosh(2r)| \geq |\sin(2t)\cosh(2r)|.$$

It follows from (3.24) and (2.8) that

$$|H_z^k(x,y)| \leq \frac{e^{-(2\gamma+d)r}}{|\sin(2t)|^{\frac{d+2\gamma}{2}}} \exp\left[\Re\left(\frac{-(1+v^2)}{2(1-v^2)}(\|x\|^2+\|y\|^2) + \frac{2v}{1-v^2}\|x\|\,\|y\|\right)\right].$$

Moreover by a simple calculations we see that

$$\left[\Re\left(\frac{-(1+v^2)}{2(1-v^2)}(\|x\|^2+\|y\|^2) + \frac{2v}{1-v^2}\|x\|\,\|y\|\right)\right] \leq 0.$$

The prove is immediately. $\square$

The following proposition is a simple application of the estimate (3.23) and the Riesz-Thorin interpolation theorem.

Proposition 7. *Let $p \in [2,\infty]$, $z = r + it$ such that $r > 0$ and $t \neq 0$. Then the operator $e^{-z\mathcal{H}_k}$ maps $L_k^{p'}(\mathbb{R}^d)$ continuously to $L_k^p(\mathbb{R}^d)$ and*

$$\|e^{-z\mathcal{H}_k} f\|_{L_k^p(\mathbb{R}^d)} \leq \left(\frac{e^{-2r}}{|\sin 2t|}\right)^{(d+2\gamma)(\frac{1}{p'}-\frac{1}{2})} \|f\|_{L_k^{p'}(\mathbb{R}^d)} \quad \textit{for all} \quad f \in L_k^{p'}(\mathbb{R}^d).$$

Proof. It follows from Proposition 6 and the fact that $e^{-z\mathcal{H}_k}$ is a bounded on $L_k^2(\mathbb{R}^d)$

$$\|e^{-z\mathcal{H}_k} f\|_{L_k^\infty(\mathbb{R}^d)} \leq \frac{e^{-(2\gamma+d)r}}{|\sin 2t|^{\frac{d+2\gamma}{2}}} \|f\|_{L_k^1(\mathbb{R}^d)}, \quad \|e^{-z\mathcal{H}_k} f\|_{L_k^2(\mathbb{R}^d)} \leq \|f\|_{L_k^2(\mathbb{R}^d)}.$$

The general case is obtained by interpolation between the case $p = 2$ and $p = \infty$. $\square$

We shall compare the operators $e^{-it\mathcal{H}_k}$ and $e^{it\Delta_k}$ by finding a link between their kernels. The kernel of $e^{it\Delta_k}$ is the Dunkl-Schrödinger kernel (cf. [20])

$$\mathfrak{I}_k(t)(x,y) = e^{-i\pi\frac{d+2\gamma}{4}\operatorname{sgn} t}\frac{1}{c_k|t|^{\frac{d+2\gamma}{2}}}\exp\left[i\,\frac{1}{4t}(\|y\|^2+|x\|^2)\right]K\left(\frac{-iy}{2t},x\right).$$

The following Proposition gives an explicit relation between the two solution operators $e^{-it\mathcal{H}_k}$ and $e^{it\Delta_k}$.

Proposition 8. *For any $f \in L^2_k(\mathbb{R}^d)$ and $v > 0$,*

$$e^{-i\,\frac{\arctan v}{2}\mathcal{H}_k}f(x) = \exp\left(-iv\frac{\|x\|^2}{2}\right)(1+v^2)^{\frac{d+2\gamma}{4}}e^{i\,\frac{v}{2}\Delta_k}f(x\sqrt{1+v^2}).$$

Proof. For $0 < t < \dfrac{\pi}{4}$, we let $\tan 2t = v$ in (3.19) and get

$$\begin{aligned} H^k_{i\frac{\arctan v}{2}}(x,y) &= \frac{e^{-i\pi\frac{d+2\gamma}{4}}}{c_k}\left(\frac{\sqrt{1+v^2}}{v}\right)^{\frac{d+2\gamma}{2}}\exp\left(-i\frac{v}{2}\|x\|^2\right)\times \\ &\quad\times\exp\left(\frac{i}{2v}(\|y\|^2+(1+v^2)\|x\|^2)\right)K\left(\frac{-iy}{v},(1+v^2)x\right) \\ &= \exp\left(-iv\frac{\|x\|^2}{2}\right)(1+v^2)^{\frac{d+2\gamma}{4}}\,\mathfrak{I}_k\left(\frac{v}{2}\right)(x\sqrt{1+v^2},y). \end{aligned}$$

Integration against $f(y)\omega_k(y)dy$, we obtain the desired equation when $f \in D(\mathbb{R}^d)$. The general case then follows by continuity in $L^2_k(\mathbb{R}^d)$. $\square$

As in the free Dunkl-Schrödinger equation (3.7), we want to prove the Strichartz estimate.

$$\|e^{-it\mathcal{H}_k}f\|_{L^q((0,2\pi),L^r_k(\mathbb{R}^d))} \leq C(d,q,r)\|f\|_{L^2(\mathbb{R}^d)} \tag{3.25}$$

with (q,r) is $\dfrac{d+2\gamma}{2}$-admissible pairs. It makes it easy to prove the following result, which implies that the estimates (3.8) for the free Dunkl-Schrödinger equation and (3.25) are actually equivalent.

