Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **37** (2011), 331–340

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

GROWTH OF SOME VARIETIES OF LEIBNIZ-POISSON ALGEBRAS

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Communicated by V. Drensky

Abstract. Let $\mathcal{V}$ be a variety of Leibniz-Poisson algebras over an arbitrary field whose ideal of identities contains the identities

$$\{\{x_1,y_1\},\{x_2,y_2\},\dots,\{x_m,y_m\}\}=0,\ \ \{x_1,y_1\}\cdot\{x_2,y_2\}\cdot\cdots\cdot\{x_m,y_m\}=0$$

for some m. It is shown that the exponent of $\mathcal{V}$ exists and is an integer.

Let K be an arbitrary field and let $A(+,\cdot,\{\ ,\ \},K)$ be a K-algebra with two binary multiplications $\cdot$ and $\{\ ,\ \}$. Let the algebra $A(+,\cdot,K)$ with multiplication $\cdot$ be a commutative associative algebra with unit and let the algebra $A(+,\{\ ,\ \},K)$ be a Leibniz algebra under the multiplication $\{\ ,\ \}$. The latter means that $A(+,\{\ ,\ \},K)$ satisfies the Leibniz identity

$$\{\{x,y\},z\}=\{\{x,z\},y\}+\{x,\{y,z\}\}.$$

2010 *Mathematics Subject Classification:* 17A32; 17B63.
Key words: Poisson algebra, Leibniz-Poisson algebra, variety of algebras, growth of variety.
*Partially supported by Grant RFFI 10-01-00209-a.

Assume that these two operations are connected by the relations ($a, b, c \in A$):

$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b,$$
$$\{c, a \cdot b\} = a \cdot \{c, b\} + \{c, a\} \cdot b.$$

Then the algebra $A(+, \cdot, \{\ ,\ \}, K)$ is called a Leibniz-Poisson algebra.

We make the convention that brackets in left-normed arrangements will be omitted:

$$\{\{\{x_1, x_2\}, x_3\}, \dots, x_n\} = \{x_1, x_2, \dots, x_n\}.$$

Let $F(X)$ be a free Leibniz-Poisson algebra freely generated by the countable set $X = \{x_1, x_2, \dots\}$. Denote by P_n the vector space in $F(X)$ consisting of the multilinear elements of degree n in the variables $x_1, \dots, x_n$.

Let $\mathcal{V}$ be a variety of Leibniz-Poisson algebras (the necessary information on varieties of PI-algebras can be found, for instance, in [1, 2]). Let $\mathrm{Id}(\mathcal{V})$ be the ideal of identities of $\mathcal{V}$. Denote

$$P_n(V) = P_n/(P_n \cap \mathrm{Id}(\mathcal{V})), \quad c_n(\mathcal{V}) = \dim P_n(\mathcal{V}).$$

Define the lower and upper exponents for the codimension sequence $\{c_n(\mathcal{V})\}_{n\geq 1}$ as follows:

$$\underline{\mathrm{EXP}}(\mathcal{V}) = \varliminf_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}, \qquad \overline{\mathrm{EXP}}(\mathcal{V}) = \varlimsup_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}.$$

These limits always exist (of course they might be infinite). If the lower and the upper limits coincide, we use the notation $\mathrm{Exp}(\mathcal{V})$.

In [3] V. M. Petrogradsky, using the necklace method developed therein, proved that the exponent of every variety of Lie algebras with nilpotent commutator subalgebra exists and is an integer. In [4] the same method was used to prove a similar result for the subvarieties of $\mathrm{var}(UT_s)$, where UT_s is the associative algebra of upper triangular matrices of size s. The method mentioned gives a good upper bound for the growth of such varieties, i.e., if $\mathcal{V}$ is a subvariety of $\mathrm{var}(UT_s)$ with $\mathrm{Exp}(\mathcal{V}) = d$, then there exists a constant β such that $c_n(\mathcal{V}) \leq n^\beta d^n$ for every n. In [5] and [6] it was proved that in this case there also exists a constant α such that $c_n(\mathcal{V}) \geq n^\alpha d^n$ for all sufficiently large n. In the present paper we use the methods of proofs applied in [5, 6, 7].

Recall that $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n, and we write $\lambda \vdash n$, if $\lambda_1 \geq \dots \geq \lambda_k > 0$ are integers such that $\lambda_1 + \dots + \lambda_k = n$.

