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WEIGHTED COMPOSITION FOLLOWED BY DIFFERENTIATION BETWEEN WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane and ψ be an analytic map on $\mathbb{D}$. Such maps induce a weighted composition operator followed by differentiation $DC_{\phi,\psi}$ acting between weighted Banach spaces of holomorphic functions. We characterize boundedness and compactness of such operators in terms of the involved weights as well as the functions ϕ and ψ.

1. Introduction. Let $H(\mathbb{D})$ be the set of all analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Moreover, we consider an analytic self-map ϕ of $\mathbb{D}$ as well as a map $\psi \in H(\mathbb{D})$. Such maps induce a so-called *weighted*

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composition operator $C_{\phi,\psi}$ by first composing a function $f \in H(\mathbb{D})$ with the map ϕ and multiplying then with ψ, that is,

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi(f \circ \phi).$$

Obviously $C_{\phi,\psi}$ is a combination of the *classical composition operator*

$$C_{\phi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi.$$

with the *multiplication operator*

$$M_{\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi f.$$

Composition operators appear naturally in a variety of problems such as e.g. the study of dynamical systems. Moreover, in the setting of the Hardy space H^2 they link operator theoretical questions with classical results in complex analysis. Therefore composition operators and weighted composition operators have been studied by many authors in many different settings. We give as an example the following list of articles on the topic [5], [3], [4], [6], [7], [8], [10], [13], [14], [15]. Since the literature is increasing steadily this can only be a sample. Differentiating weighted composition operators we obtain the following operator

$$DC_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi'(f \circ \phi) + \psi\phi'(f' \circ \phi).$$

We are interested in operators $DC_{\phi,\psi}$ acting in the following setting. Let $v : \mathbb{D} \to (0, \infty)$ be a bounded and continuous function (*weight*). Then the so-called *weighted Banach spaces of holomorphic functions* are defined by

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Endowed with the weighted sup-norm $\|\cdot\|_v$ this space is a Banach space.

Weighted Banach spaces of holomorphic functions have important applications in functional analysis (spectral theory, functional calculus), complex analysis, partial differential equations and convolution equations, as well as distribution theory. For a deep study of these spaces we refer the reader to the articles of Bierstedt-Bonet-Galbis [1] and Bierstedt-Bonet-Taskinen [2].

Our aim in this note is to characterize boundedness and compactness of operators $DC_{\phi,\psi}$ between weighted Banach spaces of holomorphic functions in terms of the involved weights as well as the inducing map. As a corollary we get a characterization of boundedness and compactness of the composition operator $C_{\phi,\psi}$ that acts between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces. This is first a generalization of the results we obtained in [17] and second a generalization of the results of Ohno [14]. Moreover, this paper is a continuation of the work we started in [18] and [19], where we studied weighted composition operators acting between weighted Bloch type spaces and between weighted Banach spaces and weighted Bloch type spaces.

2. Notation and auxiliary results. For an introduction to composition operators we refer the reader to the excellent monographs [9] and [16]. The *weighted Bloch type space* B_v is defined to be the collection of all $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_{B_v} := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty.$$

Provided we identify functions that differ by a constant, $\|\cdot\|_{B_v}$ becomes a norm and B_v a Banach space.

In the setting of weighted spaces the so-called *associated weight* plays an important role. For a weight v it is defined follows:

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H_v^{\infty\prime}}},$$

where δ_z denotes the point evaluation of z. By [2] the associated weight $\tilde{v}$ is continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = \dfrac{1}{\tilde{v}(z)}$. We say that a weight v is radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A radial, non-increasing weight is called *typical* if $\lim_{|z|\to 1} v(z) = 0$.

When studying the structure and isomorphism classes of the space H_v^∞ (see [11] and [12]), Lusky introduced the following condition $(L1)$ (renamed after the author) for radial weights:

$$(L1) \quad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

which will play a great role in this article. Among others, the standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, and the logarithmic weights $w_\beta(z) = (1-\log(1-|z|^2))^\beta$, $\beta < 0$, satisfy condition $(L1)$. Moreover, radial weights with $(L1)$ are *essential*, that is, we can find a constant $k > 0$ such that

$$\tilde{v}(z) \le v(z) \le k\tilde{v}(z) \text{ for every } z \in \mathbb{D}.$$

See [3]. We close this section stating some geometric facts of the open unit disk $\mathbb{D}$. Let $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$, $z \in \mathbb{D}$, be the Möbius transformation that interchanges a and 0. We will use the fact that the derivative of φ_a is given by

$$\varphi_a'(z) = -\frac{1-|a|^2}{(1-\overline{a}z)^2} \text{ for every } z \in \mathbb{D}.$$

Now, we define the *pseudohyperbolic metric* $\rho(z,a)$ by

$$\rho(z,a) = |\varphi_a(z)|, \; z, a \in \mathbb{D}.$$

One of the most important properties of the pseudohyperbolic metric is that it is Möbius invariant, or more precisely, that

$$\rho(\sigma(z), \sigma(a)) = \rho(z,a) \text{ for all automorphisms } \sigma \text{ of } \mathbb{D}, \; z, a \in \mathbb{D}.$$

The pseudohyberbolic metric is a true metric. In fact, it even satisfies a stronger version of the triangle inequality, stating that for any $z, a, b \in \mathbb{D}$ we have that

$$\rho(z,a) \leq \frac{\rho(z,b) + \rho(b,a)}{1 + \rho(z,b)\rho(b,a)}.$$

3. Boundedness. In order to study the boundedness of the operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ we need the following auxiliary result which is taken from [17].