Proposition 9. *Let $1 \leq q, r \leq \infty$ and assume that $\dfrac{d+2\gamma}{r}+\dfrac{2}{q}=\dfrac{d+2\gamma}{2}$. Then for $f \in L^2_k(\mathbb{R}^d)$*

$$\|e^{-it\mathcal{H}_k}f\|_{L^q((0,\frac{\pi}{4}),L^r_k(\mathbb{R}^d))} = \|e^{it\Delta_k}f\|_{L^q((0,\infty),L^r_k(\mathbb{R}^d))}.$$

P r o o f. Assuming $q, r < \infty$, we get

$$\int_0^{\frac{\pi}{4}} \left(\int_{\mathbb{R}^d} |e^{-it\mathcal{H}_k} f(x)|^r \omega_k(x) dx \right)^{\frac{q}{r}} dt =$$

$$= \int_0^{\infty} \left(\int_{\mathbb{R}^d} e^{-i\ \frac{arctan v}{2} \mathcal{H}_k} f(x)|^r \omega_k(x) dx \right)^{\frac{q}{r}} \frac{dv}{2(1+v^2)}$$

$$= \int_0^{\infty} \left(\int_{\mathbb{R}^d} \left| (1+v^2)^{\frac{d+2\gamma}{4}} e^{i\ \frac{v}{2}\Delta_k} f(x\sqrt{1+v^2}) \right|^r \omega_k(x) dx \right)^{\frac{q}{r}} \times \frac{dv}{2(1+v^2)}.$$

Using the relation $\dfrac{d+2\gamma}{r} + \dfrac{2}{q} = \dfrac{d+2\gamma}{2}$, and by a simple changement of variable we obtain

$$\int_0^{\frac{\pi}{4}} \left(\int_{\mathbb{R}^d} |e^{-it\mathcal{H}_k} f(x)|^r \omega_k(x) dx \right)^{\frac{q}{r}} dt = \int_0^{\infty} \left(\int_{\mathbb{R}^d} |e^{i\ \frac{v}{2}\Delta_k} f(x)|^r \omega_k(x) dx \right)^{\frac{q}{r}} dv.$$

The cases when q or r is infinite are similar. $\square$

Now we state the our result of this section.

Theorem 1. *Let $u_0 \in L^2_k(\mathbb{R}^d)$ and let $u(t,x) = e^{-it\mathcal{H}_k} u_0(x)$ be the solution of the problem* (3.20). *Then u is periodic in t and $u \in L^q([0,2\pi]; L^r_k(\mathbb{R}^d))$ for any sharp $\dfrac{d+2\gamma}{2}$-admissible exponent $(q,r) \neq P$, and for all pairs (q,r) $1 \le q \le 2$, $2 \le r < \dfrac{2d+4\gamma}{d+2\gamma-2}$, $d \ge 2$. Further u satisfies the inequality*

$$\|u\|_{L^q([0,2\pi];L^r_k(\mathbb{R}^d))} \le C(d,k) \|u_0\|_{L^2_k(\mathbb{R}^d)} \tag{3.26}$$

for all $u_0 \in L^2_k(\mathbb{R}^d)$ and for the above ranges of q and r.

P r o o f. First we show that there is a subsequence $z_j = x_j + it$ such that $e^{-z_j\mathcal{H}_k} u_0$ converges to $e^{-it\mathcal{H}_k} u_0$ in $L^q([0,2\pi]; L^r_k(\mathbb{R}^d))$ as $x_j \to 0$. Notice that since $u_0 \in L^2_k(\mathbb{R}^d)$, for each fixed $t > 0$ and $x_j > 0$, we have

$$\|e^{-z_j\mathcal{H}_k} u_0 - e^{-it\mathcal{H}_k} u_0\|^2_{L^2_k(\mathbb{R}^d)} = \left\| \sum_{n=0}^{\infty} |e^{-x_j(2n+d+2\gamma)} - 1| e^{it(2n+d+2\gamma)} P^k_n u_0 \right\|^2_{L^2_k(\mathbb{R}^d)}$$

$$= \sum_{n=0}^{\infty} |e^{-x_j(2n+d+2\gamma)} - 1|^2 \|P^k_n u_0\|^2_{L^2_k(\mathbb{R}^d)} \tag{3.27}$$

which converges to zero as $x_j \to 0$, by a dominated convergence argument applied to the summation. Thus $e^{-z_j\mathcal{H}_k} u_0$ converges to $e^{-it\mathcal{H}_k} u_0$ in $L^2_k(\mathbb{R}^d)$ for each fixed

$t > 0$ as $x_j \to 0$. By periodicity of $e^{-it\mathcal{H}_k}u_0$ in t, we need to consider only $0 < t \leq 2\pi$. Since $\|e^{-z_j\mathcal{H}_k}u_0 - e^{-it\mathcal{H}_k}u_0\|^2_{L^2_k(\mathbb{R}^d)} \to 0$ as $x_j \to 0$, and since

$$\|e^{-z_j\mathcal{H}_k}u_0 - e^{-it\mathcal{H}_k}u_0\|^2_{L^2_k(\mathbb{R}^d)} \leq (2\|e^{-it\mathcal{H}_k}u_0\|^2_{L^2_k(\mathbb{R}^d)})^2 = 4\|u_0\|^2_{L^2_k(\mathbb{R}^d)}$$

by dominated convergence theorem, it follows that

$$\lim_{x_j\to 0}\int_0^{2\pi}\|e^{-z_j\mathcal{H}_k}u_0 - e^{-it\mathcal{H}_k}u_0\|^2_{L^2_k(\mathbb{R}^d)}dt \to 0.$$

Thus $e^{-z_j\mathcal{H}_k}u_0$ converges to $e^{-it\mathcal{H}_k}u_0$ in $L^2([0,2\pi], L^2_k(\mathbb{R}^d))$. Hence we can extract a subsequence of this sequence, denoted also by $e^{-z_j\mathcal{H}_k}u_0$ that converges to $e^{-it\mathcal{H}_k}u_0$ for a.e. $(t,x) \in [0,2\pi] \times \mathbb{R}^d$ as $x_j \to 0$.

