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OPTIMAL INVESTMENT UNDER STOCHASTIC VOLATILITY AND POWER TYPE UTILITY FUNCTION

Abbes Benchaabane, Azzedine Benchettah

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Abstract. In this work we will study a problem of optimal investment in financial markets with stochastic volatility with small parameter. We used the averaging method of Bogoliubov for limited development for the optimal strategies when the small parameter of the model tends to zero and the limit for the optimal strategy and demonstrated the convergence of these optimal strategies.

1. Introduction. The fundamental stochastic model of optimal investment was first introduced by Merton [12] who constructed explicit solution under the assumption that the stock price follows a geometric Brownian motion and the individual preference are of special type, specifically, the utilities are either of Constant Relative Risk Aversion (CRRA) type, including the logarithmic, or of exponential type. The case of general utilities was analyzed in Karatzas at al.[8, 9] who produced the value function in closed form.

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Financial markets are sometimes quite calm and at other times much more volatile. To capture the complicated behavior of stock prices and other derivatives, it is necessary to take into consideration frequent changes of the volatility, it is more suitable to use a stochastic process to model the variation of the volatility; see [6], [5], [4] and [11].

Recently, in the book [2] and subsequent papers [1], [3] and [10], a class of volatility models has been studied in detail. Under the setup of mean reversion, two time scale methods are used. A number of interesting results were obtained.

The paper is organized as follows. In Section 2, we describe the model and give preliminary results on the value function, as well as on special cases of the Merton model. In Section 3 we introduce the class of stochastic volatility models, we discuss an approximation method based on asymptotic analysis that is effective for models in which asset prices have randomly varying volatility, we show that the optimal control for the Merton problem with constant volatility approximates the expectation of the optimal control for the some problem with stochastic volatility and obtains the desired error bounds.

2. Class of constant volatility models. We start with brief review of the Merton model. To this end, we consider an economy with two securities, a bond and a stock. The bond's price B_s is deterministic and evolves

$$\begin{cases} dB_s = rB_s ds \\ B_0 = B > 0, \end{cases} \tag{2.1}$$

with $r > 0$ being the interest rate. The stock price is modeled as a diffusion process S_s the stochastic differential equation

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ S_0 = S > 0. \end{cases} \tag{2.2}$$

The market parameters μ and σ are respectively the mean rate of return and the volatility, it is assumed that $\mu - r > 0$ and $\sigma > 0$. The process W_t is standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

Trading takes place between the bond and the stock accounts continuously in time, in the trading horizon $[t, T]$, $0 \leq t \leq T$. Denoting by u_s and $1 - u_s$ the fraction of wealth in the stock and bond (respectively) at time s, for $s \in [t, T]$. Using the price equation (2.1, 2.2) one may easily derive the equation for the sate process

$$\begin{cases} dX_s = X_s \left[(r + (\mu - r) u_s) ds + \sigma u_s dW_s \right] \\ X_t = x > 0. \end{cases} \tag{2.3}$$

'We assume that the wealth process must satisfy the state constraint

$$X_s > 0 \quad a.e.\ t \le s \le T. \tag{2.4}$$

A stochastic control process $(u_s)_{t\le s\le T}$ is called admissible if it is $\mathcal{F}_t$-progressively measurable, where $\mathcal{F}_s = \sigma(W_v;\ t \le v \le s)$, satisfies the integrability condition

$$\mathbb{E}\int_t^T u_s^2 dt < \infty, \tag{2.5}$$

and it is such that the state constraint (2.4) is satisfied, we denote by $\mathcal{U}$ the class of all admissible control processes.

For $u \in \mathcal{U}$ we define the cost function

$$V(t,x;u) = \mathbb{E}\left[\frac{(X_T^u)^p}{p} \mid X_t^u = x\right]. \tag{2.6}$$

The goal of the investor is to choose the strategy (u_s) to maximize the cost function at some given finite terminal time T, we define :

$$V(t,x) = \sup_{u\in\mathcal{U}} V(t,x;u). \tag{2.7}$$

By Bellman Principle $V(t,x)$ satisfies the nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$\frac{\partial V}{\partial t} + \sup_u \left\{\frac{1}{2}\sigma^2 u^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r + (\mu - r)u)x\frac{\partial V}{\partial x}\right\} = 0, \tag{2.8}$$

