# Serdica

## Mathematical Journal

# Сердика

## Математическо списание

Serdica Math. J. **37** (2011), 143–162

**Serdica**
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

# THE LINDELÖF NUMBER GREATER THAN CONTINUUM IS $u$-INVARIANT

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*Communicated by S. P. Gul'ko*

Abstract. Two Tychonoff spaces $X$ and $Y$ are said to be $l$-equivalent ($u$-equivalent) if $C_p(X)$ and $C_p(Y)$ are linearly (uniformly) homeomorphic. N. V. Velichko proved that countable Lindelöf number is preserved by the relation of $l$-equivalence. A. Bouziad strengthened this result and proved that any Lindelöf number is preserved by the relation of $l$-equivalence. In this paper it has been proved that the Lindelöf number greater than continuum is preserved by the relation of $u$-equivalence.

**Introduction.** Our aim is to prove the following main result of the paper.

**Theorem 0.1.** *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic and the Lindelöf number of $X$ or $Y$ greater than continuum. Then $l(X) = l(Y)$.*

2010 *Mathematics Subject Classification:* 54C35, 54D20, 54C60.
*Key words:* Function spaces, $u$-equivalence, $u$-invariant, Lindelöf number, set-valued mappings.

For the proof we need some auxiliary concepts. In the first section, we consider set-valued mappings $K$ and $K_\varepsilon$ of the space $X$ to $Y$ generated by the uniform homeomorphism of the spaces $C_p(Y)$ and $C_p(X)$, and formulate their properties. In the second section, we prove the main result. Section 3 is devoted to the proof of the auxiliary results.

**Terminology and notations.** In notation and terminology we follow R. Engelking's book [2]. The spaces considered in this paper are taken to be Tychonoff spaces. The symbols $X$, $Y$ are used only for topological spaces. $\mathbb{R}$ denotes the usual space of real numbers, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers. The symbol $\overline{k, m}$ denotes the set of all natural numbers $n$ such that $k \le n \le m$, where $k, m \in \mathbb{N}$, $k \le m$. $\mathbb{R}^X$ is a space of all real-valued functions on $X$, $C_p(X)$ is a space of all real-valued continuous functions on $X$ equipped with the topology of pointwise convergence. $\operatorname{Fin} \mathcal{F}$ is a family of all finite subsets of a set $\mathcal{F}$.

The restriction of the mapping $f$ to the subset $A$ is denoted by $f|_A$. $f^{-1}(A)$ is a preimage of the set $A$ under the mapping $f$. If $A$ is an interval, then we shall use the symbol $f^{-1}A$ instead of $f^{-1}(A)$. $|A|$ denotes the cardinality of $A$, $\operatorname{Int} A$ denotes the interior of $A$. A subset $A$ of $X$ will be called functionally closed (functionally open) if $A = f^{-1}(1)$ ($A = f^{-1}(0, 1]$ respectively) for some continuous function $f\colon X \to [0, 1]$. We say that the set $A$ is a $G_\delta$-subset of $X$ if $A$ can be represented as the intersection of some countable family of open subsets of $X$.

The cardinal number assigned to the set of all positive integers is denoted by the symbol $\aleph_0$, and the cardinal number assigned to the set of all real numbers is denoted by $c$ (continuum). The symbol $\tau$ denote infinite cardinal only. For any cardinal number $\tau$ symbol $\omega(\tau)$ denotes the initial ordinal number $\lambda$ such that $|\lambda| = \tau$. The Lindelöf number $l(X)$ of a space $X$ is the smallest infinite cardinal $\tau$ such that any open cover of $X$ contains a subcover of cardinality at most $\tau$.

For a set-valued mapping $p : X \to Y$ and sets $A \subset X$ and $B \subset Y$, the set $p(A) = \bigcup\{p(x)\colon x \in A\}$ is called the image of $A$ under $p$, and the set $p^{-1}(B) = \{x \in X\colon p(x) \cap B \neq \varnothing\}$ is called the preimage of $B$ under $p$. A set-valued mapping $p : X \to Y$ is called lower semicontinuous if for every open subset of $Y$ its preimage under $p$ is open in $X$, and $p$ is called surjective if for every $y \in Y$ there exists an $x \in X$ such that $y \in p(x)$.

## 1. Set-valued mappings concerned with uniform homeomorphisms of function spaces and their properties.

**Definition 1.1.** *Let $h\colon C_p(Y) \to C_p(X)$ be a uniform homeomorphism. Fix $x \in X$, $\delta > 0$, and finite subset $K \subset Y$, and put*

$$a(x, K, \delta) = \sup\{|h(g')(x) - h(g'')(x)| : \\ g', g'' \in C_p(Y),\ |g'(y) - g''(y)| < \delta \textit{ for all } y \in K\}.$$

This notion was introduced by S. P. Gul'ko in [3]. Next, we define

$$a(x, K, 0) = \sup\{|\, h(g')(x) - h(g'')(x)| : \\ g', g'' \in C_p(Y),\ g'(y) = g''(y) \text{ for all } y \in K\}.$$

(if the set $K$ is empty, then the supremum is taken over all $g', g'' \in C_p(Y)$). It is obvious that if $0 \le \delta_1 \le \delta_2$, then $a(x, K, \delta_1) \le a(x, K, \delta_2)$, and if $K_1 \subset K_2 \subset Y$, then $a(x, K_2, \delta) \le a(x, K_2, \delta)$ for all $\delta \ge 0$. It was proved in [3] that for every $x \in X$ there exists a nonempty finite subset $K(x) \subset Y$ such that
**1**. $a(x, K(x), \delta) < \infty$ for any $\delta > 0$,
**2**. $a(x, K', \delta) = \infty$ for every proper subset $K'$ of $K(x)$ and for any $\delta > 0$,
**3**. If $a(x, K, \delta) < \infty$ for some finite subset $K \subset Y$ and $\delta > 0$, then $K(x) \subset K$.

S. P. Gul'ko also proved that if $a(x, K, \delta_0) < \infty$ for some finite subset $K \subset Y$ and $\delta_0 > 0$, then $a(x, K, \delta) < \infty$ for all $\delta > 0$. We now prove that the set $K(x)$ has the following property, which is stronger than the property 2.

**4**. $a(x, K', 0) = \infty$ for every proper subset $K'$ of $K(x)$.

To prove this statement we need the following

**Lemma 1.2.** *If $a(x, K, 0) < \infty$, then $a(x, K, \delta) < \infty$ for all $\delta > 0$.*

Proof. Fix $x \in X$ and finite subset $K \subset Y$ such that $a(x, K, 0) < \infty$. We prove that the function $\delta \mapsto a(x, K, \delta)$ is continuous at the point 0. Let $\varepsilon > 0$. Since $h$ is a uniform homeomorphism, there exist a finite subset $K' \subset Y$ and $\delta > 0$ such that for all $g', g'' \in C_p(Y)$ we have the implication

$$(|\, g'(y) - g''(y)| < \delta \ \text{ for all } y \in K') \Rightarrow |\, h(g')(x) - h(g'')(x)| < \varepsilon.$$

Let $g', g'' \in C_p(Y)$ and $|\,g'(y) - g''(y)| < \delta$ for all $y \in K$. Since $Y$ is a Tychonoff space, there is $g \in C_p(Y)$ such that

$$g(y) = \begin{cases} g'(y) & \text{if } y \in K; \\ g''(y) & \text{if } y \in K' \setminus K. \end{cases}$$

Then $|\,g(y) - g''(y)| < \delta$ for all $y \in K'$, hence $|\,h(g)(x) - h(g'')(x)| < \varepsilon$. Now by the triangle inequality we obtain

$$|\,h(g')(x) - h(g'')(x)| \le |\,h(g')(x) - h(g)(x)| + |\,h(g)(x) - h(g'')(x)| < a(x, K, 0) + \varepsilon.$$

Passing to the supremum over all $g', g'' \in C_p(Y)$ such that $|\,g'(y) - g''(y)| < \delta$ for all $y \in K$, we have inequality $a(x, K, \delta) \le a(x, K, 0) + \varepsilon$, which implies that the function $\delta \mapsto a(x, K, \delta)$ is continuous at the point 0. Therefore there exists $\delta_0 > 0$ such that $a(x, K, \delta_0) < \infty$, hence $a(x, K, \delta) < \infty$ for all $\delta > 0$. □

For any $x \in X$ we put $a(x) = a(x, K(x), 0)$. Using this notation we have the following simple assertions.