Now we want to prove (3.26). Indeed by Fatou's lemma

$$\begin{aligned}\|e^{-it\mathcal{H}_k}u_0\|^r_{L^r_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}|e^{-it\mathcal{H}_k}u_0(x)|^r\omega_k(x)dx \\ &\leq \liminf_{x_j\to 0}\int_{\mathbb{R}^d}|e^{-z_j\mathcal{H}_k}u_0(x)|^r\omega_k(x)dx.\end{aligned}$$

Applying Fatou's lemma once again, we get

$$\begin{aligned}\int_0^{2\pi}\|u(t,.)\|^q_{L^r_k(\mathbb{R}^d)}dt &\leq \int_0^{2\pi}\left(\liminf_{x_j\to 0}\int_{\mathbb{R}^d}|e^{-z_j\mathcal{H}_k}u_0(x)|^r\omega_k(x)dx\right)^{\frac{q}{r}}dt \\ &\leq \liminf_{x_j\to 0}\int_0^{2\pi}\left(\int_{\mathbb{R}^d}|e^{-z_j\mathcal{H}_k}u_0(x)|^r\omega_k(x)dx\right)^{\frac{q}{r}}dt.\end{aligned}$$

Moreover, from the relation (3.23), the generalized Hermite-Schrödinger semigroup $e^{-z_j\mathcal{H}_k}$ verify

$$\|e^{-z_j\mathcal{H}_k}\|_{\mathcal{L}(L^1_k(\mathbb{R}^d),L^\infty_k(\mathbb{R}^d))} \leq C|t|^{-\frac{d+2\gamma}{2}} \quad \text{for} \quad |t| \leq \delta$$

and

$$\|e^{-z_j\mathcal{H}_k}\|_{\mathcal{L}(L^1_k(\mathbb{R}^d),L^\infty_k(\mathbb{R}^d))} \leq \frac{C}{|\sin(2t)|^{\frac{2\gamma+d}{2}}} \quad \text{for} \quad |t| > \delta,$$

for some $\delta > 0$. Therefore (3.3) is satisfied for $U(t) := e^{-z_j\mathcal{H}_k}$ with

$$\Theta(t) = C\left(\frac{1}{|t|}\mathbf{1}_{|t|\leq\delta} + \frac{1}{|\sin(2t)|}\mathbf{1}_{|t|>\delta}\right).$$

By a simple calculations we see that Θ belongs to $WL^1([0,2\pi])$. Now we apply

Proposition 3 ii) we obtain

$$\int_0^{2\pi} \left(\int_{\mathbb{R}^d} |e^{-z_j \mathcal{H}_k} u_0(x)|^r \omega_k(x) dx \right)^{\frac{q}{r}} dt \le (C(d,k) \|u_0\|_{L^2_k(\mathbb{R}^d)})^q$$

where the last inequality follows from (3.4) since the constant $C(d,k)$ is independent of x_j. Taking the qth root on both sides, we get the required inequality. □

Remark 2. In the classical case, a similar result can be found in [11, 15], where the authors used another methods that we can not adapt at the moment. Our method use some ideas inspired by the works [16, 23].

Now let us consider the inhomogeneous problem:

$$\begin{cases} i\dfrac{\partial u}{\partial t} - \mathcal{H}_k u = f(t,x), & (t,x) \in \mathbb{R}^{d+1}, \\ u(0,\cdot) \qquad = u_0. & \end{cases} \tag{3.28}$$

In this case, the solution is given by Duhamel's formula:

$$u(t,x) = e^{-it\mathcal{H}_k} u_0(x) - i \int_0^t e^{-i(t-s)\mathcal{H}_k} f(s,x) ds. \tag{3.29}$$

In this case the solution need not be periodic in the t variable, unless u_0 is periodic in the t variable. For the inhomogeneous equation in the periodic case we prove the following:

Theorem 2. *Let $u_0 \in L^2_k(\mathbb{R}^d)$ and $f \in L^{q'}([0,2\pi]; L^{p'}_k(\mathbb{R}^d))$ then the solution u to the problem (3.28) lies in $L^q([0,2\pi]; L^q_k(\mathbb{R}^d))$, for any sharp $\dfrac{d+2\gamma}{2}$-admissible exponent $(q,r) \neq P$, and for all pairs (q,r) $1 \le q \le 2$, $2 \le r < \dfrac{2d+4\gamma}{d+2\gamma-2}$, $d \ge 2$. Further u satisfies the inequality*

$$\|u\|_{L^q([0,2\pi];L^r_k(\mathbb{R}^d))} \le C(d,k) \left(\|u_0\|_{L^2_k(\mathbb{R}^d)} + \|f\|_{L^{q'}([0,2\pi];L^{r'}_k(\mathbb{R}^d))} \right), \tag{3.30}$$

for the above ranges of q and r.

Proof. By Theorem 1, we have

$$\|e^{-it\mathcal{H}_k} u_0(x)\|_{L^q([0,2\pi];L^p_k(\mathbb{R}^d))} \le C \|u_0\|_{L^2_k(\mathbb{R}^d)}.$$

Therefore by (3.29), it is enough to show that

$$\left\| \int_0^t e^{-i(t-s)\mathcal{H}_k} f(s,\cdot) ds \right\|_{L^q([0,2\pi];L^r_k(\mathbb{R}^d))} \le C(d,k) \|f\|_{L^{q'}([0,2\pi];L^{r'}_k(\mathbb{R}^d))}. \tag{3.31}$$

In view of Proposition 7 we have

$$\left\|\int_0^t e^{-i(t-s)\mathcal{H}_k} f(s,\cdot)\right\|_{L^r_k(\mathbb{R}^d)} \leq \int_0^t \left\|e^{-i(t-s)\mathcal{H}_k} f(s,\cdot)\right\|_{L^r_k(\mathbb{R}^d)} ds$$

$$(3.32)\qquad \leq C(d,k)\int_0^{2\pi} \frac{\|f(s,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)}}{|\sin 2(t-s)|^{(d+2\gamma)(\frac{1}{r'}-\frac{1}{2})}} ds.$$

Since $f \in L^{q'}([0,2\pi]; L^{r'}_k(\mathbb{R}^d))$, we have $\|f(s,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)} \in L^{q'}([0,2\pi])$ as a function of s. Now by an application of the Young's inequality as in Proposition 3, we see that

$$\int_0^{2\pi} \frac{\|f(s,\cdot)\|_{L^{r'}_k(\mathbb{R}^d)}}{|\sin 2(t-s)|^{(d+2\gamma)(\frac{1}{r'}-\frac{1}{2})}} ds \leq C\|f\|_{L^{q'}([0,2\pi];L^{r'}_k(\mathbb{R}^d))}.$$