with terminal condition

$$V(T,x) = \frac{x^p}{p}. \tag{2.9}$$

The special form of the chosen utility function motivates the representation

$$V(t,x) = \frac{x^p}{p}c(t).$$

This leads to the linear ordinary differential equation for $c(t)$,

$$\frac{\partial c}{\partial t} + \alpha p \sup_u \left\{-\frac{1}{2}\sigma^2 u^2 (p-1) + r + (\mu - r)u\right\} = 0,$$

with $c(T) = 1$. The supremum is attained at

$$u_t^* = \frac{\mu - r}{\sigma^2(1-p)}. \tag{2.10}$$

The corresponding maximum expected utility is given by

$$V(t,x) = V(t,x,u^*) = \frac{x^p}{p} \exp\left[(T-t)p\left(r + \frac{(\mu - r)^2}{2\sigma^2(1-p)}\right)\right]. \tag{2.11}$$

The following result was proved in Karatzas [9]

Theorem 1. *The optimal value of The value $V(t,x,u)$ is given by* (2.11)*, which is the unique increasing and concave solution of the Hamilton-Jacobi-Bellman equation* (2.8) *and* (2.9)*.*

3. Class of stochastic volatility models. We consider now the Merton optimal portfolio, under the assumption of stochastic volatility. Then the wealth process evolves according to the following system

$$\begin{cases} dX_s = X_s\left[(r + (\mu - r)u_s)ds + f(Y_s)u_s dW_s^x\right] \\ X_0 = x > 0. \end{cases} \tag{3.1}$$

The wealth process must also satisfy the state constraint

$$X_s > 0 \ a.e. \quad s \in [t,T].$$

The volatility $f(.)$ is driven by another stochastic process Y_s given by

$$\begin{cases} dY_s = -\alpha Y_s dt + \beta dW_s^y \\ Y_0 = y \in \mathbb{R}. \end{cases} \tag{3.2}$$

The process W_s^x and W_s^y are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$. The function $f : \mathbb{R} \to \mathbb{R}^+$ satisfy the global Lipschitz condition

$$|f(y) - f(\overline{y})| \leq K\,|y - \overline{y}|\,, \tag{3.3}$$

and the linear growth condition

$$f(y) \leq K(1+y), \tag{3.4}$$

for every y, $\overline{y} \in \mathbb{R}$, K is positive constant. Moreover,the volatility coefficient $f(y)$ satisfies $f(y) \geq N > 0$ for some constant N. The control u_s is said to be admissible if it is $\mathcal{F}_s$-progressively measurable processes, where $\mathcal{F}_s = \{\sigma(W_v^x, W_v^y); t \leq v \leq s\}$, satisfying condition $E\int_t^T f^2(Y_s)u_s^2 ds < \infty$ and is such that the above state constraint is satisfied. We denote by $\mathcal{A}$ the set of admissible policies. The Merton problem consists in choosing a strategy u, which maximize a given utility function at some final time T. In particular the problem can be described in terms of the value function

$$V(t,x,y) = \sup_{u\in\mathcal{A}} V(t,x,y;u), \tag{3.5}$$

where

$$V(t,x,y;u) = \mathbb{E}\left(\frac{(X_T^u)^p}{p} \mid X_t = x, Y_t = y\right). \tag{3.6}$$

By Bellman Principle $V(t,x,y)$ satisfies the nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t} + \sup_u \left\{\frac{1}{2} f^2(y)u^2x^2\frac{\partial^2 V}{\partial x^2} + (r + (\mu - r)u)x\frac{\partial V}{\partial x}\right\} \\ &\quad + \frac{1}{2}\beta^2\frac{\partial^2 V}{\partial y^2} - \alpha y\frac{\partial V}{\partial y}, \end{aligned}$$

with terminal condition

$$V(T,x,y) = \frac{x^p}{p}. \tag{3.7}$$

Using the transformation

$$V(t,x) = \frac{x^p}{p}c(t,y),$$

we obtain the following equation for c,

$$\begin{aligned} 0 &= \frac{\partial c}{\partial t} + pc\sup_u \left\{-\frac{1}{2}(p-1)f^2(y)u^2 + (r + (\mu - r)u)\right\} \\ &\quad + \frac{1}{2}\beta^2\frac{\partial^2 c}{\partial y^2} - \alpha y\frac{\partial c}{\partial y}, \end{aligned}$$

with $c(T,y) = 1$. The supremum is attained at

$$u_t^* = \frac{\mu - r}{f^2(y)(1-p)}. \tag{3.8}$$

For the proof of the following result we refer the reader to [13].