**(K1)** If $g', g'' \in C_p(Y)$ and $g'|_{K(x)} = g''|_{K(x)}$, then $|\,h(g')(x) - h(g'')(x)| \le a(x)$.

**(K2)** For any proper subset $K' \subset K(x)$ and any real $b$ there exist functions $g', g'' \in C_p(Y)$ such that $g'|_{K'} = g''|_{K'}$ and $|\,h(g')(x) - h(g'')(x)| > b$.

Besides, this mapping surjectively maps the space $X$ onto $Y$ (see Lemma 3.2 on page 158), i.e., for any $y \in Y$ there exists $x \in X$ such that $y \in K(x)$.

For every $x \in X$ and every $\varepsilon > 0$ we define nonempty finite set $K_\varepsilon(x) \subset Y$ satisfying the following conditions:

**(KE1)** $a(x, K_\varepsilon(x), 0) \le \varepsilon$;

**(KE2)** $a(x, K', 0) > \varepsilon$ for every proper subset $K'$ of $K_\varepsilon(x)$.

It is easy to check that such a set always exists. Indeed, since $h$ is uniformly continuous, it follows that there exist $\delta > 0$ and a finite set $K \subset Y$ such that for all $g', g'' \in C_p(Y)$ we have the implication $(|\,g'(y) - g''(y)| < \delta$ for all $y \in K) \Rightarrow |\,h(g')(x) - h(g'')(x)| \le \varepsilon$. Then $a(x, K, 0) \le \varepsilon$. Reducing the set $K$ until it satisfies the condition (KE2), we obtain the set $K_\varepsilon(x)$.

There can be several sets satisfying properties (KE1) and (KE2); then we denote by $K_\varepsilon(x)$ anyone of them. By the property 3 of $K(x)$ we have $K(x) \subset K_\varepsilon(x)$ for every $\varepsilon > 0$, and by the property 4 we have $K(x) = K_a(x)$ for any $a \ge a(x)$. Thus $K(x)$ is the smallest of all sets $K_\varepsilon(x)$.

The following lemma is analogous to result obtained by O. G. Okunev [4] for $t$-equivalence.

**Lemma 1.3.** *Let $x_0 \in X$, $\varepsilon > 0$, $U$ is an open subset of $Y$ such that $K(x_0) \cap U \neq \varnothing$. Then there is an open neighborhood $V$ of $x_0$ such that $K_\varepsilon(x) \cap U \neq \varnothing$ for any $x \in V$.*

Proof. We can assume that $K(x_0) \cap U = \{y_0\}$. Put $K' = K(x_0) \setminus \{y_0\}$. By the property 2 of $K(x)$ there exist functions $g_1, g_2 \in C_p(Y)$ coinciding on $K'$ such that $|\,h(g_1)(x_0) - h(g_2)(x_0)| > \varepsilon + a(x)$. Since $Y$ is completely regular, it follows that there exists a function $g_0 \in C_p(Y)$ coinciding with $g_1$ on $Y \setminus U$ such that $g_0(y_0) = g_2(y_0)$. Then $g_0|_{K(x_0)} = g_2|_{K(x_0)}$ and $|\,h(g_0)(x_0) - h(g_2)(x_0)| \leq a(x)$. By the triangle inequality we obtain that

$$|\,h(g_1)(x_0) - h(g_0)(x_0)| \geq |\,h(g_1)(x_0) - h(g_2)(x_0)| - |\,h(g_0)(x_0) - h(g_2)(x_0)| > \varepsilon.$$

Let us prove that the set $V$ defined by the formula $V = \{x \in X : |\,h(g_1)(x) - h(g_0)(x)| > \varepsilon\}$ is the required open neighborhood of $x_0$. Assume the contrary. Let $x \in V$ be a point such that $K_\varepsilon(x) \cap U = \varnothing$. Then $g_1$ coincides with $g_0$ on $K_\varepsilon(x)$. Therefore $|\,h(g_1)(x) - h(g_0)(x)| \leq \varepsilon$, a contradiction to the assumption that $x \in V$. □

The last theorem yields the following corollaries.

**Corollary 1.4.** *Let $x_0 \in X$, $\varepsilon > 0$, $k \in \mathbb{N}$, and let $U$ be an open subset of $Y$ such that $|K(x_0) \cap U| \geq k$. Then there is an open neighborhood $V$ of $x_0$ such that $|K_\varepsilon(x) \cap U| \geq k$ for all $x \in V$.*

The proof is trivial.

**Corollary 1.5.** *Let $U$ be an open subset of $Y$. Then $K^{-1}(U)$ is a $G_\delta$-set in $X$.*

Proof. Let $K^{-1}(U) \neq \varnothing$. Since $K(x) \subset K_m(x)$ for all $m \in \mathbb{N}$ and there is a natural number $n$ such that $K(x) = K_n(x)$, it follows that $K^{-1}(U) = \bigcap_{m\in\mathbb{N}} K_m^{-1}(U)$. By Corollary 1.4 we have $K^{-1}(U) \subset \operatorname{Int} K_m^{-1}(U)$ for every $m \in \mathbb{N}$, consequently, $K^{-1}(U) = \bigcap_{m\in\mathbb{N}} \operatorname{Int} K_m^{-1}(U)$. □

It is well known (see Lemma 3.5 on page 161) that every uniform homeomorphism $h$ between $C_p$-spaces can be extended to a uniform homeomorphism between the spaces of all real-valued functions. We shall denote this new homeomorphism also by $h$.

**Definition 1.6.** *Fix a point $x \in X$, $\delta > 0$, and a finite subset $K \subset Y$, and put*

$$\bar{a}(x, K, \delta) = \sup\{|\, h(g')(x) - h(g'')(x)| : \\ g', g'' \in \mathbb{R}^Y,\ |g'(y) - g''(y)| < \delta \text{ for all } y \in K\},$$

$$\bar{a}(x, K, 0) = \sup\{|\, h(g')(x) - h(g'')(x)| : \\ g', g'' \in \mathbb{R}^Y,\ g'(y) = g''(y) \text{ for all } y \in K\}.$$

**Lemma 1.7.** *Let $h\colon \mathbb{R}^Y \to \mathbb{R}^X$ be a uniform homeomorphism such that $h(C_p(Y)) = C_p(X)$. Then $a(x, K, \delta) = \bar{a}(x, K, \delta)$ for all $x \in X$, any finite set $K \subset Y$, and $\delta \geq 0$.*