Denote by $\mathcal{V}_s$ the variety of Leibniz-Poisson algebras defined by all multilinear identities of the form

$$\begin{aligned}(1)\quad &\{\{x_{11},y_{11}\},\{x_{12},y_{12}\},\dots,\{x_{1\lambda_1},y_{1\lambda_1}\}\}\cdot\{\{x_{21},y_{21}\},\{x_{22},y_{22}\},\dots,\\ &\{x_{2\lambda_2},y_{2\lambda_2}\}\}\cdot\ \cdots\ \cdot\{\{x_{k1},y_{k1}\},\{x_{k2},y_{k2}\},\dots,\{x_{k\lambda_k},y_{k\lambda_k}\}\}=0,\quad \lambda\vdash s.\end{aligned}$$

For instance, all identities defining the varieties $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_4$ are the following:

$\mathcal{V}_1$:

$$\{x,y\}=0;$$

$\mathcal{V}_2$:

$$\{\{x_1,y_1\},\{x_2,y_2\}\}=0,$$
$$\{x_1,y_1\}\cdot\{x_2,y_2\}=0;$$

$\mathcal{V}_3$:

$$\{\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\}\}=0,$$
$$\{\{x_1,y_1\},\{x_2,y_2\}\}\cdot\{x_3,y_3\}=0,$$
$$\{x_1,y_1\}\cdot\{x_2,y_2\}\cdot\{x_3,y_3\}=0;$$

$\mathcal{V}_4$:

$$\{\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\},\{x_4,y_4\}\}=0,$$
$$\{\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\}\}\cdot\{x_4,y_4\}=0,$$
$$\{\{x_1,y_1\},\{x_2,y_2\}\}\cdot\{\{x_3,y_3\},\{x_4,y_4\}\}=0,$$
$$\{\{x_1,y_1\},\{x_2,y_2\}\}\cdot\{x_3,y_3\}\cdot\{x_4,y_4\}=0,$$
$$\{x_1,y_1\}\cdot\{x_2,y_2\}\cdot\{x_3,y_3\}\cdot\{x_4,y_4\}=0.$$

Note that

$$P_n(\mathcal{V}_s)\cong P_n(F(X)/\mathrm{Id}(\mathcal{V}_1))\bigoplus_{c=1}^{s-1}P_n(\mathrm{Id}(\mathcal{V}_c)/\mathrm{Id}(\mathcal{V}_{c+1})),$$

where

$$P_n(F(X)/\mathrm{Id}(\mathcal{V}_1))=\langle x_1\cdot x_2\cdot\ \cdots\ \cdot x_n\rangle_K,$$

and the space $P_n(\mathrm{Id}(\mathcal{V}_c)/\mathrm{Id}(\mathcal{V}_{c+1}))$ is a direct sum of the linear hull of the elements of the form:

$$\begin{aligned}&P_n(\mathrm{Id}(\mathcal{V}_c)/\mathrm{Id}(\mathcal{V}_{c+1}))=\bigoplus_{\lambda\vdash c}\Big\langle t_{i_1}\cdot\ \cdots\ \cdot t_{i_l}\cdot\\ &\cdot\{\{x_{11},x_{12},\dots,x_{1a_{11}}\},\{x_{21},x_{22},\dots,x_{2a_{12}}\},\dots,\{x_{\lambda_1 1},x_{\lambda_1 2},\dots,x_{\lambda_1 a_{1\lambda_1}}\}\}\cdot\\ (2)\quad &\{\{y_{11},y_{12},\dots,y_{1a_{21}}\},\{y_{21},y_{22},\dots,y_{2a_{22}}\},\dots,\{y_{\lambda_2 1},y_{\lambda_2 2},\dots,y_{\lambda_2 a_{2\lambda_2}}\}\}\cdot\ \cdots\ \cdot\\ &\{\{z_{11},z_{12},\dots,z_{1a_{k1}}\},\{z_{21},z_{22},\dots,z_{2a_{k2}}\},\dots,\{z_{\lambda_k 1},z_{\lambda_k 2},\dots,z_{\lambda_k a_{k\lambda_k}}\}\}\,|\\ &\{x_{i_1j_1},y_{i_2j_2},\dots,z_{i_3j_3},t_{i_1},\dots,t_{i_l}\}=\{x_1,x_2,\dots,x_n\},\ l\geq 0,\ a_{ij}\geq 2\Big\rangle_K.\end{aligned}$$

We may interchange $x_{i3},\dots,x_{ia_{1i}}$, $i=1,\dots,\lambda_1$, $y_{i3},\dots,y_{ia_{2i}}$, $i=1,\dots,\lambda_2,\dots$, $z_{i3},\dots,z_{ia_{ki}}$, $i=1,\dots,\lambda_k$, since interchanging two neighboring elements produces an extra element in $\mathrm{Id}(\mathcal{V}_{c+1})$. Denote this property by $(*)$.