Proposition 1. *The value function V is given by*

$$V(t,x,y) = \frac{x^p}{p} c(t,y)$$

where $c\colon\ [0,T] \times \mathbb{R} \to \mathbb{R}^+$ solves the linear parabolic equation

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\beta^2 \dfrac{\partial^2 c}{\partial y^2} - \alpha y \dfrac{\partial c}{\partial y} + pc\left(r + \dfrac{(\mu - r)^2}{2f^2(y)(1-p)}\right) = 0 \\ c(T,y) = 1. \end{cases} \tag{3.9}$$

3.1. Singular perturbation analysis. We introduce the scaling

$$\alpha = \frac{1}{\epsilon}, \text{ and } \beta = \delta^\epsilon / \sqrt{\epsilon}, \tag{3.10}$$

where ϵ is small positive parameter, then the processes $X_s^\epsilon, Y_s^\epsilon$ can be treated as "slow" and "fast" variable. We consider a singularly perturbed stochastic control problem on a finite time interval $[0,T]$ for a system described by the variable $(X_s^\epsilon, Y_s^\epsilon)$, then the wealth process X_s^ϵ satisfies

$$dX_s^\epsilon = X_s^\epsilon \left[(r + (\mu - r)u_s)ds + f(Y_s^\epsilon) u_s dW_s^x \right], \quad X_0^\epsilon = x, \tag{3.11}$$

where

$$\epsilon dY_s^\epsilon = -Y_s^\epsilon ds + \delta^\epsilon \sqrt{\epsilon} dW_s^y, \quad Y_0^\epsilon = y. \tag{3.12}$$

Then the value function of the investor is

$$V^\epsilon(t,x,y) = \sup_{u \in \mathcal{A}} V^\epsilon(t,x,y;u). \tag{3.13}$$

where

$$V^\epsilon(t,x,y;u) = \mathbb{E}\left(\frac{(X_T^\epsilon)^p}{p} \mid X_t^\epsilon = x, Y_t^\epsilon = y \right). \tag{3.14}$$

Our aim is to study the control problem below if δ^ϵ is constant, which is related to the Bogoliubov average principle. In such a case the fast variable may be oscillatory and it does not converge in probability see [7].

The distribution of Y_t in (3.2) is the Gaussian distribution $\mathcal{N}\left(0, \frac{\beta^2}{2\alpha}\right)$. Denoting its variance by $v^2 = \frac{\beta^2}{2\alpha}$. The operator $\alpha\mathcal{L}_0$ is the infinitesimal generator of the Ornestrin-Uhlenbeck process Y_t with $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = v^2\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}. \tag{3.15}$$

The process Y_t has an invariant distribution $\mathcal{N}(0, v^2)$, which admits the density $\phi(y)$ obtained by solving the adjoint equation

$$\mathcal{L}_0^*\phi = 0, \tag{3.16}$$

where $\mathcal{L}_0^*$ denotes the adjoint of $\mathcal{L}_0$. The density is explicitly given by

$$\phi(y) = \frac{1}{\sqrt{2\pi v^2}}\exp\left(-\frac{y^2}{2v^2}\right) \tag{3.17}$$

Let $\langle . \rangle$ denote the average with respect to this invariant distribution

$$\langle g \rangle = \int_{-\infty}^{+\infty} g(y)\phi(y)dy. \tag{3.18}$$

Given a bounded function g, by the ergodic theorem, the long time average of $g(Y_t)$ is close to the average distribution

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t g(Y_s)ds = \langle g \rangle. \tag{3.19}$$

We take advantage of fast mean reversion in volatility and write

$$\alpha = \frac{1}{\epsilon} \text{ and } \beta = \frac{\sqrt{2}v}{\sqrt{\epsilon}}, \tag{3.20}$$

the constant $\epsilon > 0$ is small to represent fast mean-reversion. Now $1/\epsilon$ is a measure of the local speed of process and we are interested in asymptotic expansions in the limit $\epsilon \downarrow 0$. The equation (3.9) becomes

$$\begin{cases} \frac{1}{\epsilon}\mathcal{L}_0 c^\epsilon + \mathcal{L}_2 c^\epsilon = 0 \\ c^\epsilon(T, y) = 1, \end{cases} \tag{3.21}$$

where

$$\mathcal{L}_2 c^\epsilon = \frac{\partial c^\epsilon}{\partial t} + p\left(r + \frac{(\mu - r)^2}{2f^2(y)(1-p)}\right)c^\epsilon. \tag{3.22}$$

