Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **37** (2011), 277–282

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DIRAC TYPE CONDITION AND HAMILTONIAN GRAPHS

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Communicated by V. Drensky

Abstract. In 1952, Dirac introduced the degree type condition and proved that if G is a connected graph of order $n \geq 3$ such that its minimum degree satisfies $\delta(G) \geq n/2$, then G is Hamiltonian. In this paper we investigate a further condition and prove that if G is a connected graph of order $n \geq 3$ such that $\delta(G) \geq (n-2)/2$, then G is Hamiltonian or G belongs to four classes of well-structured exceptional graphs.

1. Introduction. We consider only finite undirected graphs without loops or multiple edges. For a graph G, let $V(G)$ be the vertex set and let $E(G)$ be the edge set of G. The complete graph of order n is denoted by K_n and the empty graph of order n is denoted by $\overline{K}_n$. For two vertices u and v, let $d(u,v)$ be the length of the shortest path between vertices u and v in G, i.e., $d(u,v)$ is the distance between u and v. The minimum degree of the graph G is denoted by $\delta(G)$. For a subgraph H of the graph G and a subset S of $V(G)$, let $N_H(S)$ be the set of vertices in H that are adjacent to some vertex in S and let the

2010 *Mathematics Subject Classification:* 05C38, 05C45.
Key words: Dirac type condition, sufficient condition, Hamiltonian graph.

cardinality of $N_H(S)$ be $d_H(S)$. Furthermore, we denote by $G-H$ the subgraph of G induced by $V(G)-V(H)$. For each integer $m \geq 3$, let $C_m = x_1x_2\cdots x_mx_1$ denote a cycle of order m and let P be a path of a section of C_m (if $|V(P)| = m$, then $P = C_m$). Define

$$N_P^+(u) = \{x_{i+1} : x_i \in N_P(u)\}, N_P^-(u) = \{x_{i-1} : x_i \in N_P(u)\},$$

and define $N_P^{\pm}(u) = N_P^+(u) \cup N_P^-(u)$, where subscripts are taken modulo m.

If no ambiguity can arise we sometimes write $N(u)$ instead of $N_G(u)$, δ instead of $\delta(G)$, etc. Other notations can be found in [1, 3].

In 1952, Dirac proved the following well-known result on Hamiltonian graphs.

Theorem 1 (Dirac [2]). *If G is a connected graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then G is Hamiltonian.*

Recently we also have investigated some Hamiltonian graphs under other sufficient conditions such as neighborhood union conditions [4, 5].

In this paper, our purpose is to present the following result, which extends the above Theorem 1.

Theorem 2. *If G is a connected graph of order $n \geq 3$ and $\delta(G) \geq (n-2)/2$, then G is Hamiltonian or G has one of the following four types:*

$$G_{(n-2)/2} \vee (\overline{K}_{(n-2)/2} \cup K_2), \quad G_{(n-2)/2} \vee \overline{K}_{(n+2)/2},$$

$$G_{(n-1)/2} \vee \overline{K}_{(n+1)/2}, \quad w : (K_h \cup K_t^-).$$

Here h is an integer, G_h denotes an arbitrary graph of order h, and $\overline{K}_h$ is the empty graph with h vertices and without edges. The join operator $A \vee B$ of two graphs A and B is the graph constructed from A and B by adding all edges joining the vertices of A and the vertices of B. The graph $A \cup B$ denotes the disjoint union of the graphs A and B. The graph $w : (K_h \cup K_t^-)$ is defined by the properties: K_t^- is a graph of order t and minimum degree $\delta(K_t^-) \geq t-2$, w is a cut vertex which is adjacent to at least $(n-2)/2$ vertices of the disjoint graphs K_h and K_t^-, $(n-2)/2 \leq h \leq t \leq n/2$.

Corollary 3. *If G is a 2-connected graph of order $n \geq 3$ and $\delta(G) \geq (n-2)/2$, then G is Hamiltonian or*

$$G \in \{G_{(n-2)/2} \vee (\overline{K}_{(n-2)/2} \cup K_2), G_{(n-2)/2} \vee \overline{K}_{(n+2)/2}, G_{(n-1)/2} \vee \overline{K}_{(n+1)/2}.$$

2. The proof of main Theorem.

Proof of Theorem 2. Assume that G satisfies the condition of Theorem 2 and it is not Hamiltonian. Then let $C_m = x_1x_2\cdots x_mx_1$ be the longest cycle of G and let H be a component of $G - C_m$. We consider two cases.

Case 1. G is 2-connected.