For instance, for the varieties $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_4$ their $P_n(\mathcal{V}_i)$, $i=1,2,3,4$, are the following:

$$P_n(\mathcal{V}_1)=\langle x_1\cdot x_2\cdot\ \cdots\ \cdot x_n\rangle_K;$$

$$P_n(\mathcal{V}_2)=P_n(\mathcal{V}_1)\oplus\langle x_{a_1}\cdot\ \cdots\ \cdot x_{a_m}\cdot\{x_{i_1},\dots,x_{i_s}\}\rangle_K,$$

where $s\geq 2$, $\{x_{a_1},\dots,x_{a_m},x_{i_1},\dots,x_{i_s}\}=\{x_1,\dots,x_n\}$ as sets,

$$a_1<\dots<a_m,\quad i_3<\dots<i_s;$$

$$P_n(\mathcal{V}_3)=P_n(\mathcal{V}_2)\oplus\langle x_{a_1}\cdot\ \cdots\ \cdot x_{a_m}\cdot\{\{x_{i_1},\dots,x_{i_s}\},\{x_{j_1},\dots,x_{j_t}\}\}\rangle_K$$
$$\oplus\langle x_{b_1}\cdot\ \cdots\ \cdot x_{b_p}\cdot\{x_{i_1},\dots,x_{i_s}\}\cdot\{x_{j_1},\dots,x_{j_t}\}\rangle_K,$$

where $s\geq 2$, $t\geq 2$,

$$a_1<\dots<a_m,\quad b_1<\dots<b_p,\quad i_3<\dots<i_s,\quad j_3<\dots<j_t;$$

$$P_n(\mathcal{V}_4)=P_n(\mathcal{V}_3)\oplus\langle x_{a_1}\cdot\cdots\cdot x_{a_m}\cdot\{\{x_{i_1},\dots,x_{i_s}\},\{x_{j_1},\dots,x_{j_t}\},\{x_{k_1},\dots,x_{k_u}\}\}\rangle_K$$
$$\oplus\langle x_{b_1}\cdot\ \cdots\ \cdot x_{b_p}\cdot\{\{x_{i_1},\dots,x_{i_s}\},\{x_{j_1},\dots,x_{j_t}\}\}\cdot\{x_{k_1},\dots,x_{k_u}\}\rangle_K$$
$$\oplus\langle x_{c_1}\cdot\ \cdots\ \cdot x_{c_q}\cdot\{x_{i_1},\dots,x_{i_s}\}\cdot\{x_{j_1},\dots,x_{j_t}\}\cdot\{x_{k_1},\dots,x_{k_u}\}\rangle_K,$$

where $s\geq 2$, $t\geq 2$, $u\geq 2$,

$$a_1<\dots<a_m,\quad b_1<\dots<b_p,\quad c_1<\dots<c_q,$$

$$i_3<\dots<i_s,\quad j_3<\dots<j_t,\quad k_3<\dots<k_u.$$

Let S_{km} be the symmetric group of degree km. Denote by S^*_{km} the following subset in S_{km}:

$$S^*_{km}=\{\sigma\mid\sigma\in S_{km},\ \sigma(im+1)<\sigma(im+2)<\dots<\sigma(im+m),\ i=0,\dots,k-1\}.$$

Obviously, $|S^*_{km}|=\dfrac{(km)!}{(m!)^k}$.