We present here the formal asymptotic expansion computed which leads to a (first-order in $\sqrt{\epsilon}$) approximation $c^\epsilon \approx c_0 + \sqrt{\epsilon} c_1$. We start by writing c^ϵ in powers of $\sqrt{\epsilon}$

$$c^\epsilon = c_0 + \sqrt{\epsilon} c_1 + \epsilon c_2 + \epsilon\sqrt{\epsilon} c_3 + \cdots \tag{3.23}$$

Substituting (3.23) in (3.21) leads to

$$0 = \frac{1}{\epsilon}\mathcal{L}_0 c_0 + (\mathcal{L}_2 c_0 + \mathcal{L}_0 c_2) + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_0 c_1 \tag{3.24}$$

$$+\sqrt{\epsilon}\,(\mathcal{L}_2 c_1 + \mathcal{L}_0 c_3) + \cdots \tag{3.25}$$

We shall next obtain expressions of $c_n,\ n \geq 0$, by successively equating any order term in (3.24) to zero, we must have

$$\begin{cases} \mathcal{L}_0 c_0 = 0, \\ \mathcal{L}_0 c_1 = 0, \\ \mathcal{L}_0 c_n + \mathcal{L}_2 c_{n-2} = 0, \quad \forall n \geq 2. \end{cases} \tag{3.26}$$

We will need to solve the Poisson equation associated with $\mathcal{L}_0$:

$$\mathcal{L}_0 \chi + g = 0, \tag{3.27}$$

which requires the solvability condition (centering condition)

$$\langle g \rangle = 0, \tag{3.28}$$

in order to admit solutions. Properties of this equation and its solutions are recalled in (Appendix A)

Consider the first equation

$$\mathcal{L}_0 c_0 = 0. \tag{3.29}$$

The operator $\mathcal{L}_0$ is the generator of an ergodic Markov process and takes derivatives with respect to y, any function independent of y satisfies this equation. Therefore we seek solutions which are independent of y : $c_0 = c_0(t)$ with the terminal condition $c_0(T) = 1$.

Consider next

$$\mathcal{L}_2 c_0 + \mathcal{L}_0 c_2 = 0, \tag{3.30}$$

$\mathcal{L}_2$ does not involve the derivative with respect to y; but the variable y is present through $f(y)$, we have a Poisson equation (in y) for c_2. The solvability condition is that $\mathcal{L}_2 c_0$ must be centered with respect to the invariant distribution of the process Y_t. Therefore

$$\langle \mathcal{L}_2 c_0 \rangle (t,y) = 0. \tag{3.31}$$

The averaged operator is

$$\begin{aligned}
\langle \mathcal{L}_2 c_0 \rangle (t,y) &= \int_{-\infty}^{+\infty} (\mathcal{L}_2 c_0)(t,u)\phi(u)du \\
&= \int_{-\infty}^{+\infty} \left(\frac{\partial c_0}{\partial t}(t) + p\left(r + \frac{(\mu-r)^2}{2f^2(y)(1-p)} \right) c_0(t) \right) \phi(u)du \\
&= \frac{\partial c_0}{\partial t}(t) + pc_0 \left(r + \frac{(\mu-r)^2}{2(1-p)} \left(\int_{-\infty}^{+\infty} \frac{1}{f^2(u)} \phi(u)du \right) \right) \\
&= \frac{\partial c_0}{\partial t}(t) + pc_0 \left(r + \frac{(\mu-r)^2}{2(1-p)} \left\langle \frac{1}{f^2} \right\rangle \right).
\end{aligned}$$

If we set

$$\sigma^* = \frac{1}{\sqrt{\langle 1/f^2 \rangle}}, \tag{3.32}$$

then

$$c_0 = \exp\left[(T-t)p\left(r + \frac{(\mu-r)^2}{2(\sigma^*)^2(1-p)} \right) \right]. \tag{3.33}$$

Similarly of c_0 in order to calculate c_1, we must have

$$\mathcal{L}_0 c_1 = 0 \tag{3.34}$$

$$\mathcal{L}_2 c_1 + \mathcal{L}_0 c_3 = 0 \tag{3.35}$$

it follows that

$$c_1 = \exp\left[(T-t)p\left(r + \frac{(\mu-r)^2}{2(\sigma^*)^2(1-p)} \right) \right], \tag{3.36}$$

then the first-order maximum expected utility is

$$\begin{aligned}
V^*(t,x) &= \frac{x^p}{p} \exp\left[(T-t)p\left(r + \frac{(\mu-r)^2}{2(\sigma^*)^2(1-p)} \right) \right] \\
&\quad + \sqrt{\epsilon}\frac{x^p}{p} \exp\left[(T-t)p\left(r + \frac{(\mu-r)^2}{2(\sigma^*)^2(1-p)} \right) \right].
\end{aligned}$$

Theorem 2. *Under the boundedness assumption on the function of the volatility f, at any fixed time $t > T$, and fixed point $x >, y \in \mathbb{R}$*

$$|V^{\epsilon}(t,x,y) - V^{*}(t,x)| = o(\epsilon^{1-\eta}), \tag{3.37}$$

for any $\eta > 0$.