P r o o f. It follows from the definition that $a(x, K, \delta) \leq \bar{a}(x, K, \delta)$. Let us prove the reverse inequality. Let $\delta > 0$. Take $\varepsilon > 0$ and two functions $g_1, g_2 \in \mathbb{R}^Y$ such that

$$|\, g_1(y) - g_2(y)| < \delta \text{ for all } y \in K. \tag{1.1}$$

Since $h$ is a uniform homeomorphism, it follows that there exist a finite set $K' \subset Y$ and $\Delta > 0$ such that for all $g', g'' \in \mathbb{R}^Y$ wehave the implication

$$(|\, g'(y) - g''(y)| < \Delta \ \text{ for all } y \in K') \Rightarrow |\, h(g')(x) - h(g'')(x)| < \varepsilon/2. \tag{1.2}$$

There are functions $g_0', g_0'' \in C_p(Y)$ such that $g_0'|_{K \cup K'} \equiv g_1|_{K \cup K'}$ and $g_0''|_{K \cup K'} \equiv g_2|_{K \cup K'}$. Then $|\, g_0'(y) - g_0''(y)| < \delta$ for all $y \in K$. Observe that from (1.2) it follows that $|\, h(g_1)(x) - h(g_0')(x)| < \varepsilon/2$ and $|\, h(g_2)(x) - h(g_0'')(x)| < \varepsilon/2$, and – by virtue of the triangle inequality – we have $a(x, K, \delta) \geq |\, h(g_0')(x) - h(g_0'')(x)| > |\, h(g_1)(x) - h(g_2)(x)| - \varepsilon$. Passing to the supremum over all $g_1, g_2 \in \mathbb{R}^Y$ satisfying condition (1.1) we obtain inequality $a(x, K, \delta) \geq \bar{a}(x, K, \delta) - \varepsilon$. Since $\varepsilon$ being an arbitrary positive number, this implies that $a(x, K, \delta) = \bar{a}(x, K, \delta)$. Equality $a(x, K, 0) = \bar{a}(x, K, 0)$ is proved analogously. □

## 2. Main result.

**Theorem 2.1.** *Let $X$ and $Y$ be $u$-equivalent, $\tau$ a cardinal not less than the continuum, and $l(X) \leq \tau$. Then $l(Y) \leq \tau$.*

P r o o f. Since any uniform homeomorphism between $C_p$-spaces can be extended to a uniform homeomorphism between the spaces of all real-valued functions, one can assume without loss of generality that there is a uniform homeomorphism $h$ of $\mathbb{R}^Y$ onto $\mathbb{R}^X$ satisfying the following conditions:

1. $h(C_p(Y)) = C_p(X)$;
2. $h$ takes zero function $0_Y \in \mathbb{R}^Y$ to zero function $0_X \in \mathbb{R}^X$.

To prove the theorem we shall need some notation.

Let $p\colon X \to Y$ be a set-valued mapping of $X$ to $Y$ and let $U \subset Y$ be an arbitrary set. Put

$$p^*(U) = \{x \in X : p(x) \subset U\}.$$

By $\mathcal{T}$ we shall denote the family of all open subsets of $Y$. Let $\mathcal{U}$ be an open cover of $Y$, $\tau$ an infinite cardinal. A cover $\mathcal{U}$ will be called $\tau$-trivial if it contains a subcover of cardinality at most $\tau$. Otherwise it will be called $\tau$-nontrivial. This notion was introduced by A. Bouziad in [1]. Put

$$[\mathcal{U}]_\tau = \left\{\bigcup \mathcal{U}' : \mathcal{U}' \subset \mathcal{U},\ |\mathcal{U}'| \le \tau\right\}.$$

We say that the set $A$ is an $F_\tau$-subset of $X$ if $A$ can be represented as the union of some family, of cardinality at most $\tau$, of closed subsets of $X$. The complements of $F_\tau$-subsets will be called $G_\tau$-subsets. If $\tau = \aleph_0$, then we shall write $F_\sigma$ and $G_\delta$ instead of $F_{\aleph_0}$ and $G_{\aleph_0}$ respectively. The symbol $\mathcal{F}_\tau$ denotes the family of all $F_\tau$-subsets of $X$, $\mathcal{G}_\tau$ is a family of all $G_\tau$-subsets of $X$. The family of all subsets $A$ of $X$ such that $l(A) \le \tau$ will be denoted by $\mathcal{L}_\tau$.

Let $l(X) \le \tau$, where $\tau \ge c$. Assume that $l(Y) > \tau$ to obtain a contradiction. It means that there exists $\tau$-nontrivial open cover $\mathcal{U}$ of $Y$. Without loss of generality we can assume that $\mathcal{U}$ is closed under the operation of finite union and $\mathcal{U} \subset \mathcal{B}$, where $\mathcal{B}$ is a base of $Y$ consisting of all functionally open subsets of $Y$. It is well known that the family $\mathcal{B}$ is also closed under the operation of finite union (see [2], page 43).

Define a mapping

$$U\colon \operatorname{Fin}\mathcal{F}_\tau \to [\mathcal{U}]_\tau, \qquad U = U(\mathcal{F}), \qquad \text{where} \qquad \mathcal{F} \in \operatorname{Fin}\mathcal{F}_\tau,$$

using set-valued mappings defined in the previous section. For any $x \in X$ put $\rho(x) = |K(x)|$. For every set $F \subset X$ we define a number

$$\rho(F) = \min\{\rho(x) : x \in F\},$$

which will be called the level of the set $F$.

Further, for any $U \in \mathcal{T}$ and any natural numbers $k$ and $m$ put

$$U_m^{[k]} = \operatorname{Int}\{\, x \in X : |\, K_m(x) \cap U\,| \geq k \,\}\,.$$

Let $\mathcal{F} = \{F_1, \ldots, F_n\} \subset \mathcal{F}_\tau$. For any nonempty set $A \subset \{1, \ldots, n\}$ we put

$$F_A = \bigcap_{i \in A} F_i, \qquad \overline{\mathcal{F}} = \{\, F_A \colon A \subset \{1, \ldots, n\},\ F_A \neq \varnothing \,\}.$$

Let $F \in \overline{\mathcal{F}}$, $m \in \mathbb{N}$ and $k = \rho(F)$. Then the family

$$\mathcal{U}_m^{[k]} = \left\{ U_m^{[k]} \colon U \in \mathcal{U} \right\}$$

is an open cover of $F$. Indeed, since the family $\mathcal{U}$ is closed under the operation of finite union, it follows that for every $x_0 \in F$ there is $U \in \mathcal{U}$ such that $K(x_0) \subset U$. As $\rho(F) = k$, it follows that $|K(x_0) \cap U| \geq k$ and by Corollary 1.4 there exists an open neighborhood $V$ of $x_0$ such that $|K_m(x) \cap U| \geq k$ for all $x \in V$. Then $x_0 \in V \subset U_m^{[k]}$, hence $\mathcal{U}_m^{[k]}$ is an open cover of $F$. From the condition $l(X) \leq \tau$ it follows that $F \in \mathcal{L}_\tau$; therefore the cover $\mathcal{U}_m^{[k]}$ contains a subcover $\left\{ U_m^{[k]} \colon U \in \mathcal{U}_{F,m} \right\}$ of $F$, where $\mathcal{U}_{F,m} \subset \mathcal{U}$ and $|\mathcal{U}_{F,m}| \leq \tau$. Put

$$U(\mathcal{F}) = \bigcup_{F \in \overline{\mathcal{F}}} \bigcup_{m \in \mathbb{N}} \left( \bigcup \mathcal{U}_{F,m} \right).$$

Obviously, $U(\mathcal{F}) \in [\mathcal{U}]_\tau$, and if $\mathcal{F}_1 \subset \mathcal{F}_2$, then $U(\mathcal{F}_1) \subset U(\mathcal{F}_2)$. The mapping $U$ we shall call the constructor. A similar construction was used by N. V. Velichko in [5].