Let $\mathcal{V}$ be a fixed subvariety of $\mathcal{V}_s$. Then

$$P_n(\mathcal{V})\cong\sum_{c=0}^{s-1}\bigoplus_{\lambda\vdash c}W_{c,\lambda,n}(\mathcal{V}),\tag{3}$$

where

$$W_{0,n}(\mathcal{V}) = P_n(F(X)/\mathrm{Id}(\mathcal{V}\cap\mathcal{V}_1)),$$

$$\bigoplus_{\lambda\vdash c} W_{c,\lambda,n}(\mathcal{V}) = P_n\left(\mathrm{Id}(\mathcal{V}\cap\mathcal{V}_c)/\mathrm{Id}(\mathcal{V}\cap\mathcal{V}_{c+1})\right),\ c = 1,\dots,s-1.$$

Remark. Let the element f belong to the space $W_{c,\lambda,n}$. Then f has the following general form:

$$\cdots\{x_1,x_2,\dots\}\cdots\{x_3,x_4,\dots\}\cdots\{x_{2c-1},x_{2c},\dots\}, \tag{4}$$

where the dots connecting the bracket monomials $\{x_1,x_2,\dots\},\dots,\{x_{2c-1},x_{2c},\dots\}$ are used to represent brackets $\{\ ,\ \}$ and multiplications $\cdot$ arranged in some way. In order to simplify the notation, we shall use the presentation of the form (4) when the way in which the operations $\{\ ,\ \}$ and $\cdot$ are arranged is not important for our considerations, and we want to emphasize on the explicit form of the entries in the brackets $\{x_{2i-1},x_{2i},\dots,\}$.

For a variety $\mathcal{V}$ we introduce the following numerical characteristics. Fix arbitrary positive integers k and n with $1\le k\le s$. Say that the nonnegative integer m enjoys the property $Q(n,k,\mathcal{V})$ if there are a number c and a partition $\lambda\vdash c$ such that the space $W_{c,\lambda,n}(\mathcal{V})$ contains a collection of linearly independent elements either of the form

$$\begin{aligned} a_\sigma = q\cdot t_1\cdots\{t_{i_1},x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(m)}\}\cdots \\ \cdots\{t_{i_2},x_{\sigma(m+1)},x_{\sigma(m+2)},\dots,x_{\sigma(2m)}\}\cdots \\ \cdots\{t_{i_k},x_{\sigma((k-1)m+1)},x_{\sigma((k-1)m+2)},\dots,x_{\sigma(km)}\}\cdots t_c,\ \sigma\in S^*_{km}, \end{aligned} \tag{5}$$

or of the form

$$\begin{aligned} a_\sigma = x_{\sigma(1)}\cdot x_{\sigma(2)}\cdot\ \cdots\ \cdot x_{\sigma(m)}\cdot\ \cdots\ \cdot t_1\cdots \\ \cdots\{t_{i_1},x_{\sigma(m+1)},x_{\sigma(m+2)},\dots,x_{\sigma(2m)}\}\cdots \\ \cdots\{t_{i_{k-1}},x_{\sigma((k-1)m+1)},x_{\sigma((k-1)m+2)},\dots,x_{\sigma(km)}\}\cdots t_c,\ \sigma\in S^*_{km}, \end{aligned} \tag{6}$$

where $q = y_1\cdot\ \cdots\ \cdot y_l$ is a (possibly empty) monomial, $t_1,\dots,t_c$ are brackets $\{,\}$ containing at least two variables and $q,t_1,\dots,t_c$ are equal for all elements a_σ, $\sigma\in S^*_{km}$. Define $m_n(k,\mathcal{V})$ as follows: if there is no nonnegative integer less than n which enjoys $Q(n,k,V)$, then put $m_n(k,\mathcal{V}) = -1$; otherwise, define $m_n(k,\mathcal{V})$ as the greatest of the numbers enjoying $Q(n,k,\mathcal{V})$. Introduce another characteristic of $\mathcal{V}$ as follows:

$$d(\mathcal{V}) = \max\{k \mid \overline{\lim_{n\to\infty}}\, m_n(k,\mathcal{V}) = +\infty,\ k = 1,\dots,s\}. \tag{7}$$

Lemma 1. *Let $\mathcal{V} \subseteq \mathcal{V}_s$. Then for every $r \in \{1,2,\dots,d(\mathcal{V})\}$ and every n the following inequality holds:*

$$n - r m_n(r,\mathcal{V}) \le 2(s-1) + r - 1. \tag{8}$$

Proof. Regard S_{am} as a subgroup of S_{bm} for $b \ge a$. Then we have an embedding $S^*_{am} \subseteq S^*_{bm}$. Thus, $m_n(r,\mathcal{V}) \ge m_n(d(\mathcal{V}),\mathcal{V})$ for every $r \in \{1,2,\dots,d(\mathcal{V})\}$ and every n. Consequently,

$$\overline{\lim_{n\to\infty}}\, m_n(r,\mathcal{V}) = +\infty,\ r \in \{1,2,\dots,d(\mathcal{V})\},$$

since $\overline{\lim\limits_{n\to\infty}}\, m_n(d(\mathcal{V}),\mathcal{V}) = +\infty$.