This completes the proof of the theorem. □

We consider the Cauchy problem for the Dunkl-Schrödinger equation with the forced generalized harmonic oscillator:

$$(3.33)\qquad \begin{cases} i\dfrac{du}{dt} - \mathcal{H}_k u = V(t,x)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u(0,\cdot) = u_0 \in L^2(\mathbb{R}^d). \end{cases}$$

Theorem 3. *Let $d\geq 2$ and $\frac{1}{p}+\frac{2}{r}=1$, and assume that $\|V\|_{L^\infty([0,\infty);L^p_k(\mathbb{R}^d))}$ is sufficiently small where $p \in \left[\frac{d+2\gamma}{2},\infty\right]$. Then there exists a unique global solution of* (3.33) *belonging to $C([0,\infty);L^2_k(\mathbb{R}^d)) \cap L^2_{loc}([0,\infty);L^r_k(\mathbb{R}^d))$.*

Proof. We use the standard contraction mapping argument. By Duhamel's formula

$$u(t,x) = e^{-it\mathcal{H}_k}u_0 - i\int_0^t e^{-i(t-s)\mathcal{H}_k} V(s,\cdot)u(s,\cdot)(x)ds.$$

For $2 \leq r < \frac{2d+4\gamma}{d+2\gamma-2}$, by Theorem 1, there exists a positive constant C such that

$$(3.34)\qquad \|e^{-it\mathcal{H}_k}f\|_{L^2([0,2\pi];L^r_k(\mathbb{R}^d))} \leq C\|f\|_{L^2_k(\mathbb{R}^d)},$$

and, by duality, this yields

$$(3.35)\qquad \left\|\int_0^t e^{is\mathcal{H}_k}G(s,\cdot)ds\right\|_{L^2_k(\mathbb{R}^d)} \leq C\|G\|_{L^2([0,2\pi];L^{r'}_k(\mathbb{R}^d))}, \quad t \in [0,2\pi].$$

Now, by various applications of Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^t e^{-i(t-s)\mathcal{H}_k}VF(s,\cdot)(x)dsG(t,x)\omega_k(x)dxdt &=\\
&\int_0^{2\pi}\int_0^t\int_{\mathbb{R}^d} e^{-i(t-s)\mathcal{H}_k}VF(s,\cdot)(x)G(t,x)\omega_k(x)dxdsdt &=\\
&\int_0^{2\pi}\int_0^t\int_{\mathbb{R}^d} e^{is\mathcal{H}_k}VF(s,\cdot)e^{-it\mathcal{H}_k}G(t,x)\omega_k(x)dxdsdt &=\\
&\int_{\mathbb{R}^d}\int_0^t e^{is\mathcal{H}_k}VF(s,\cdot)(x)ds\int_0^{2\pi} e^{-it\mathcal{H}_k}G(t,x)dt\omega_k(x)dx,
\end{aligned}$$

where the second equality follows using the orthogonality of the generalized Hermite functions. Thus, by the Cauchy-Schwarz inequality followed by two applications of (3.35) and duality,

$$\left\|\int_0^t e^{-i(t-s)\mathcal{H}_k}V(s,\cdot)F(s,\cdot)(x)ds\right\|_{L^2([0,2\pi];L^r_k(\mathbb{R}^d))} \leq C^2\|VF\|_{L^2([0,2\pi],L^{r'}_k(\mathbb{R}^d))}. \tag{3.36}$$

We define the Banach space $X_k = C([0,2\pi];L^2_k(\mathbb{R}^d))\cap L^2([0,2\pi];L^r_k(\mathbb{R}^d))$ via the norm

$$\|u\|_{X_k} = \sup_{t\in[0,2\pi]}\|u(t,\cdot)\|_{L^2_k(\mathbb{R}^d)} + \|u\|_{L^2([0,2\pi];L^r_k(\mathbb{R}^d))},$$

and the nonlinear map $\mathcal{L}_k : X_k \to X_k$ by

$$\mathcal{L}_kF := e^{-it\mathcal{H}_k}u_0 - i\int_0^t e^{-i(t-s)\mathcal{H}_k}V(s,\cdot)F(s,\cdot)(x)ds.$$

By (3.21) and the conservation of the L^2_k norm (3.22) we see that

$$\|e^{-it\mathcal{H}_k}u_0\|_{X_k} \leq (C+1)\|u_0\|_{L^2_k(\mathbb{R}^d)},$$

and combining (3.35) and (3.36), we also have

$$\left\|i\int_0^t e^{-i(t-s)\mathcal{H}_k}V(s,\cdot)F(s,\cdot)(x)ds\right\|_{X_k} \leq (C+C^2)\|V\|_{L^\infty([0,\infty);L^p_k(\mathbb{R}^d))}\|F\|_{X_k};$$

here we have used the fact that

$$\|VF\|_{L^2([0,2\pi];L^{r'}_k(\mathbb{R}^d))} \leq \|V\|_{L^\infty([0,2\pi];L^p_k(\mathbb{R}^d))}\|F\|_{L^2([0,2\pi];L^r_k(\mathbb{R}^d))}, \quad \frac{1}{p}+\frac{2}{r}=1.$$

Thus we see that $\mathcal{L}_k$ maps $\{f : \|f\|_{X_k} \leq 2(C+1)\|u_0\|_{L^2_k(\mathbb{R}^d)}\}$ into itself provided $(C+C^2)\|V\|_{L^\infty([0,2\pi];L^p_k(\mathbb{R}^d))} \leq \frac{1}{2}$. This also guarantees that

$$\|\mathcal{L}_k(f-g)\|_{X_k} \leq \frac{1}{2}\|f-g\|_{X_k}, \tag{3.37}$$

so that by the contraction mapping principle, there exists a solution. Now although the L^2_k-norm may have increased in size, we know that it is at least finite, so by iterating the process, replacing u_0 with $u(2n\pi,\cdot)$, $n \in \mathbb{N}$, we obtain a global solution.