Proof. We first introduce some additional notation. Define the error Z^{ϵ} in the approximation of c^{ϵ} in (3.23) by

$$c^{\epsilon} = c_0 + \sqrt{\epsilon}c_1 + \epsilon c_2 + \epsilon\sqrt{\epsilon}c_3 - Z^{\epsilon}$$

Setting

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon}\mathcal{L}_0 c^{\epsilon} + \mathcal{L}_2 c^{\epsilon},$$

we can write,

$$\begin{aligned} \mathcal{L}^{\epsilon} Z^{\epsilon} &= \mathcal{L}^{\epsilon}\left[c_0 + \sqrt{\epsilon}c_1 + \epsilon c_2 + \epsilon\sqrt{\epsilon}c_3 - c^{\epsilon}\right] \\ &= \frac{1}{\epsilon}\mathcal{L}_0 c_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_0 c_1 \\ &\quad + (\mathcal{L}_0 c_2 + \mathcal{L}_2 c_0) + \sqrt{\epsilon}(\mathcal{L}_0 c_3 + \mathcal{L}_2 c_1) \\ &\quad + (\epsilon\mathcal{L}_2 c_2 + \epsilon\sqrt{\epsilon}\mathcal{L}_2 c_3) \\ &= \epsilon(\mathcal{L}_2 c_2 + \sqrt{\epsilon}\mathcal{L}_2 c_3)) \\ &= (\epsilon + \epsilon\sqrt{\epsilon})\mathcal{L}_2 c_2 =: G^{\epsilon} \end{aligned} \tag{3.38}$$

because $\mathcal{L}^{\epsilon}$ solves the original equation (3.21): $\mathcal{L}^{\epsilon} c^{\epsilon} = 0$ and we choose c_0, c_1, c_2, c_3 to cancel the first four terms.

Using that

$$\begin{aligned} \mathcal{L}_2 c_0 &= \mathcal{L}_2 c_0 - \langle \mathcal{L}_2 c_0 \rangle \\ &= \frac{1}{2}\frac{p(\mu - r)^2}{(1-p)}\left(1/f^2(y) - 1/(\sigma^*)^2\right)c_0, \end{aligned}$$

we choose

$$c_2(t,y) = -\frac{1}{2}\frac{p(\mu - r)^2}{(1-p)}c_0(t)\chi(y), \tag{3.39}$$

with $\chi(y)$ is a solution of the Poisson equation

$$\mathcal{L}_0\chi(y) = 1/f^2(y) - 1/(\sigma^*)^2.$$

So that

$$\begin{aligned} \mathcal{L}_0 c_2(t,y) &= -\frac{1}{2}\frac{p(\mu-r)^2}{(1-p)}\left(1/f^2(y) - 1/(\sigma^*)^2\right)c_0 \\ &= -\mathcal{L}_2 c_0(t). \end{aligned}$$

Obviously

$$\mathcal{L}_2 c_2 = -\frac{1}{4}\left(\frac{p(\mu-r)^2}{(1-p)}\right)^2 c_0(t)\chi(y)\left(1/f^2(y) - 1/(\sigma^*)^2\right),$$

By boundedness assumption of f,there exist a constant $K>0$

$$|\mathcal{L}_2 c_2| \le K\chi(y). \tag{3.40}$$

At the terminal time T, we have

$$\begin{aligned} Z^\epsilon(T,y) &= \epsilon c_2(T,y) + \epsilon\sqrt{\epsilon}c_3(T,y) \\ &= (\epsilon+\epsilon\sqrt{\epsilon})c_2(T,y) \\ &= -\frac{1(\epsilon+\epsilon\sqrt{\epsilon})}{2}\frac{p(\mu-r)^2}{(1-p)}\chi(y) \end{aligned}$$

It follows that there exist a constant $K>0$

$$|Z^\epsilon(T,y)| \le K(\epsilon+\epsilon\sqrt{\epsilon})\chi(y)$$

where we have used the terminal conditions $c^\epsilon(T,y) - c_0(T,y) = c_1(T,y) = 0$.