We note one important property of the constructor.

(*) For every $\mathcal{F} \in \operatorname{Fin} \mathcal{F}_\tau$, any $F \in \overline{\mathcal{F}}$, and any $x \in F$ the following inequality holds:

$$|\, K(x) \cap U\,(\mathcal{F})| \geq \rho(F). \tag{2.1}$$

Indeed, for any $x \in F$ there exist a natural number $m$ and a set $U \in \mathcal{U}_{F,m}$ such that $K_m(x) = K(x)$ and $x \in U_m^{[k]}$; hence $|\, K(x) \cap U\,(\mathcal{F})| \geq |\, K(x) \cap U| \geq k = \rho(F)$.

Let us recall some important properties of the set-valued mappings $K$ and $K_m$ defined in the previous section.

**(P1)** If $g', g'' \in \mathbb{R}^Y$ and $g'|_{K_m(x)} = g''|_{K_m(x)}$, then $|h(g')(x) - h(g'')(x)| \le m$. In particular, if $g'|_{K_m(x)} \equiv 0$, then $|h(g')(x)| \le m$.

**(P2)** If $g', g'' \in \mathbb{R}^Y$ and $g'|_{K(x)} = g''|_{K(x)}$, then $|h(g')(x) - h(g'')(x)| \le a(x) < \infty$. In particular, if $g'|_{K(x)} \equiv 0$, then $|h(g')(x)| \le a(x) < \infty$.

For each $V \subset Y$ consider the function $e_V \in \mathbb{R}^Y$ defined by the formula

$$e_V(y) = \begin{cases} 0, & y \in V, \\ 1, & y \notin V. \end{cases}$$

Denote by $\mathcal{C}$ the family of all functionally closed subsets of $Y$. Every functionally open set $V \subset Y$ admits a decomposition

$$V = \bigcup_{n\in\mathbb{N}} F_n\,, \quad \text{where } F_n \in \mathcal{C} \ \text{ and } F_n \subset F_{n+1} \quad \text{for all } n \in \mathbb{N} \tag{2.2}$$

(see Lemma 3.4 on page 160). Further, by decomposition of functionally open set $V$ we mean a sequence $(F_n)_{n\in\mathbb{N}}$ satisfying condition (2.2). If there is a decomposition $(F_n)_{n\in\mathbb{N}}$ of $V$ satisfying the following condition:

$$K_1^*(V) \setminus K_1^*(F_n) \neq \varnothing \ \text{ for all } n \in \mathbb{N}\,, \tag{2.3}$$

then we say that the set $V$ is *adequate*. A similar notion was introduced by A. Bouziad in [1].

For every open set $V \in \mathcal{T}$ put

$$G(V) = \left\{ x \in X\colon \sup_{m\in\mathbb{N}} |h(me_V)(x)| < \infty \right\},$$

$$F(V) = \left\{ x \in X\colon \sup_{m\in\mathbb{N}} |h(me_V)(x)| = \infty \right\}.$$

Analogous mappings were used by A. Bouziad in [1].

**Lemma 2.2.** *The mapping $G$ has the following properties:*

**(S1)** $K^*(V) \subset G(V)$ *for any* $V \in \mathcal{T}$*;*

**(S2)** *For any expanding sequence* $(U_n)_{n\in\mathbb{N}}$ *of the sets* $U_n \in \mathcal{T}$ *such that*

(2.4)
$$X = \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} G\,(U_n)$$

*the following condition holds:*

$$Y = \bigcup_{n \in \mathbb{N}} U_n \,.$$

P r o o f. Let us verify that condition (S1) is satisfied. Take $V \in \mathcal{T}$ and $x \in K^*(V)$. Then $K(x) \subset V$, hence $me_V|_{K(x)} \equiv 0$ for any natural number $m$ and by (P2) we have $|\,h(me_V)(x)| \leq a(x) < \infty$, therefore $\sup_{m\in\mathbb{N}} |\,h(me_V)(x)| \leq a(x) < \infty$, which implies that $x \in G(V)$.

Let us show that condition (S2) is fulfilled. Let $(U_n)_{n\in\mathbb{N}}$ be an expanding sequence of the sets $U_n \in \mathcal{T}$ such that equality (2.4) is valid. Assume that $Y \neq \bigcup_{n\in\mathbb{N}} U_n$. Put $U = \bigcup_{n\in\mathbb{N}} U_n$. Take $y \in Y \setminus U$. Choose a finite subset $K' = \{x_1, \dots, x_p\} \subset X$ and $\delta > 0$ such that for any two functions $f', f'' \in \mathbb{R}^X$ the following implication holds:

$$\big(\,|f'(x_i) - f''(x_i)| \leq \delta \quad \text{for all } i \in \overline{1,p}\,\big) \Rightarrow |\,h^{-1}(f')(y) - h^{-1}(f'')(y)| < 1.$$

Such a choice is possible because the mapping $h^{-1}$ is uniformly continuous. Then, as shown in [3], for any two functions $f', f'' \in \mathbb{R}^X$ and every natural number $n$ the following implication holds:

$$\big(\,|f'(x_i) - f''(x_i)| \leq n\delta \ \text{ for all } i \in \overline{1,p}\,\big) \Rightarrow |\,h^{-1}(f')(y) - h^{-1}(f'')(y)| < n.$$

In particular,

(2.5)
$$\big(\,|\,h(g)(x_i)| \leq n\delta \ \text{ for all } i \in \overline{1,p}\,\big) \Rightarrow |\,g(y)| < n$$

for any $g \in \mathbb{R}^Y$.

From equality (2.4) it follows that there is a natural number $N$ such that $x_i \in G\,(U_N)$ for all $i \in \overline{1,p}$. Put

$$M = \max_{i\in\overline{1,p}} \sup_{m\in\mathbb{N}} |h(me_{U_N})(x_i)|.$$

Obviously, $M < \infty$. Pick a natural number $n \in \mathbb{N}$ such that $n \geq M/\delta$. Then $|\,h(ne_{U_N})(x_i)| \leq M \leq n\delta$ for all $i \in \overline{1,p}\,$. From this inequality and condition (2.5) it follows that $|ne_{U_N}(y)| < n$, hence $|e_{U_N}(y)| < 1$, therefore $y \in U_N \subset U$. Thus we obtain a contradiction. $\square$

**Lemma 2.3.** *Let $\{U_t\}_{t\in T} \subset \mathcal{U}$ and $|T| \le \tau$. Then there is a family $\{V_s\}_{s\in S} \subset [\mathcal{U}]_\tau$ closed under the operation of finite union and satisfying the following conditions:*

1. $|S| \le \tau$;
2. *each set $V_s$ is adequate;*
3. $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$.