To see that the solution is unique in $L^2_{loc}([0,\infty);L^r_k(\mathbb{R}^d))$, suppose that u_1 and u_2 are solutions. Then by (3.36) as before, we see that

$$\|u_1-u_2\|_{L^2([2n\pi,2(n+1)\pi];L^r_k(\mathbb{R}^d))} \leq \frac{1}{2}\|u_1-u_2\|_{L^2([2n\pi,2(n+1)\pi];L^r_k(\mathbb{R}^d))}$$

for all $n \geq 0$, so they are in fact the same. $\square$

4. Critical nonlinear Dunkl-Schrödinger equations with and without harmonic potential. In this section we collect some notations which we need in the sequel.

Notations. We denote by $H^s_k(\mathbb{R}^d)$ (Dunkl-Sobolev space) ($s \in \mathbb{R}$) Hilbert space of elements $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $(1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_D(u) \in L^2_k(\mathbb{R}^d)$. $H^s_k(\mathbb{R}^d)$ is equipped with the norm

$$\|u\|_{H^s_k(\mathbb{R}^d)} = \|(1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_D(u)\|_{L^2_k(\mathbb{R}^d)}.$$

The Hilbert space $\Sigma_k := H^1_{k,W}(\mathbb{R}^d)\bigcap \mathcal{F}_D(H^1_{k,W}(\mathbb{R}^d))$ equipped with the norm

$$\|u\|_{\Sigma_k} := \|u\|_{H^1_k(\mathbb{R}^d)} + \|xu\|_{L^2_k(\mathbb{R}^d)}, \tag{4.1}$$

where

$$H^1_{k,W}(\mathbb{R}^d) := \Big\{u \in H^1_k(\mathbb{R}^d) : \quad u(\sigma_\alpha) = u, \quad \text{for all } \alpha \in R_+.\Big\}$$

Let u be a solution of the nonlinear Dunkl-Schrödinger equation:

$$\begin{cases} i\partial_t u(t,x) + \Delta_k u(t,x) = \lambda u|u|^{\frac{4}{d+2\gamma}}, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad \lambda \in \mathbb{R} \\ u(0,\cdot) = u_0. \end{cases} \tag{4.2}$$

In [21], we have proved that if u_0 in Σ_k, then there exists T such that $u \in C(]-T,T[;\Sigma_k)$.

Fix $w > 0$ and definefor $|t| < \dfrac{\arctan(wT)}{w}$

$$v(t,x) = \frac{1}{(\cos(wt))^{\frac{d+2\gamma}{2}}}\, e^{-i\,\frac{w\tan(wt)}{4}\,\|x\|^2} u\left(\frac{\tan(wt)}{w}, \frac{x}{\cos(wt)}\right). \tag{4.3}$$

Then v solves the following

$$\begin{cases} i\partial_t v(t,x) + \Delta_k v(t,x) &= \dfrac{w^2\|x\|^2}{4} v(t,x) + \lambda v(t,x)|v(t,x)|^{\frac{4}{d+2\gamma}}, \\ & \qquad\qquad\qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v(0,\cdot) &= u_0 \end{cases} \tag{4.4}$$

Reciprocally, if v solves (4.4), then u, defined by

$$u(t,x) = \frac{1}{(1+w^2t^2)^{\frac{d+2\gamma}{4}}}\, e^{i\,\frac{w^2 t}{1+w^2t^2}\,\frac{\|x\|^2}{4}} v\left(\frac{\arctan(wt)}{w}, \frac{x}{\sqrt{1+w^2t^2}}\right) \tag{4.5}$$

solves (4.2). The transforms (4.3) and (4.5) do not alter the initial data u_0, and are isometric on $L^2_k(\mathbb{R}^d)$.

The aim of this section is the follow:

Proposition 10. *Let $u_0 \in \Sigma_k$. Then there exist $T_{\min}, T_{\max} > 0$ and there exists a unique, maximal solution*

$$u \in C(]-T_{\min}, T_{\max}[; \Sigma_k) \cap C^1(]-T_{\min}, T_{\max}[; H_k^{-1}(\mathbb{R}^d))$$

of problem (4.4). It is maximal in the sense that is $T_{\max} < \infty$, then $\|u(t,\cdot)\|_{H^1_k(\mathbb{R}^d)} \to \infty$ as $t \uparrow T_{\max}$, and if $T_{\min} < \infty$ then $\|u(t,\cdot)\|_{H^1_k(\mathbb{R}^d)} \to \infty$ as $t \downarrow -T_{\min}$. In addition,

(1) *Conservation of mass: $\|u(t,\cdot)\|_{L^2_k(\mathbb{R}^d)} = \|u_0\|_{L^2_k(\mathbb{R}^d)}$.*

(2) *Conservation of first part of the energy:*

$$E^1_k(u) = \left\|\frac{1}{2} wx\sin(wt)u(t,\cdot) - i\cos(wt)\nabla_k u(t,\cdot)\right\|^2_{L^2_k(\mathbb{R}^d)} + \frac{(d+2\gamma)\lambda}{d+2\gamma+2}\cos^2(wt)\|u(t,\cdot)\|^{\frac{2d+4\gamma+4}{d+2\gamma}}_{L_k^{\frac{2d+4\gamma+4}{d+2\gamma}}(\mathbb{R}^d)}.$$

(3) *Conservation of second part of the energy:*

$$E_k^2(u) = \left\| \frac{1}{2} wx\cos(wt)u(t,\cdot) + i\sin(wt)\nabla_k u(t,\cdot) \right\|^2_{L_k^2(\mathbb{R}^d)} + \frac{(d+2\gamma)\lambda}{d+2\gamma+2}\sin^2(wt)\|u(t,\cdot)\|^{\frac{2d+4\gamma+4}{d+2\gamma}}_{L_k^{\frac{2d+4\gamma+4}{d+2\gamma}}(\mathbb{R}^d)}.$$