Because of the smoothness of G^ϵ and H^ϵ, and the regularity of the coefficients of the diffusion (X^ϵ, Y^ϵ), we have the probabilistic representation of the solution of equation (3.38), $\mathcal{L}^\epsilon Z^\epsilon = G^\epsilon$ with terminal condition H^ϵ :

$$\begin{aligned} Z^\epsilon(t,y) &= \mathbb{E}_{t,y}\Bigg[Z^\epsilon(T,Y_T^\epsilon)\exp\left(\int_t^T p\left(r+\frac{(\mu-r)^2}{2f^2(Y_s^\epsilon)(1-p)}\right)ds\right) \\ &\quad + \int_t^T G^\epsilon(s,Y_s^\epsilon)\exp\left(\int_t^s p\left(r+\frac{(\mu-r)^2}{2f^2(Y_\lambda^\epsilon)(1-p)}\right)d\lambda\right)\Bigg]ds, \end{aligned}$$

we obtain the upper bounds, for some constant $K>0$

$$|Z^\epsilon(t,y)| \le K(\epsilon+\epsilon\sqrt{\epsilon})\left(|\chi(y)| + \mathbb{E}\left[|\chi(Y_s^\epsilon)| \mid Y_t^\epsilon = y\right]\right)$$

where $t \le s \le T$.

It follows from (A.2) in Appendix, and classical a priory estimates on the moments of the process Y_t^ϵ which are uniform in ϵ and by simple time change $t = \epsilon t^{'}$ that there exists a constant $K > 0$ (which may depend on y) such that

$$\mathbb{E}\left[|\chi(Y_s^\epsilon)| \mid Y_t^\epsilon = y\right] \leq K < \infty,$$

Obviously, for (t, y) fixed with $t < T$:

$$|Z^\epsilon(t, y)| \leq K(\epsilon + \epsilon\sqrt{\epsilon}),$$

and therefore also for (t, y) fixed with $t < T$:

$$\begin{aligned} \left|c_0 + \sqrt{\epsilon}c_1 - c^\epsilon\right| &= \left|(\epsilon + \epsilon\sqrt{\epsilon})c_2(t, y) - Z^\epsilon(t, y)\right| \\ &\leq K(\epsilon + \epsilon\sqrt{\epsilon}) \end{aligned}$$

since c_2 evaluated for $t < T$ can also be bounded using (3.39) and (A.2).

It follows that, for (t, x, y) fixed with $t < T$

$$|V^\epsilon(t, x, y) - V^*(t, x)| \leq K(\epsilon + \epsilon\sqrt{\epsilon}),$$

then

$$\lim_{\epsilon \longrightarrow 0} \frac{|V^\epsilon(t, x, y) - V^*(t, x)|}{\epsilon^{1-\eta}} = 0,$$

for any $\eta > 0$. □

Appendix A. Solution of the Poisson equation. Let χ solve

$$\mathcal{L}_0\chi + g = 0, \tag{A.1}$$

with $\mathcal{L}_0$ defined as in (3.15) and with g satisfying the centering condition

$$\langle g \rangle = 0,$$

where the averaging is done with respect to the invariant distribution associated with the infinitesimal generator $\mathcal{L}_0$ (see (3.18) for an explicit formula). We note that the Poisson equation (A.1) is

$$\mathcal{L}_0\chi = \frac{v^2}{\phi}(\phi\chi^{'})^{'} = -g,$$

so that, using the explicit form of the differential operator $\mathcal{L}_0$, one can easily deduce that

$$\phi(y)\chi'(y) = \frac{-1}{v^2}\int_{-\infty}^{y} g(z)\phi(z)dz = \frac{1}{v^2}\int_{y}^{\infty} g(z)\phi(z)dz,$$

with ϕ being the probability density of the invariant distribution $\mathcal{N}(0, v^2)$ associated with $\mathcal{L}_0$. From this it follows that if g is bounded

$$|\chi(y)| \le N_2(1 + \log(1 + |y|)). \tag{A.2}$$

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Abbes Benchaabane
Laboratoire Raphaël Salem
UMR CNRS 6085
University of Rouen
France
and
Department of Mathematics
Faculty of Sciences
University of Guelma, Algeria
e-mail: abbesbenchaabane@gmail.com

Azzedine Benchettah
Laboratoire of Numerical Analysis,
Optimization and Statistics
University of Annaba, Algeria
e-mail: abenchetah@hotmail.com

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