In this case, there must exist vertices $u, v \in V(H)$ such that $x_i \in N_{C_m}(u)$ and $x_j \in N_{C_m}(v)$. (If $|V(H)| = 1$, then $u = v$.) Now we claim that $d(x_{i+1}) + d(x_{j+1}) \le n - |V(H)|$. Otherwise, if the claim is false, let P and R denote, respectively, the path $x_{i+1}x_{i+2}\cdots x_j$ of C_m and the path $x_{j+1}x_{j+2}\cdots x_i$ of C_m. Then, clearly, none of $N_P^+(x_{j+1}) \cup N_R^-(x_{j+1})$ are adjacent to x_{i+1}. (For example, if $x_k \in N_P^+(x_{j+1})$ is adjacent to x_{i+1}, let T be a path in H which has two end-vertices adjacent to x_i and x_j, respectively. Then the cycle

$$x_iTx_jx_{j-1}\cdots x_kx_{k-1}\cdots x_{i+1}x_{k-1}x_{j+1}x_{j+2}\cdots x_i$$

is longer than C_m, a contradiction.) Clearly $|N_P^+(x_{j+1}) \cup N_R^-(x_{j+1})| = |N_{C_m}(x_{j+1})| - |\{x_{j+1}\}|$. Also, none of $N_{G-C_m}(x_{j+1})$ are adjacent to x_{i+1}, and both x_{i+1}, x_{j+1} are not adjacent to any vertex of $\{x_{i+1}\} \cup V(H)$. Hence we can check that

$$\begin{aligned} d(x_{i+1}) &\le |V(G)| - |N_P^+(x_{j+1}) \cup N_R^-(x_{j+1})| - |N_{G-C_m}(x_{j+1})| - |\{x_{i+1}\} \cup V(H)| \\ &\le |V(G)| - (|N_{C_m}(x_{j+1})| - |\{x_{j+1}\}|) - |N_{G-C_m}(x_{j+1})| - |\{x_{i+1}\} \cup V(H)| \\ &\le n - (|N(x_{j+1})| - |\{x_{j+1}\}|) - |\{x_{i+1}\}| - |V(H)|, \end{aligned}$$

and this implies that

$$d(x_{i+1}) + d(x_{j+1}) \le n - |V(H)|. \tag{1}$$

On the other hand, by the condition of Theorem 2 we have $d(x_{i+1}) + d(x_{j+1}) \ge n - 2$. Together with the inequality (1), we have $|V(H)| \le 2$. Now we consider the subcases $|V(H)| = 1$ and $|V(H)| = 2$.

Subcase 1.1. $|V(H)| = 2$.

In this case, if $u \in V(H)$, then we have $|j - i| \ge 3$ for all pairs $x_i, x_j \in N_{C_m}(H)$ such that $\{x_{i+1}, x_{i+2}, \cdots x_{j-1}\} \cap N(u) = \emptyset$. (Otherwise, if $|j - i| \le 2$ for some $x_i, x_j \in N_{C_m}(H)$, by $|V(H)| = 2$, it is easy to construct a cycle longer than C_m, a contradiction.) Thus, we can check that

$$|N(u)| \le |V(C_m)|/3 + |V(H) \setminus \{u\}| \le (n-2)/3 + 1,$$

i.e., $d(u) \le (n-2)/3+1$. Clearly, $n \ge |V(C_m)|+|V(H)| \ge 8$, so $d(u) \le (n-3)/2$, and this contradicts the assumption of Theorem 2 that $d(u) \ge (n-2)/2$.

Subcase 1.2. $|V(H)| = 1$.

In this case, if $u \in V(H)$, then we have $|j-i| \ge 2$ for all pairs $x_i, x_j \in N_{C_m}(H)$ with $\{x_{i+1}, x_{i+2}, \cdots x_{j-1}\} \cap N(u) = \emptyset$, so $|N(u)| \le |V(C_m)|/2 \le (n-1)/2$. Together with the assumption of Theorem 2 that $d(u) \ge (n-2)/2$, we have $(n-2)/2 \le d(u) \le (n-1)/2$. Then we consider the following subcases.

Subcase 1.2.1. $d(u) = (n-2)/2$.

Then, since C_m is the longest cycle of G, the vertex u is not adjacent to two consecutive x_i, x_{i+1} on C_m. By $d(u) = (n-2)/2$, $|V(C_m)| \ge 2d(u) \ge n-2$, so $|V(G-C_m)| \le 2$.