Proof. The results follows from the relation between the problems (4.2) and (4.4) and the known results for the critical nonlinear Dunkl-Schrödinger equation proved in [21]. $\square$

5. The nonlinear Schrödinger equation with a quadratic potential. The main results of this Section is in sprit of the classical case (cf. [3, 4, 10, 13, 17]). Consider a real valued potential $U \in \mathcal{E}(\mathbb{R}^d)$ such that

$$U \geq 0 \quad \text{and} \quad D^\mu U \in L_k^\infty(\mathbb{R}^d)\ , \qquad \text{for all } \mu \in \mathbb{N}^d \text{ such that } |\mu| \geq 2.$$

The model case being $U(x) = \|x\|^2$. We define the operator A_k on $L_k^2(\mathbb{R}^d)$ by

$$\begin{cases} D(A_k) := \{u \in H_k^1(\mathbb{R}^d) :\ U|u|^2 \in L_k^1(\mathbb{R}^d) \quad \text{and} \quad \Delta_k u - Uu \in L_k^2(\mathbb{R}^d)\}, \\ A_k u := \Delta_k u - Uu, \quad \text{for } u \in D(A_k). \end{cases}$$

We consider the nonlinear Dunkl-Schrödinger equation

$$\begin{cases} iu_t + A_k u = Vu + f(u(\cdot),\cdot), \\ u(0,\cdot) = u_0; \end{cases} \tag{5.1}$$

where V and f are as follows:

$V : \mathbb{R}^d \to \mathbb{R}$ such that $V \in L_k^p(\mathbb{R}^d) + L_k^\infty(\mathbb{R}^d)$ for some $p \geq 1$, $p > \dfrac{d+2\gamma}{2}$.

$f : \mathbb{C} \times \mathbb{R}^d \to \mathbb{C}$ be a is measurable in x and continuous in $z \in \mathbb{C}$, and that $f(0,x) = 0$, almost everywhere on $\mathbb{R}^d$. Assume that there exist constants C and $\mu \in \left[0, \dfrac{4}{d+2\gamma-2}\right)$ such that

$$|f(z_1,x) - f(z_2,x)| \leq C(1+|z_1|+|z_2|)^\mu |z_1 - z_2|, \quad \text{for almost all } x \in \mathbb{R}^d \text{ and all } z_1, z_2 \in \mathbb{C}.$$

Set

$$F(z,x) = \int_0^{|z|} f(s,x)ds, \ \text{for all } z \in \mathbb{C} \text{ and almost all } x \in \mathbb{R}^d.$$

Finally, set

$$g(u) = Vu + f(u(\cdot), \cdot)$$

$$G(u) = \int_{\mathbb{R}^d} \left[\frac{1}{2}V(x)|u(x)|^2 + F(u(x), x)\right] \omega_k(x)dx,$$

and

$$E_k(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_k u(x)|^2 + U(x)|u(x)|^2\omega_k(x)dx - G(u).$$

Concerning the existence of solutions of (5.1) for initial data in $L^2_k(\mathbb{R}^d)$, we proceed as in the Dunkl-Schrödinger equation without potential (cf. [21]), we prove the following result:

Proposition 11. *Let g be as above, and assume further that $\mu < \frac{4}{d+2\gamma}$. Let $r = \max\left\{\mu+2, \frac{2p}{p-1}, \frac{2q}{q-1}\right\}$ and let (q,r) be the corresponding sharp $\frac{d+2\gamma}{2}$-admissible pair. Then, for every $u_0 \in L^2_k(\mathbb{R}^d)$, there exists a unique function $u \in C(\mathbb{R}, L^2_k(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^r_k(\mathbb{R}^d))$ with $u_t \in L^q_{loc}(\mathbb{R}, (D(A_k))')$, solution of (5.1). In addition, we have $\|u(t)\|_{L^2_k(\mathbb{R}^d)} = \|u_0\|_{L^2_k(\mathbb{R}^d)}$, for all $t \in \mathbb{R}$, and $u \in L^a_{loc}(\mathbb{R}, L^b_k(\mathbb{R}^d))$, for every sharp $\frac{d+2\gamma}{2}$-admissible pair (a,b).*

On the follow we assume that $W = \mathbb{Z}_2^d$.

Now, we consider the nonlinear Dunkl-Schrödinger equation on $\mathbb{R}^d$:

$$\text{(5.2)} \qquad \begin{cases} i\partial_t u(t,x) + \Delta_k u(t,x) &= V(x)u + \lambda u|u|^p, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,\cdot) &= u_0 \end{cases}$$

where V is of the form:

$$\text{(5.3)} \qquad V(x) = \sum_{j=1}^{d} \delta_j \omega_j^2 x_j^2 : \quad d \geq 1,\ \omega_j > 0,\ \delta_j \in \{-1,0,1\},\ \delta_1 = -1.$$

The last assumptions means that we do not consider positive potentials as in the previous sections.

We denote

$$\mathcal{H}_{V,k} = -\Delta_k + V, \quad U^k_V(t) = e^{-it\mathcal{H}_{V,k}}.$$

Proposition 12. *The semigroup $(U^k_V(t))_{t\in\mathbb{R}}$ is a unitary on $L^2_k(\mathbb{R}^d)$, satisfying the dispersive estimate (3.3). Then there exists $\Theta \in WL^1(\mathbb{R})$ such that* *for any $T \in \bar{\mathbb{R}}_+$, and any sharp $\frac{d+2\gamma}{2}$-admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ we have*

$$\|U_V^k(t)f\|_{L^q(]-T,T[;L_k^r(\mathbb{R}^d))} \leq C\|\Theta\mathbf{1}_{]-2T,2T[}\|_{WL^1(\mathbb{R})}^{\frac{1}{q}}\|f\|_{L_k^2(\mathbb{R}^d)},$$

$$\left\|\int_0^t U_V^k(t-s)F(s)ds\right\|_{L^q(]-T,T[;L_k^r(\mathbb{R}^d))} \leq C\|\Theta\mathbf{1}_{]-2T,2T[}\|_{WL^1(\mathbb{R})}^{\frac{1}{q}+\frac{1}{\tilde{q}}}\|F\|_{L^{\tilde{q}'}(]-T,T[;L_k^{\tilde{r}'}(\mathbb{R}^d))}.$$