Proof. Let $V_0 = \bigcup_{t\in T} U_t$. Since the cover $\mathcal{U}$ is $\tau$-nontrivial, there exists $y_1 \in Y \setminus V_0$. Choose $x_1 \in X$ such that $y_1 \in K_1(x_1)$, i.e., $K_1(x_1) \not\subseteq V_0$ (such an element exists since the mapping $x \mapsto K(x)$ is surjective), and choose a set $V_1 \in \mathcal{U}$ such that $K_1(x_1) \subset V_1$ (such a set exists since the set $K_1(x_1)$ is finite and the family $\mathcal{U}$ is closed under the operation of finite union). Assume that $x_1, \ldots, x_k$ and $V_1, \ldots, V_k$ are already chosen, where $k \in \mathbb{N}$. The set $Y \setminus \bigcup_{i=0}^{k} V_i$ is nonempty, hence there is an element $x_{k+1} \in X$ such that $K_1(x_{k+1}) \not\subseteq \bigcup_{i=0}^{k} V_i$ and there is a set $V_{k+1} \in \mathcal{U}$ such that $K_1(x_{k+1}) \subset V_{k+1}$. We obtain two sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(V_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $K_1(x_n) \not\subseteq \bigcup_{i=0}^{n-1} V_i$, $V_n \in \mathcal{U}$, and $K_1(x_n) \subset V_n$ for any natural number $n$. Put $V = \bigcup_{n\in\mathbb{N}} V_n$. Let $(W_s)_{s\in S}$ be the family of all finite unions of sets in $(U_t)_{t\in T}$. For each $s \in S$ put $V_s = W_s \cup V$. Clearly, the family $(V_s)_{s\in S} \subset [\mathcal{U}]_\tau$ is closed under the operation of finite union, $|S| \le \tau$, and $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$. It remains to verify that each set $V_s$ is adequate. Let $s \in S$. Fix a decomposition $(F_n^s)_{n\in\mathbb{N}}$ of the set $W_s$ and decomposition $\left(F_n^k\right)_{n\in\mathbb{N}}$ of the set $V_k$, where $k \in \mathbb{N}$. The sequence $(G_n^s)_{n\in\mathbb{N}}$, where $G_n^s = F_n^s \cup F_n^1 \cup \ldots \cup F_n^n$, is a required decomposition of the set $V_s$, since $(x_n)_{n\in\mathbb{N}} \subset K_1^*(V_s)$ and $x_{n+1} \notin K_1^*(G_n^s)$ for all $n \in \mathbb{N}$. $\square$

**Lemma 2.4.** *Let $\{V_s\}_{s\in S}$ be a family of adequate functionally open subsets of $Y$ closed under the operation of finite union $|S| \le \tau$. Then $F\left(\bigcup_{s\in S} V_s\right)$ is an $F_\tau$-subset of $X$.*

Proof. Put $V = \bigcup_{s\in S} V_s$. Let $(F_n^s)_{n\in\mathbb{N}}$ be a decomposition of $V_s$ satisfying conditions $F_n^s \in \mathcal{C}$, $F_n^s \subset F_{n+1}^s$, and $K_1^*(V_s) \setminus K_1^*(F_n^s) \neq \varnothing$ for all $n \in \mathbb{N}$. For any natural number $n$ and any $s \in S$ we can find a function $g_n^s \in C_p(Y)$ (see Lemma 3.5 on page 161) such that

$$g_n^s|_{F_n^s} \equiv 0\,, \qquad g_n^s|_{Y\setminus V_s} \equiv 1.$$

For any $x \in K_1^*(V_s)$ and $k, n \in \mathbb{N}$ put

$$U_{k,n}^s(x) = \left\{ x' \in X\colon \left| h\left(ng^s_{k+N(x,s)}\right)(x') - h\left(ng^s_{k+N(x,s)}\right)(x) \right| < k \right\},$$

where $N(x,s)$ is the smallest natural number $N$ such that $K_1(x) \subset F_N^s$. Then $U_{k,n}^s(x)$ is an open neighborhood of the point $x$ in $X$. Put

$$A_s = \bigcap_{m\in\mathbb{N}} \bigcup_{k\geq m} \bigcap_{n\in\mathbb{N}} \bigcup_{x\in K_1^*(V_s)} U_{k,n}^s(x), \quad B_s = \{x \in X\colon K(x) \cap (V \setminus V_s) \neq \varnothing\},$$

$$A = \bigcap_{s\in S} (A_s \cup B_s).$$

Since each set $\bigcup_{x\in K_1^*(V_s)} U_{k,n}^s(x)$ is open in $X$, by Corollary 3.7 on page 161 we have that $\bigcup_{k\geq m} \bigcap_{n\in\mathbb{N}} \bigcup_{x\in K_1^*(V_s)} U_{k,n}^s(x)$ is a $G_c$-subset of $X$ for any natural number $n$, which implies that $A_s$ is a $G_c$-set. Since $B_s$ is a $G_\delta$-subset of $X$ (see Lemma 3.8 on page 161), it follows that $A$ is a $G_\tau$-subset of $X$. Here we have used the fact that $\tau \geq c$. We shall prove that $G(V) = A$. Since $F(V) = X \setminus G(V)$, this will be sufficient to prove the lemma. We first prove that

$$F(V) \subset X \setminus A. \tag{2.6}$$

Take $x' \in F(V)$. Since $K(x')$ is a finite set and the family $\{V_s\}_{s\in S}$ is closed under the operation of finite union, there exists $s \in S$ such that $K(x') \cap V \subset V_s$, i.e., $x' \notin B_s$. It remains to prove that $x' \notin A_s$. There exists a natural number $m_0$ satisfying the condition $K(x') \cap V \subset F_{m_0}^s$. Then

$$e_V|_{K(x')} = e_{V_s}|_{K(x')} = g_n^s|_{K(x')} \tag{2.7}$$

for any $n \geq m_0$. Since $x' \in F(V)$, for any $k \in \mathbb{N}$ there is a natural number $n_k$ such that

$$\left| h\left(n_k e_V\right)(x') \right| \geq k + a(x') + 1. \tag{2.8}$$

Take an arbitrary natural number $k \geq m_0$. We verify that $x' \notin \bigcup_{x\in K_1^*(V_s)} U_{k,n_k}^s(x)$.

From (2.7) and (P2) it follows that $|\, h\left(n_k e_V\right)(x') - h\left(n_k g_n^s\right)(x')| \leq a(x')$ for any natural numbers $n, k \geq m_0$ and this together with (2.8) gives the inequality $|\, h\left(n_k g_n^s\right)(x')| \geq k + 1$. Take an arbitrary $x \in K_1^*(V_s)$. It remains to show that $x' \notin U^s_{k,\,n_k}(x)$. Since $g^s_{k+N(x,\,s)}|_{K_1(x)} \equiv 0$, from (P1) it follows that $|\,h(n_k g^s_{k+N(x,\,s)})(x)| \le 1$. Then

$$\left|h\left(n_k g^s_{k+N(x,\,s)}\right)(x') - h\left(n_k g^s_{k+N(x,\,s)}\right)(x)\right|$$
$$\ge \left|h\left(n_k g^s_{k+N(x,\,s)}\right)(x')\right| - \left|h\left(n_k g^s_{k+N(x,\,s)}\right)(x)\right| \ge (k+1) - 1 = k.$$

Hence, $x' \notin U^s_{k,\,n_k}(x)$. Inclusion (2.6) is proved.