Suppose that there exist N and $r_0 \in \{1,2,\dots,d(\mathcal{V})\}$ for which (8) fails:

$$N - r_0 m_N(r_0,\mathcal{V}) \ge 2(s-1) + r_0.$$

Then all collections of a_σ, $\sigma \in S^*_{r_0\widetilde{m}}$, of the form (5), (6) with $\widetilde{m} = m_N(r_0,\mathcal{V}) + 1$ in $W_{c,\lambda,N}$ are linearly dependent modulo the ideal of identities of $\mathcal{V}$ for every $c \ge r_0 - 1$. Suppose that $n \ge N$ and $m \ge \widetilde{m}$. Then the elements a_σ, $\sigma \in S^*_{r_0 m}$, of the form (5), (6) are linearly dependent modulo $\mathrm{Id}(\mathcal{V})$ since by $(*)$ already the elements

$$q \cdot t_1 \cdots \{t_{i_1}, x_{\sigma(1)}, \dots, x_{\sigma(\widetilde{m})}, x_{r_0\widetilde{m}+1}, \dots, x_{r_0\widetilde{m}+m-\widetilde{m}}\} \cdots$$
$$\cdots \{t_{i_2}, x_{\sigma(\widetilde{m}+1)}, \dots, x_{\sigma(2\widetilde{m})}, x_{r_0\widetilde{m}+m-\widetilde{m}+1}, \dots, x_{r_0\widetilde{m}+2(m-\widetilde{m})}\} \cdots$$
$$\cdots \{t_{i_{r_0}}, x_{\sigma((r_0-1)\widetilde{m}+1)}, \dots, x_{\sigma(r_0\widetilde{m})}, x_{r_0\widetilde{m}+(r_0-1)(m-\widetilde{m})+1}, \dots, x_{r_0\widetilde{m}+r_0(m-\widetilde{m})}\} \cdots t_c,$$
$$\sigma \in S^*_{r_0\widetilde{m}},$$

and

$$x_{\sigma(1)} \cdot \ \cdots \ \cdot x_{\sigma(\widetilde{m})} \cdot x_{r_0\widetilde{m}+1} \cdot \cdots \cdot x_{r_0\widetilde{m}+m-\widetilde{m}} \cdot \ \cdots \ \cdot t_1 \cdots$$
$$\cdots \{t_{i_1}, x_{\sigma(\widetilde{m}+1)}, \dots, x_{\sigma(2\widetilde{m})}, x_{r_0\widetilde{m}+m-\widetilde{m}+1}, \dots, x_{r_0\widetilde{m}+2(m-\widetilde{m})}\} \cdots$$
$$\cdots \{t_{i_{r_0-1}}, x_{\sigma((r_0-1)\widetilde{m}+1)}, \dots, x_{\sigma(r_0\widetilde{m})}, x_{r_0\widetilde{m}+(r_0-1)(m-\widetilde{m})+1}, \dots, x_{r_0\widetilde{m}+r_0(m-\widetilde{m})}\} \cdots t_c,$$

$$\sigma \in S^*_{r_0\widetilde{m}}.$$

are linearly dependent. Thus, $m_n(r_0, \mathcal{V}) < \widetilde{m}$ for every $n \geq N$. Therefore, we arrive at contradiction with $\overline{\lim}_{n\to\infty} m_n(r_0, \mathcal{V}) = +\infty$. The lemma is proved. □

Lemma 2. *Assume that the variety $\mathcal{V}$ of Leibniz-Poisson algebras satisfies the following multilinear identities*

$$\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}\} = 0, \tag{9}$$

$$\{x_1, y_1\} \cdot \{x_2, y_2\} \cdot \; \cdots \; \cdot \{x_m, y_m\} = 0 \tag{10}$$

for some m. Then there exists s such that $\mathcal{V}$ is a subvariety of $\mathcal{V}_s$.