First, let $|V(G-C_m)| = 1$. In this case, since G does not have a Hamiltonian cycle C_n and $d(u) = (n-2)/2$, it is easy to obtain $N(u) = \{x_i, x_{i+3}, x_{i+5}, x_{i+7}, \cdots x_{i-2}\}$ on C_m, i.e., there exists only one pair of consecutive neighbor vertices x_i, x_{i+3} of u on C_n such that $|(i+3)-i| = 3$ and for all other two consecutive neighbor vertices x_{i+k}, x_{i+h} of u with $\{x_{i+k+1}, x_{i+k+2}, \cdots x_{i+h-1}\} \cap N(u) = \emptyset$, we have $|(i+k)-(i+h)| = 2$.

In this case, since G does not have a Hamiltonian cycle C_n, we derive that

$$G \in G_{(n-2)/2} \vee (\overline{K}_{(n-2)/2} \cup K_2),$$

where

$$V(G_{(n-2)/2}) = \{x_i, x_{i+3}, x_{i+5}, x_{i+7}, \cdots x_{i-2}\},$$

$$\overline{K}_{(n-2)/2} = \{x_{i+4}, x_{i+6}, \cdots x_{i-1}, u\}, \quad K_2 = \{x_{i+1}, x_{i+2}\}.$$

Now, let $|V(G-C_m)| = 2$ and let $v \in V(G-C_m-u)$. Since G does not have a Hamiltonian cycle C_n and $d(u) = (n-2)/2$, it is easy to obtain $N(u) = N(v) = \{x_i, x_{i+1}, \cdots x_{i-2}\}$. This implies

$$G \in G_{(n-2)/2} \vee \overline{K}_{(n+2)/2},$$

where

$$V(G_{(n-2)/2}) = \{x_i, x_{i+2}, \cdots x_{i-2}\}, \overline{K}_{(n+2)/2} = \{x_{i+1}, x_{i+3}, \cdots x_{i-1}, u, v\}.$$

Subcase 1.2.2. $d(u) = (n-1)/2$.

In this case, since C_m is the longest cycle of G, the vertex u is not adjacent to two consecutive x_i, x_{i+1} on C_m, so $|V(C_m)| \ge 2d(u) \ge n-1$. This implies

$|V(G - C_m)| = 1$. In this case, it is easy to obtain $N(u) = \{x_i, x_{i+2}, \cdots x_{i-2}\}$ on C_m, so we have

$$G \in G_{(n-1)/2} \vee \overline{K}_{(n+1)/2},$$

where

$$V(G_{(n-1)/2}) = \{x_i, x_{i+2}, \cdots x_{i-2}\}, \overline{K}_{(n+1)/2} = \{x_{i+1}, x_{i+3}, \cdots x_{i-1}, u\}.$$

Case 2. The connectivity of G is 1.

In this case, let w be a cut vertex of G. Then, since $d(x) \geq (n-2)/2$ for each vertex x in G, we have that $G - w$ has two components. (Otherwise, let $G - w$ have at least three components H_1, H_2 and H_3, and let $|V(H_1)| = \min\{|V(H_1)|, |V(H_2)|, |V(H_3)|\}$. Then we have $d(y) \leq n/3 - |\{y\}| + |\{w\}| < (n-2)/2$ for each vertex y in H_1, a contradiction.) Let H_1, H_2 be the components of $G - w$, i.e., $G - w = H_1 \cup H_2$. We denote G by $w : (H_1 \cup H_2)$. Clearly, we have

$$(n-2)/2 \leq \min\{|V(H_1)|, |V(H_2)|\} \leq \max\{|V(H_1)|, |V(H_2)|\} \leq n/2.$$

Without loss of generality we may assume that

$$|V(H_1)| = h = \min\{|V(H_1)|, |V(H_2)|\}$$

$$|V(H_2)| = t = \max\{|V(H_1)|, |V(H_2)|\}.$$

(i) When $h = (n-2)/2$, then H_1 is the complete subgraph K_h, the vertex w is adjacent to all vertices of H_1, and each vertex of H_2 is not adjacent to at least one vertex of H_2. (ii) When $h = (n-1)/2$, then H_1 and H_2 are both complete subgraphs. However, we can write $w : (H_1 \cup H_2) = w : (K_h \cup K_t^-)$, where $(n-2)/2 \leq h \leq t \leq n/2$, and each vertex of K_t^- is not adjacent to at least one vertex of K_t^-.

Therefore, the proof is complete. □

Acknowledgements. The author is very grateful to the anonymous referee for the helpful comments and suggestions.

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Received November 18, 2010
Revised September 26, 2011