Let us prove the reverse inclusion $X \setminus A \subset F(V)$. Let $x' \notin A$. We shall show that $x' \in F(V)$. Choose $s \in S$ such that $x' \notin A_s \cup B_s$. Then $K(x') \cap V \subset V_s$. Fix $m_0 \in \mathbb{N}$ such that $x' \notin \bigcup_{k \ge m_0} \bigcap_{n \in \mathbb{N}} \bigcup_{x \in K_1^*(V_s)} U^s_{k,\,n}(x)$ and take an arbitrary natural number $k \ge m_0$. Then there is $n_k \in \mathbb{N}$ such that $x' \notin \bigcup_{x \in K_1^*(V_s)} U^s_{k,\,n_k}(x)$. Choose $q \in \mathbb{N}$ such that $K(x') \cap V \subset F^s_q$ and an element $x_0 \in K_1^*(V_s)$ satisfying the condition $K_1(x_0) \nsubseteq F^s_q$. Such an element exists because the set $V_s$ is adequate. Then $N(x_0, s) > q$ and

$$K(x') \cap V = K_1(x') \cap V_s \subset F^s_q \subset F^s_{k+N(x_0,\,s)}.$$

Put $i = k + N(x_0, s)$. Since $x' \notin U^s_{k,\,n_k}(x_0)$, we have $|h\,(n_k g^s_i)\,(x') - h\,(n_k g^s_i)\,(x_0)| \ge k$. Besides, $|h\,(n_k g^s_i)\,(x_0)| \le 1$. Hence, by the triangle inequality we obtain that

$$\left|h\,(n_k g^s_i)\,(x')\right| \ge k - 1.$$

Since $e_V|_{K(x')} = e_{V_s}|_{K(x')} = g^s_i|_{K(x')}$, we have $|\,h(n_k g^s_i)(x') - h(n_k e_V)(x')| \le a(x')$. Then, again applying the triangle inequality we obtain

$$|\,h(n_k e_V)(x')| \ge |\,h(n_k g^s_i)(x')| - |\,h(n_k g^s_i)(x') - h(n_k e_V)(x')| \ge k - 1 - a(x'),$$

hence, $\sup_{m \in \mathbb{N}} |\,h(m e_V)(x')| = \infty$. $\square$

Lemmas 2.4 and 2.3 yield the following corollary.

**Corollary 2.5.** *For any $U \in [\mathcal{U}]_\tau$ there exists $V \in [\mathcal{U}]_\tau$ such that $U \subset V$ and $F(V)$ is an $F_\tau$-subset of $X$.*

We shall now construct an expanding sequence $(V_n)_{n \in \mathbb{N}}$ such that $V_n \in [\mathcal{U}]_\tau$. Simultaneously with it we shall construct a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \in \operatorname{Fin} \mathcal{F}_\tau$ and $\mathcal{F}_{n'} \subset \mathcal{F}_{n''}$ for any two natural numbers $n' < n''$.

Let $\mathcal{F}_0 = \{X\}$. Choose a set $V_1 \in [\mathcal{U}]_\tau$ such that

$$U(\mathcal{F}_0) \subset V_1 \qquad \text{and} \qquad F(V_1) \in \mathcal{F}_\tau$$

(it is possible by Corollary 2.5), and put $\mathcal{F}_1 = \{X,\ F(V_1)\}$. Choose a set $V_2 \in [\mathcal{U}]_\tau$ such that

$$V_1 \cup U(\mathcal{F}_1) \subset V_2 \qquad \text{and} \qquad F(V_2) \in \mathcal{F}_\tau.$$

Assume that we have already defined the sets $V_i \in [\mathcal{U}]_\tau$ and $\mathcal{F}_i \in \operatorname{Fin} \mathcal{F}_\tau$ for every natural number $i \leq k$ satisfying the following conditions:

1. $F(V_i) \in \mathcal{F}_\tau, \qquad 1 \leq i \leq k;$
2. $V_i \cup U(\mathcal{F}_i) \subset V_{i+1}, \qquad 1 \leq i \leq k-1$, where $\mathcal{F}_i = \{X, F(V_1), \ldots, F(V_i)\}$, $1 \leq i \leq k$.

Choose a set $V_{k+1} \in [\mathcal{U}]_\tau$ satisfying the following conditions:

$$V_k \cup U(\mathcal{F}_k) \subset V_{k+1} \qquad \text{and} \qquad F(V_{k+1}) \in \mathcal{F}_\tau. \tag{2.9}$$

Put $\mathcal{F}_{k+1} = \{X, F(V_1), \ldots, F(V_{k+1})\}$. The sequences $(V_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ are defined.

We shall prove by induction with respect to $n$ the following assertion.

**Assertion 2.6.** *For any natural number $n$ and each set $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ such that $F(V_{j_1}) \cap \ldots \cap F(V_{j_k}) \neq \varnothing$ the following inequality holds:*

$$\rho(F(V_{j_1}) \cap \ldots \cap F(V_{j_k})) \geq k+1. \tag{2.10}$$

Proof. We shall show that $\rho(F(V_n)) \geq 2$. For any $x \in X$ by inequality (2.1) we have

$$|K(x) \cap V_n| \geq |K(x) \cap V_1| \geq |K(x) \cap U(\mathcal{F}_0)| \geq \rho(X) \geq 1.$$

Therefore, if $\rho(x) = 1$ for some $x \in X$, then $K(x) \subset V_n$, hence, by (S1) we have $x \notin F(V_n)$. This implies that $\rho(F(V_n)) \geq 2$. In particular, this yields that the assertion is valid for $n = 1$.

Assume that Assertion 2.6 holds for every natural number $n \leq N$. We shall prove that it holds for $n = N+1$. It suffice to show that for each subset $\{j_1, \ldots, j_k\} \subset \{1, \ldots, N\}$ such that $F = F(V_{j_1}) \cap \ldots \cap F(V_{j_k}) \cap F(V_{N+1}) \neq \varnothing$

the following inequality holds: $\rho(F) \geq k+2$. Put $F' = F(V_{j_1}) \cap \ldots \cap F(V_{j_k})$, then $F = F' \cap F(V_{N+1})$. By induction hypothesis we have inequality $\rho(F') \geq k+1$. Assume that $\rho(F) = k+1$ to obtain a contradiction.

Take an element $x \in F$ such that $|K(x)| = k+1$. Since $F' \in \overline{\mathcal{F}_N}$, we see that from (2.9) and (2.1) it follows that

$$|K(x) \cap V_{N+1}| \geq |K(x) \cap U(\mathcal{F}_N)| \geq \rho(F') \geq k+1.$$

Hence, $K(x) \subset V_{N+1}$ and condition (S2) implies that $x \notin F(V_{N+1})$. Therefore $x \notin F$. This contradiction completes the proof of Assertion 2.6.

In particular, inequality (2.10) implies that for any $x \in X$ there exists a natural number $k$ such that $x \notin F(V_n)$ for all $n > k$, i.e., that $x \in G(V_n)$. In other words, equality (2.4) holds. By Lemma 2.2 we obtain $Y = \bigcup_{n \in \mathbb{N}} V_n$. Since $V_n \in [\mathcal{U}]_\tau$ for any $n \in \mathbb{N}$, we see that the cover $\mathcal{U}$ of $Y$ is $\tau$-trivial, a contradiction. Hence, $l(Y) \leq \tau$. □

**Corollary 2.7.** *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic, and let $l(X), l(Y) \geq c$. Then $l(X) = l(Y)$.*

**Corollary 2.8.** *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic. Then $l(X) \leq c$ if and only if $l(Y) \leq c$.*

The statement of Theorem 0.1 follows from Corollaries 2.7 and 2.8.