Proof. As V is of the form (5.3), then we have:

$$U_V^k(t)f := e^{-it\mathcal{H}_{V,k}}f = \prod_{j=1}^{d}\left(\frac{1}{2ig_j(t)}\right)^{1/2}\int_{\mathbb{R}^d} S_k(t,x,y)f(y)\omega_k(y)dy \tag{5.4}$$

where

$$S_k(t,x,y) = \prod_{n=1}^{d} e^{\frac{i}{g_n(t)}\left(\frac{x_n^2+y_n^2}{2}h_n(t)\right)}K_{\alpha_n}\left(\frac{-ix_n}{g_n(t)}, y_n\right)$$

and the functions g_n, h_n and K_{α_n} are given by:

$$(g_n(t), h_n(t)) = \begin{cases} \left(\dfrac{\sin h(2w_n t)}{w_n}, \cos h(2w_n t)\right), & \text{if } \delta_n = -1, \\ (t,1), & \text{if } \delta_n = 0, \\ \left(\dfrac{\sin(2w_n t)}{w_n}, \cos(2w_n t)\right), & \text{if } \delta_n = 1 \end{cases} \tag{5.5}$$

$$K_{\alpha_n}(z_n,t_n) = j_{\alpha_n-\frac{1}{2}}(iz_nt_n) + \frac{z_nt_n}{2\gamma+1}j_{\alpha_n+\frac{1}{2}}(iz_nt_n), \quad z_n,\ t_n \in \mathbb{C},\ \alpha_n \geq 0 \tag{5.6}$$

where for $\beta \geq \frac{-1}{2}$, j_β is the normalized Bessel function defined by

$$j_\beta(z) = 2^\beta\Gamma(\beta+1)\frac{J_\beta(z)}{z^\beta} = \Gamma(\beta+1)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(\beta+n+1)},$$

with J_β is the Bessel function of first kind and index β.

Thus we have, for some $\delta > 0$,

$$\|U_V^k(t)\|_{\mathcal{L}(L_k^1(\mathbb{R}^d),L_k^\infty(\mathbb{R}^d))} \leq C|t|^{-\frac{d+2\gamma}{2}} \quad \text{for } |t| \leq \delta.$$

Now for $|t| > \delta$, the "worst" possible case is when, say, $\delta_1 = -1$ and $\delta_j = 1$ for $j \geq 2$. Then

$$\|U_V^k(t)\|_{\mathcal{L}(L_k^1(\mathbb{R}^d), L_k^\infty(\mathbb{R}^d))} \leq C \left(e^{-\omega_1|t|} \prod_{j=2}^{d} \frac{1}{|\sin(2\omega_j t)|} \right)^{\frac{1}{2}} \quad \text{for } |t| > \delta.$$

Therefore (3.3) is satisfied with

$$\Theta(t) = C \left[\frac{1}{|t|} \mathbf{1}_{|t|\leq\delta} + \left(e^{-\omega_1|t|} \prod_{j=2}^{d} \frac{1}{|\sin(2\omega_j t)|} \right)^{\frac{1}{d+2\gamma}} \mathbf{1}_{|t|>\delta} \right].$$

By a simple calculations we see that $\Theta \in WL^1(\mathbb{R})$, finally we apply Proposition 3, we obtain the result. $\square$

Our result of this section is the follow:

Theorem 4. *Let $d \geq 3$, $\lambda \in \mathbb{R}$, $p \leq \dfrac{4}{d+2\gamma-2}$, and V satisfying* (5.3). *We assume that $u_0 \in L_k^2(\mathbb{R}^d)$.*

i) *If $p < \frac{4}{d+2\gamma}$, then* (5.2) *has a unique solution*

$$C \cap L^\infty(\mathbb{R}; L_k^2(\mathbb{R}^d)) \cap L_{loc}^{\frac{4p+8}{p(d+2\gamma)}}(\mathbb{R}; L_k^{p+2}(\mathbb{R}^d)).$$

In addition, the L_k^2-norm of $u(t,.)$ is independent of time.

ii) *If $p \leq \dfrac{4}{d+2\gamma}$, then exists $\delta > 0$ such that if $\|u_0\|_{L_k^2(\mathbb{R}^d)} < \delta$, then* (5.2) *has a unique solution $u \in C(\mathbb{R}; L_k^2(\mathbb{R}^d)) \cap L^{p+2}(\mathbb{R}; L_k^{p+2}(\mathbb{R}^d))$. In addition, the L_k^2-norm of $u(t,\cdot)$ is independent of time, and there is scattering: there exist unique $u_-, u_+ \in L_k^2(\mathbb{R}^d)$ such that*

$$\|e^{-it\mathcal{H}_{V,k}} - u_\pm\|_{L_k^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

P r o o f. The first part of Theorem 4 is straightforward : one can mimic the proof given in the case of the nonlinear Dunkl-Schrödinger equation with no potential (cf. [21]). We recall the main argument. Duhamel's formula writes

$$u(t) = U_V^k(t)u_0 - i\lambda \int_0^t U_V^k(t-s)(|u|^p u)(s)ds. \tag{5.7}$$

Define $F(u)(t)$ as the right hand side of (5.7). The idea is to use a fixed point argument in the space given in Theorem. Introduce the following Lebesgue exponents:

$$r = p+2, \quad q = \frac{4p+8}{(d+2\gamma)p}; \quad s = \frac{2p(p+2)}{4-(d+2\gamma-2)p}.$$

Then (q,r) is the sharp $\frac{d+2\gamma}{2}$-admissible pair and

$$\frac{1}{r'} = \frac{p}{r} + \frac{1}{r}; \quad \frac{1}{q'} = \frac{p}{s} + \frac{1}{q}.$$