Proof. Let $s = (m-1)^2 + 1$. We claim that for an arbitrary partition λ of s, the variety $\mathcal{V}$ satisfies the identity (1). If $\lambda_1 \geq m$ holds for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of s, then (1) will be a consequence of (9). If $\lambda_1 < m$ then $\lambda_1^{'} \geq m$ holds for the partition $\lambda^{'} = (\lambda_1^{'}, \lambda_2^{'}, \ldots, \lambda_l^{'})$ conjugate to λ. In this case the identity (1) follows from (10). The lemma is proved. □

We introduce a partial order on the set of disjoint subsets of $\{1, 2, \ldots, n\}$. Given $A, B \subset \{1, 2, \ldots, n\}$ with $A \cap B = \emptyset$, we say that $A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

Theorem 1. *Let $\mathcal{V}$ be a variety of Leibniz-Poisson algebras over an arbitrary field whose ideal of identities contains the identities* (9) *and* (10) *for some m. Then there exist constants N, α, β and an integer d, $d \in \{1, 2, \ldots, s\}$, such that for every $n \geq N$ we have the double inequality*

$$n^\alpha d^n \leq c_n(\mathcal{V}) \leq n^\beta d^n. \tag{11}$$

Proof. Lemma 2 implies that $\mathcal{V}$ is a subvariety of $\mathcal{V}_s$ for some s. We shall verify that (11) holds for the variety $\mathcal{V}$ with $d = d(\mathcal{V})$ defined in (7).

Lemma 1 implies that for every n

$$0 \leq n - dm_n \leq 2(s-1) + d - 1 \leq 3(s-1),$$

where $d = d(\mathcal{V})$, $m_n = m_n(d(\mathcal{V}))$. Then

$$c_n(\mathcal{V}) \geq |S^*_{dm_n}| = \frac{(dm_n)!}{[m_n!]^d} \geq \frac{(n-3s+3)!}{[(\frac{n}{d})!]^d} \geq \frac{1}{n^{3s-3}} \frac{n!}{[(\frac{n}{d})!]^d}.$$

It remains to apply the Stirling formula and to deduce the lower bound for $c_n(\mathcal{V})$ in (11).

Now we shall verify the upper bound. If $d(\mathcal{V}) = s$, then (11) holds for $\mathcal{V}$ since $c_n(\mathcal{V}) \leq c_n(\mathcal{V}_s) \leq n^\beta s^n$ for some constant β and all n.

Suppose $d(\mathcal{V}) \leq s-1$ and let $k = d(\mathcal{V})+1$. The definition of $d(\mathcal{V})$ implies that for the given k there exists m such that the collections of elements of the form (5) and (6) are linearly dependent for every $c \geq k-1$ and every n starting with some number N. Fix an arbitrary value of c with $k-1 \leq c \leq s-1$ and fix an arbitrary partition λ of c. Then for every $n \geq N$ the space $W_{c,\lambda,n}(\mathcal{V})$ is the linear span of the elements of the form (2) which, applying $(*)$, cannot be reduced to

$$(12)\quad q \cdot g_1 \cdots \{g_{i_1}, x_{11}, \ldots, x_{1m}\} \cdots \{g_{i_2}, x_{21}, \ldots, x_{2m}\} \cdots \{g_{i_k}, x_{k1}, \ldots, x_{km}\} \cdots g_c$$

or to

$$(13)\quad x_{11} \cdot \cdots \cdot x_{1m} \cdot \cdots \cdot g_1 \cdots \{g_{i_1}, x_{21}, \ldots, x_{2m}\} \cdots \{g_{i_{k-1}}, x_{k1}, \ldots, x_{km}\} \cdots g_c,$$

where $q = y_1 \cdot \cdots \cdot y_l$ is a (possibly empty) monomial and $t_1, \ldots, t_c$ are brackets $\{\ ,\ \}$ containing at least two variables; moreover, (12) and (13) have decreasing sequence of subsets like

$$\{x_{11}, x_{12}, \ldots, x_{1m}\} > \{x_{21}, x_{22}, \ldots, x_{2m}\} > \cdots > \{x_{k1}, x_{k2}, \ldots, x_{km}\}.$$