**Problem 2.9** *Are there spaces $X$ and $Y$ such that $l(X) = c$, $l(Y) < c$ and $C_p(X)$ is uniformly homeomorphic to $C_p(Y)$?*

## 3. Auxiliary statements used in the proof.

**Theorem 3.1.** *Let $h\colon C_p(Y) \to C_p(X)$ be a uniform homeomorphism. Then there is a uniform homeomorphism $\bar{h}\colon \mathbb{R}^Y \to \mathbb{R}^X$ such that $\bar{h}(g) = h(g)$ for all $g \in C_p(Y)$.*

P r o o f. Let $\widetilde{K}_n(x) = \bigcup_{m=1}^n K_{1/m}(x)$, $\widetilde{K}(x) = \bigcup_{m=1}^\infty K_{1/m}(x)$, where $x \in X$. For the mapping $H = h^{-1}\colon C_p(X) \to C_p(Y)$ we define such mappings as defined in section 1 for $h$. For any $y \in Y$, $\delta > 0$, and any finite subset $L \subset X$ put

$$b(y, L, \delta) = \sup\{|H(f')(y) - H(f'')(y)| : \\ f', f'' \in C_p(X),\ |f'(x) - f''(x)| < \delta \text{ for all } x \in L\}.$$

We also put

$$b(y, L, 0) = \sup\{|\,H(f')(y) - H(f'')(y)| : \\ f', f'' \in C_p(X),\ f'(x) = f''(x) \text{ for all } x \in L\}.$$

As in the case of the mapping $h$, for every $y \in Y$ there exist finite sets $L(y) \subset X$ and $L_\varepsilon(y) \subset X$ for any $\varepsilon > 0$ satisfying the following conditions:

**1**. $b(y, L(y), \delta) < \infty$ for all $\delta \geq 0$;
**2**. $b(y, L', \delta) = \infty$ for all $\delta \geq 0$, where $L'$ is a proper subset of $L(y)$;
**3**. If $b(y, L, \delta) < \infty$ for some finite set $L \subset X$ and $\delta \geq 0$, then $L(y) \subset L$;
**4**. $b(y, L_\varepsilon(y), 0) \leq \varepsilon$;
**5**. $b(y, L', 0) > \varepsilon$, where $L'$ is a proper subset of $L_\varepsilon(y)$;
**6**. $L(y) \subset L_\varepsilon(y)$.

Let $\widetilde{L}_n(y) = \bigcup_{m=1}^{n} L_{1/m}(y)$, $\widetilde{L}(y) = \bigcup_{m=1}^{\infty} L_{1/m}(y)$, where $y \in Y$.

For the proof we need two lemmas.

**Lemma 3.2.** *$y \in \bigcup_{x \in L(y)} K(x)$ for any $y \in Y$.*

Proof. Let $K = \bigcup_{x \in L(y)} K(x)$. Assume that $y \notin K$ to obtain a contradiction. Let $\delta = \max\{a(x) \colon x \in L(y)\}$, $b = b(y, L(y), \delta)$. Take a function $g \in C_p(Y)$ such that $g|_K \equiv 0$ and $g(y) = b + 1$. Since $g|_{K(x)} \equiv 0$, we have $|\,h(g)(x)| \leq a(x) \leq \delta$ for any $x \in L(y)$. Then $b + 1 = |g(y)| \leq b(y, L(y), \delta) = b$. This contradiction completes the proof. □

We now define a mapping $\bar{h} \colon \mathbb{R}^Y \to \mathbb{R}^X$. Let $g \in \mathbb{R}^Y$ and $x \in X$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions on $Y$ such that $g_n|_{\widetilde{K}_n(x)} = g|_{\widetilde{K}_n(x)}$ for each $n \geq n_0$, where $n_0$ is some natural number. We shall prove that the sequence $(h(g_n)(x))_{n \in \mathbb{N}}$ has a limit. Take $\varepsilon > 0$ and put $N = \max([1/\varepsilon] + 1, n_0)$, where $[x]$ denotes the integer part of $x$. Then $g_n|_{\widetilde{K}_N(x)} = g_m|_{\widetilde{K}_N(x)}$ for all $n, m \geq N$, hence, $|\,h(g_n)(x) - h(g_m)(x)| \leq 1/N < \varepsilon$. We obtain that the sequence $(h(g_n)(x))_{n \in \mathbb{N}}$ is fundamental (Cauchy sequence), hence it has a limit. We define a mapping $\bar{h}$ by the formula

$$\bar{h}(g)(x) = \lim_{n \to \infty} h(g_n)(x).$$

We have to prove that the definition does not depend on the choice of the sequence $(g_n)_{n \in \mathbb{N}}$. Let $(g'_n)_{n \in \mathbb{N}}$ be another sequence of continuous functions on $Y$ such that $g'_n|_{\widetilde{K}_n(x)} = g|_{\widetilde{K}_n(x)}$ starting from some $n_1$, and let $a = \lim_{n \to \infty} h(g_n)(x)$,

$b = \lim_{n\to\infty} h(g'_n)(x)$. From the sequences $\{g_n\}$ and $\{g'_n\}$, we construct another sequence $\{g''_n\}$ defined by the formula

$$g''_n = \begin{cases} g_n & \text{if } n \text{ is odd;} \\ g'_n & \text{if } n \text{ is even.} \end{cases}$$

As shown above, there is a limit of the sequence $(h(g''_n)(x))_{n\in\mathbb{N}}$ which we denote by $c$. Then

$$c = \lim_{n\to\infty} h(g''_n)(x) = \lim_{n\to\infty} h(g''_{2n})(x) = \lim_{n\to\infty} h(g''_{2n-1}),$$

which implies that $a = b = c$. Obviously, if $g \in C_p(Y)$, then $\bar{h}(g) = h(g)$.

We now define a mapping $\bar{H}\colon \mathbb{R}^X \to \mathbb{R}^Y$. Let $f \in \mathbb{R}^X$ and $y \in Y$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $X$ such that $f_n|_{\widetilde{L}_n(y)} = f|_{\widetilde{L}_n(y)}$ starting from some $n_0$. Similarly, we can prove that there is a limit of the sequence $(h^{-1}(f_n)(y))_{n\in\mathbb{N}}$. Consider the mapping $H$ defined by the formula $\bar{H}(f)(y) = \lim_{n\to\infty} h^{-1}(f_n)(y)$. It can be proved analogously that the definition is correct and $\bar{H}(f) = h^{-1}(f)$ for all $f \in C_p(X)$.

**Lemma 3.3.** *The mappings $\bar{h}\colon \mathbb{R}^Y \to \mathbb{R}^X$ and $\bar{H}\colon \mathbb{R}^X \to \mathbb{R}^Y$ are uniformly continuous.*

Proof. Take $x \in X$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 4/\varepsilon$. Then for each natural number $n \geq N$ we have $a(x, \widetilde{K}_n(x), 0) \leq 1/N < \varepsilon/4$. Since the mapping $\delta \mapsto a(x, K, \delta)$ is continuous at zero, there exists $\delta > 0$ such that

$$a(x, \widetilde{K}_N(x), \delta) < \varepsilon/2\,. \tag{3.1}$$

Let $g', g'' \in \mathbb{R}^Y$ and $|g'(y) - g''(y)| < \delta$ for any $y \in \widetilde{K}_N(x)$. We shall consider the sequences $(g'_n)_{n\in\mathbb{N}}$, $(g''_n)_{n\in\mathbb{N}} \subset C_p(Y)$ such that $g'_n|_{\widetilde{K}_n(x)} = g'|_{\widetilde{K}_n(x)}$ and $g''_n|_{\widetilde{K}_n(x)} = g''|_{\widetilde{K}_n(x)}$ for all $n \in \mathbb{N}$. Then $|\,h(g'_N)(x) - h(g'_n)(x)| \leq 1/N < \varepsilon/4$ and $|\,h(g''_N)(x) - h(g''_n)(x)| \leq 1/N < \varepsilon/4$ for all $n \geq N$. It is clear that $\lim_{n\to\infty} h(g'_n)(x) = \bar{h}(g')(x)$ and $\lim_{n\to\infty} h(g''_n)(x) = \bar{h}(g'')(x)$. Hence, passing to the limit in the last inequalities as $n \to \infty$, we obtain inequalities $|\,h(g'_N)(x) - \bar{h}(g')(x)| < \varepsilon/4$ and $|\,h(g''_N)(x) - \bar{h}(g'')(x)| < \varepsilon/4$. In addition, $|\,g'_N(y) - g''_N(y)| < \delta$ for all $y \in \widetilde{K}_N(x)$, therefore, from (3.1) it follows that $|\,h(g'_N)(x) - h(g''_N)(x)| < \varepsilon/2$. Then

$$\begin{aligned} &|\,\bar{h}(g')(x) - \bar{h}(g'')(x)| \\ &= |\,(\bar{h}(g')(x) - h(g'_N)(x)) + (h(g'_N)(x) - h(g''_N)(x)) + (h(g''_N)(x) - \bar{h}(g'')(x))| \\ &\qquad < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \end{aligned}$$