Lemma 1. *Let v be a radial weight satisfying $(L1)$. Then there is $C_v > 0$ such that for every $f \in H_v^\infty$*

$$|f^{(n)}(z)| \leq \frac{C_v}{(1-|z|^2)^n v(z)} \|f\|_v$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

Now, we are able to characterize the boundedness of the operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$.

Theorem 2. *Let w be an arbitrary weight and v be a radial weight satisfying condition $(L1)$. Then the operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is bounded if and only if the following conditions are satisfied:*

(a) $\sup_{z\in\mathbb{D}} \dfrac{w(z)|\psi'(z)|}{v(\phi(z)} < \infty,$

(b) $\sup_{z\in\mathbb{D}} \dfrac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} < \infty.$

Proof. Let us first assume that $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is bounded. We fix a point $a \in \mathbb{D}$ and put

$$f_a(z) := \varphi_{\phi(a)}(z)g_a(z) \text{ for every } z \in \mathbb{D},$$

where g_a belongs to the unit ball of H_v^∞ and satisfies $g_a(\phi(a)) = \dfrac{1}{\tilde{v}(\phi(a))}$. Thus, $\|f_a\|_v \le 1$ for every $a \in \mathbb{D}$. Hence $f_a(\phi(a)) = 0$ and $f_a'(z) = \varphi'_{\phi(a)}(z)g_a(z) + \varphi_{\phi(a)}(z)g_a'(z)$ for every $z \in \mathbb{D}$. This yields $f_a'(\phi(a)) = \dfrac{1}{(1-|\phi(a)|^2)\tilde{v}(\phi(a))}$, and therefore, by the boundedness of the operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$, we obtain

$$\begin{aligned} w(a)|(DC_{\phi,\psi}f_a)(a)| &= w(a)|\psi'(a)f_a(\phi(a)) + \psi(a)\phi'(a)f_a'(\phi(a))| \\ &= \frac{w(a)|\psi(a)||\phi'(a)|}{(1-|\phi(a)|^2)\tilde{v}(\phi(a))} \\ &\le \|DC_{\phi,\psi}\|\|f_a\|_v \le \|DC_{\phi,\psi}\| < \infty. \end{aligned}$$

Since $a \in \mathbb{D}$ is arbitrary and v is essential, condition (b) follows.

To show (a) we consider a fixed point $a \in \mathbb{D}$ and a function f_a which belongs to the unit ball of H_v^∞ and satisfies $f_a(\phi(a)) = \dfrac{1}{\tilde{v}(\phi(a))}$. We then get

$$\begin{aligned} w(a)|(DC_{\phi,\psi}f_a)(a)| &= w(a)|\psi'(a)f_a(\phi(a)) + \psi(a)\phi'(a)f_a'(\phi(a))| \\ &= \left|\frac{w(a)\psi'(a)}{\tilde{v}(\phi(a))} + \psi(a)\phi'(a)w(a)f_a'(\phi(a))\right| \\ &\le \|DC_{\phi,\psi}\| < \infty. \end{aligned}$$

Since $|f_a'(\phi(a))| \le \dfrac{C_v}{(1-|\phi(a)|^2)\tilde{v}(\phi(a))}$, (b) holds and v is essential, we obtain condition (a).

Conversely, applying Lemma 1 yields for every $f \in H_v^\infty$:

$$\begin{aligned}\|C_{\phi,\psi} f\|_{B_w} &\leq \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f(\phi(z))| + \sup_{z\in\mathbb{D}} w(z)|\psi(z)||\phi'(z)||f'(\phi(z))| \\ &\leq \sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{\tilde{v}(\phi(z))}\|f\|_v + C_v \sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)||\phi'(z)|}{(1-|\phi(z)|^2)\tilde{v}(\phi(z))}\|f\|_v.\end{aligned}$$

The claim follows. □

Corollary 3. *Let w be an arbitrary weight and v be a radial weight satisfying condition $(L1)$. Then the operator $C_{\phi,\psi} : H_v^\infty \to B_w$ is bounded if and only if the following conditions are satisfied*

(a) $\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} < \infty,$

(b) $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{v(\phi(z))} < \infty.$

4. Compactness. Next, we turn our attention to the compactness of the operator. For the study of the compactness of the operator $DC_{\phi,\psi}$ we need the following result. To show this we might easily adapt the proof of Proposition 3.11 in [9].