Indeed, by the linear dependence of (5) and (6), identities of the form

$$t_1 \cdots \{t_{i_1}, x_{(k-1)m+1}, \ldots, x_{km}\} \cdots \{t_{i_{k-1}}, x_{m+1}, \ldots, x_{2m}\} \cdots \{t_{i_k}, x_1, \ldots, x_m\} \cdots t_c$$

$$= \sum_{\sigma \in S^*_{km} \setminus \{e\}} \alpha_\sigma t_1 \cdots \{t_{i_1}, x_{\sigma((k-1)m+1)}, \ldots, x_{\sigma(km)}\} \cdots \{t_{i_k}, x_{\sigma(1)}, \ldots, x_{\sigma(m)}\} \ldots t_c,$$

$$x_{(k-1)m+1} \cdot \ \cdots \ \cdot x_{km} \cdot t_1 \cdots \{t_{i_{k-2}}, x_{m+1}, \ldots, x_{2m}\} \cdots \{t_{i_{k-1}}, x_1, \ldots, x_m\} \cdots t_c$$

$$= \sum_{\sigma \in S^*_{km} \setminus \{e\}} \alpha_\sigma x_{\sigma((k-1)m+1)} \cdot \ \cdots \ \cdot x_{\sigma(km)} \cdot t_1 \cdots \{t_{i_{k-1}}, x_{\sigma(1)}, \ldots, x_{\sigma(m)}\} \cdots t_c,$$

hold in $W_{c,\lambda,n}(\mathcal{V})$, where e is the identity permutation.

Thus the basis for $W_{c,\lambda,n}(\mathcal{V})$ can be chosen from elements (2), so that these basis elements cannot be reduced to the form (12) and (13). In [3] it was shown that the number of such basis elements in $W_{c,\lambda,n}$ does not exceed $n^\beta(k-1)^n$ for some constant β. Taking into account the decomposition (3), we obtain the upper bound. The theorem is proved. $\square$

Let char $K = 0$. The space $P_n(\mathcal{V})$ carries the structure of a left S_n-module, where S_n is the symmetric group on n letters. Let χ_λ be the character of the irreducible representation of the symmetric group corresponding to the partition λ of n. The module $P_n(\mathcal{V})$ is completely reducible and so the cocharacter sequence of $\mathcal{V}$ admits the decomposition

$$\chi_n(\mathcal{V}) = \sum_{\lambda \vdash n} m_\lambda(\mathcal{V})\chi_\lambda, \tag{14}$$

where $m_\lambda(\mathcal{V})$ is the multiplicity of the irreducible character χ_λ, $\lambda \vdash n$.

Given an arbitrary variety $\mathcal{V} \subseteq \mathcal{V}_s$ define the following numerical values:

$$q_n(k, \mathcal{V}) = \max\{\lambda_k \mid \lambda \vdash n,\ m_\lambda(\mathcal{V}) > 0\},$$

$$d_0(\mathcal{V}) = \max\{k \mid \overline{\lim_{n\to\infty}}\, q_n(k, \mathcal{V}) = +\infty,\ k = 1, \dots, s\}.$$

The proof of the next lemmas and the theorem is similar to those in [5, 6].

Lemma 3. *If the Young diagram of the partition λ of n contains more than $4(s-1)^2$ cells outside the first s rows, then the multiplicity $m_\lambda(\mathcal{V}_s)$ of χ_λ in the cocharacter sequence of $\mathcal{V}_s$ is equal to* 0.

Lemma 4. *Given a subvariety $\mathcal{V}$ of $\mathcal{V}_s$ with $d_0(\mathcal{V}) \geq 1$, we have*

$$n - rq_n(r, \mathcal{V}) \leq 2(s-1) + r - 1$$

for all $r \in \{1, 2, \dots, d_0(\mathcal{V})\}$ and all n.

Lemma 5. *Let $\mathcal{V} \subseteq \mathcal{V}_s$. Then there exist N, α and β such that the codimension sequence $c_n(\mathcal{V})$ satisfies the double inequality*

$$n^\alpha (d_0(\mathcal{V}))^n \leq c_n(\mathcal{V}) \leq n^\beta (d_0(\mathcal{V}))^n$$

for every $n \geq N$.

Theorem 2. *Let $\mathcal{V}$ be a variety of Leibniz-Poisson algebras over a field of characteristic zero whose ideal of identities contains the identities* (9) *and* (10) *for some m. Also assume that d is a positive integer. Then the following conditions are equivalent:*

(*i*) $\operatorname{Exp}(\mathcal{V}) \leq d$;

(ii) There exists a constant C such that $m_\lambda(\mathcal{V}) = 0$ in the sum (14) *if*

$$n - (\lambda_1 + \lambda_2 + \cdots + \lambda_d) > C.$$

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Received March 12, 2012