The proof for $\bar{H}$ is analogous. $\square$

We now prove that $\bar{H} = \bar{h}^{-1}$. Let $g \in \mathbb{R}^Y$, $y \in Y$. We shall show that $\bar{H}(\bar{h}(g))(y) = g(y)$. For any natural numbers $n, m$ put

$$\widetilde{K}_{n,m}(y) = \bigcup_{x \in \widetilde{L}_n(y)} \widetilde{K}_m(x).$$

Take a sequence $(f_n)_{n\in\mathbb{N}} \subset C_p(X)$ such that $f_n|_{\widetilde{L}_n(y)} = \bar{h}(g)|_{\widetilde{L}_n(y)}$ for every natural number $n$. Put $g_n = h^{-1}(f_n) \in C_p(Y)$. Then $\bar{H}(\bar{h}(g))(y) = \lim_{n\to\infty} g_n(y)$. Since the mapping $\delta \mapsto b(y, L, \delta)$ is continuous at zero, for any natural number $n$ there is $\delta_n > 0$ such that for any two functions $g', g'' \in C_p(Y)$ the following implication holds:

$$\text{(3.2)} \quad \Big(|\, h(g')(x) - h(g'')(x)| < \delta_n \text{ for all } x \in \widetilde{L}_n(y)\Big) \Rightarrow |\, g'(y) - g''(y)| < 2/n\,.$$

Take a sequence $(g'_m)_{m\in\mathbb{N}} \subset C_p(Y)$ such that $g'_m|_{\widetilde{K}_{m,m}(y)} = g|_{\widetilde{K}_{m,m}(y)}$ for all natural number $m$. Then for each $x \in \widetilde{L}(y)$ there is natural number $m_x$ such that for any $m \geq m_x$ we have $g'_m|_{\widetilde{K}_m(x)} = g|_{\widetilde{K}_m(x)}$; hence, $\lim_{m\to\infty} h(g'_m)(x) = \bar{h}(g)(x)$ for each $x \in \widetilde{L}(y)$. Therefore, for any natural number $n$ there is $m_n \in \mathbb{N}$ such that $|\, h(g'_{m_n})(x) - \bar{h}(g)(x)| < \delta_n$ for each $x \in \widetilde{L}_n(y)$; hence,

$$|\, h(g'_{m_n})(x) - h(g_n)(x)| = |\, h(g'_{m_n})(x) - f_n(x)| = |\, h(g'_{m_n})(x) - \bar{h}(g)(x)| < \delta_n$$

for each $x \in \widetilde{L}_n(y)$. From (3.2) it follows that $|\, g'_{m_n}(y) - g_n(y)| < 2/n$. Since $y \in \widetilde{K}_{1,1}(y)$ by Lemma 3.2, we obtain the equality $g'_m(y) = g(y)$ for every natural number $m$, which implies that $|\, g(y) - g_n(y)| < 2/n$. Passing to the limit in this inequality as $n \to \infty$, we obtain that $g(y) = \bar{H}(\bar{h}(g))(y)$. It can be proved analogously that $\bar{h}(\bar{H}(f))) = f$ for any $f \in \mathbb{R}^X$, which implies that $\bar{H} = \bar{h}^{-1}$. This completes the proof of Theorem 3.1. $\square$

**Lemma 3.4.** *Let $U$ be a functionally open subset of $X$. Then there is an expanding sequence $(F_n)_{n\in\mathbb{N}}$ of functionally closed subset of $X$ such that $U = \bigcup_{n\in\mathbb{N}} F_n$.*

P r o o f. Let $f\colon X \to [0,1]$ be a continuous function such that $U = f^{-1}(0,1]$. Put $F_n = f^{-1}(\frac{1}{n}, 1]$ for every $n \in \mathbb{N}$. It is easy to verify that each set $F_n$ is functionally closed and $U = \bigcup_{n\in\mathbb{N}} F_n$. $\square$

**Lemma 3.5.** *Let $U$ and $V$ be functionally closed subset of $X$. Then there is a continuous function $f\colon X \to [0,1]$ such that $f^{-1}(0) = U$, $f^{-1}(1) = V$.*

Proof. See [2], page 43.

**Lemma 3.6.** *Let $S$ and $T$ be nonempty sets and let $\{X_{s,\,t}\}_{(s,\,t)\in S\times T}$ be a family of subsets of $X$. Then*

$$\bigcup_{s\in S}\bigcap_{t\in T} X_{s,\,t} = \bigcap_{f\in T^S}\bigcup_{s\in S} X_{s,\,f(s)}.$$

Proof. Put $A = \bigcup_{s\in S}\bigcap_{t\in T} X_{s,\,t}$, $B = \bigcap_{f\in T^S}\bigcup_{s\in S} X_{s,\,f(s)}$.

Let $x \in A$. Then there is $s_0 \in S$ such that $x \in X_{s_0,\,t}$ for all $t \in T$. Let $f \in T^S$. Then $x \in X_{s_0,\,f(s_0)}$, hence $x \in \bigcup_{s\in S} X_{s,\,f(s)}$, which implies that $x \in B$, i.e., that $A \subset B$.

Let $x \notin A$. Then for each $s \in S$ there is $t = f(s) \in T$ such that $x \notin X_{s,\,f(s)}$; hence $x \notin \bigcup_{s\in S} X_{s,\,f(s)}$ and $x \notin B$, i.e., $B \subset A$. □

The previous lemma implies the following corollary.

**Corollary 3.7.** *If in the condition of the previous lemma we require that $S$ and $T$ should be countable and each set $X_{s,\,t}$ should be open in $X$, then the set $\bigcup_{s\in S}\bigcap_{t\in T} X_{s,\,t}$ is a $G_c$-subset of $X$.*

**Lemma 3.8.** *The set $B_s = \{x \in X\colon K(x) \cap (V \setminus V_s) \neq \varnothing\}$ is a $G_\delta$-subset of $X$.*

Proof. Let $(F_n^s)_{n\in\mathbb{N}}$ be a decomposition of $V_s$ satisfying the following conditions:

$$F_n^s \in \mathcal{C} \text{ and } F_n^s \subset F_{n+1}^s \text{ for all } n \in \mathbb{N}.$$

Put $U_n = V \setminus F_n^s$. Then $V \setminus V_s = \bigcap_{n\in\mathbb{N}} U_n$, where each $U_n$ is open and $U_n \supset U_{n+1}$ for all $n \in \mathbb{N}$. Let $C_s = \bigcap_{n\in\mathbb{N}} K^{-1}(U_n)$. We shall show that $B_s = C_s$. The inclusion $B_s \subset C_s$ is obvious. Let $x \in C_s$. Since $K(x)$ is finite, there is $y \in K(x)$ such that $y \in U_n$ for all $n$ in some infinite subset of $\mathbb{N}$. Hence, $y \in \bigcap_{n\in\mathbb{N}} U_n$ and $x \in B_s$. By Corollary 1.5 on page 147, the set $K^{-1}(U_n)$ is a $G_\delta$-subset of $X$ for all $n \in \mathbb{N}$. This implies that $B_s$, as a countable intersection of $G_\delta$-sets, is a $G_\delta$-set. □

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*Received June 14, 2011*