The main remark to prove the first point of Theorem is that if $p < \dfrac{4}{d+2\gamma}$, we have $\dfrac{1}{q} < \dfrac{1}{s}$ and Hölder's inequality in time yields

$$\|u\|_{L^s(I;L^r_k(\mathbb{R}^d))} \leq |I|^{\frac{1}{s}-\frac{1}{q}} \|u\|_{L^q(I;L^r_k(\mathbb{R}^d))} = |I|^{\frac{4-(d+2\gamma)p}{4p}} \|u\|_{L^q(I,L^r_k(\mathbb{R}^d))}.$$

The positive power of $|I|$ yields contraction in $L^\infty(I; L^2_k(\mathbb{R}^d)) \cap L^q(I; L^r_k(\mathbb{R}^d))$ for small time intervals, and the conservation of the L^2_k norm of the solution shows global existence at the L^2_k level. This proves i).

If $p = \dfrac{4}{d+2\gamma}$, then $s = q$, and Proposition 12 yields

$$\|u\|_{L^{\frac{2d+4\gamma+4}{d+2\gamma}}(I;L_k^{\frac{2d+4\gamma+4}{d+2\gamma}}(\mathbb{R}^d))} \leq C\|u_0\|_{L^2_k(\mathbb{R}^d)} + C\|u\|^{\frac{d+2\gamma+4}{d+2\gamma}}_{L^{\frac{2d+4\gamma+4}{d+2\gamma}}(I;L_k^{\frac{2d+4\gamma+4}{d+2\gamma}}(\mathbb{R}^d))},$$

for some constant C independent of the time interval I. The idea is then to use a bootstrap argument, for $\|u_0\|_{L^2_k(\mathbb{R}^d)}$ sufficiently small. This completes the proof of the first part of Theorem. Note that in the small data case, we used the fact that we have global in time Strichartz estimates, due to the repulsive character of the potential, $\delta_1 = -1$ in (5.3), we obtain the results. □

REFERENCES

[1] T. H. Baker, P. J. Forrester. Non symmetric Jack polynomials and integral kernels. *Duke Math. J.* **95** (1998), 1–50.

[2] S. Ben Saïd, B. Ørsted. The wave equation for Dunkl operators. *Indag. Math.* **16** (2005), 351–391.

[3] R. Carles. Semi-classical analysis for nonlinear Schrödinger equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[4] T. Cazenave, F. Weissler. Some remarks on the nonlinear Schrödinger equation in the critical case. Lect. Notes in Math. vol. **1394** Springer-Verlag, Berlin, 1989, 18–29.

[5] T. Kawazoe, H. Mejjaoli. Uncertainty principles for the Dunkl transform. *Hiroshima Math. J.* **40**, *2* (2010), 241–268.

[6] J. F. van Diejen, L. Vinet. Calogero-Sutherland-Moser Models. CRM Series in Mathematical Physics, Springer-Verlag, 2000.

[7] C. F. Dunkl. Differential-difference operators associated to reflection group. *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[8] C. F. Dunkl. Hankel transforms associated to finite reflection groups. *Contemp. Math.* **138** (1992), 123–138.

[9] M. F. E. de Jeu. The Dunkl transform. *Invent. Math.* **113** (1993), 147–162.

[10] R. P. Feynman, A. R. Hibbs. Quantum mechanics and path integrals. International Series in Pure and Applied Physics, Maidenhead, Berksh., McGraw-Hill Publishing Company Ltd., 1965, 365 p.

[11] D. Fujiwara. A construction of the fundamental solution for the Schrödinger equation. *J. Anal. Math.* **35** (1979), 41–96.

[12] L. Gallardo, K. Trimèche. An L^p version of Hardy's theorem for the Dunkl transform. *J. Aust. Math. Soc.* **77** (2004), 371–385.

[13] H. J. Ginibro, G. Velo. On a class of nonlinear Schrödinger equations. *J. Funct. Anal.* **32**, (1979), 1–72.

[14] K. Hikami. Dunkl operators formalism for quantum many-body problems associated with classical root systems. *J. Phys. Soc. Japan* **65** (1996), 394–401.

[15] L. Hörmander. Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.* **219**, *3* (1995), 413–449.

[16] J. L. Journe, A. Soffer, C. D. Sogge. Decay estimates for Schrödinger operators. *Comm. Pure Appl. Math.* **XLIV** (1991), 573–604.

[17] T. Kato. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* **46** (1987), 113–129.

[18] M. Keel, T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.* **120** (1998), 955–980.

[19] H. Mejjaoli. Strichartz estimates for the Dunkl wave equation and applications. *J. Math. Anal. Appl.* **346**, *1* (2008), 41–54.

[20] H. Mejjaoli. Dispersion phenomena in Dunkl-Schrödinger equation and applications. *Serdica Math. J.* **35**, *1* (2009), 25–60.

[21] H. Mejjaoli. Global well-posedness and scattering for a class of nonlinear Dunkl-Schrödinger equations. *Nonlinear Anal.* **72**, *3–4* (2010), 1121–1139.

[22] H. Mejjaoli. On the wave and Schrödinger equations associated to the generalized harmonic oscillator and applications. Preprint, 2010.

[23] A. K. Nandakumaran, P. K. Ratnakumar. Schrödinger equation and the oscillatory. *J. Funct. Anal.* **224** (2005), 371–385.

[24] M. Rösler. Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.* **192** (1998), 519–542.

[25] R. S. Strichartz. Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* **44** (1977), 705–774.

[26] K. Trimèche. An analogue of Beurling-Hörmander's theorem for the Dunkl transform. *Global J. of Pure and Appl. Math.* **5** (2007), 342–357.

Department of Mathematics
College of Sciences
King Faisal University
P.O. 380 Ahsaa 31982
Kingdom of Saudi Arabia
e-mail: `hmejjaoli@kfu.edu.sa`
`hatem.mejjaoli@ipest.rnu.tn`

Received April 27, 2011
Revised August 06, 2011