Proposition 4. *The operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is compact if and only if it is bounded and if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in H_v^∞ such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, $DC_{\phi,\psi} f_n \to 0$ in H_w^∞.*

In this section we will assume the following situation. We first consider a holomorphic function ν on $\mathbb{D}$ that is non-vanishing, strictly positive and decreasing on $[0,1)$ such that $\lim_{r\in[0,1),r\to 1} \nu(r) = 0$. Then we define the corresponding weight by

$$v(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}. \tag{4.1}$$

We will illustrate this setting by giving examples of weights of the type desribed in (4.1).

1. The holomorphic function $\nu(z) = (1-z)^\alpha$, $\alpha > 0$, for every $z \in \mathbb{D}$, gives the *standard weight* $v(z) = (1-|z|^2)^\alpha$ for every $z \in \mathbb{D}$.

2. The *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$, $\alpha > 0$, for every $z \in \mathbb{D}$ are generated by the holomorphic functions $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$.

3. The *logarithmic weights* $v(z) = (1 - \log(1 - |z|^2))^\beta$, $\beta < 0$, for every $z \in \mathbb{D}$

Next, we fix a point $a \in \mathbb{D}$ and introduce a function

$$v_a(z) := \nu(\overline{a}z) \text{ for every } z \in \mathbb{D}.$$

Since ν is a holomorphic function, so is v_a. Additionally we assume that there is a constant $M > 0$ such that

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{|v_a(z)|} \leq M. \tag{4.2}$$

Here are some examples satisfying (4.2):

1. For $v(z) = 1 - |z|^2$ we have that

$$\frac{v(z)}{|v_a(z)|} = \frac{1-|z|^2}{|1-\overline{a}z|} \leq \frac{1-|z|^2}{1-|z|} \leq 1 + |z| \leq 2$$

   for every $z \in \mathbb{D}$.

2. We study $v(z) = \dfrac{1}{1-\log(1-|z|^2)}$ for every $z \in \mathbb{D}$. This weight has the desired property since $|1 - \log(1 - \overline{a}z)| \leq 1 - \log(1 - |z|)$ for every $z \in \mathbb{D}$ and the function $\dfrac{1-\log(1-|z|)}{1-\log(1-|z|^2)}$ is continuous and tends to 1 if $|z| \to 1$.

Theorem 5. *Let w be an arbitrary weight and v be a weight in* (4.1) *such that v satisfies additionally conditions* ($L1$) *and* (4.2). *The operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is compact if and only if*

(*a*) $\displaystyle\lim_{r\to1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0,$

(*b*) $\displaystyle\lim_{r\to1} \sup_{|\phi(z)|>r} \frac{w(z)|\psi'(z)|}{v(\phi(z))} = 0.$

Proof. First, we assume that the operator $DC_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is compact. We start proving condition (a). Let $(z_n)_n \subset \mathbb{D}$ be a sequence with $|\phi(z_n)| \to 1$ such that

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = \lim_{n\to\infty} \frac{w(z_n)|\phi'(z_n)||\psi(z_n)|}{v(\phi(z_n))(1-|\phi(z_n)|^2)}.$$

By passing to a subsequence we may assume that there is $n_0 \in \mathbb{N}$ such that $|\phi(z_n)|^n \geq \frac{1}{2}$ for every $n \geq n_0$. For every $n \in \mathbb{N}$ we consider functions

$$f_n(z) := z^n \varphi_{\phi(z_n)}(z) g_{\phi(z_n)}(z) \text{ for every } z \in \mathbb{D},$$

where $g_{\phi(z_n)}$ is a function in the unit ball of H_v^∞ such that $|g_{\phi(z_n)}(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$. Hence $f_n(\phi(z_n)) = 0$ for every $n \in \mathbb{N}$. Obviously, $(f_n)_n \subset H_v^\infty$ is a bounded sequence that tends to zero uniformly on the compact subsets of $\mathbb{D}$. Hence by Proposition 4 we have that $\|DC_{\phi,\psi} f_n\|_w \to 0$ if $n \to \infty$. Moreover,

$$f_n'(z) = nz^{n-1}\varphi_{\phi(z_n)}(z)g_{\phi(z_n)}(z) + z^n\varphi'_{\phi(z_n)}(z)g_{\phi(z_n)}(z) + z^n\varphi_{\phi(z_n)}(z)g'_{\phi(z_n)}(z).$$

Therefore $f_n'(\phi(z_n)) = \dfrac{\phi(z_n)^n}{1-|\phi(z_n)|^2\tilde{v}(\phi(z_n))}$ and, for $n \geq n_0$

$$\begin{aligned}\frac{1}{2}\frac{w(z_n)|\phi'(z_n)||\psi(z_n)|}{(1-|\phi(z_n)|^2)\tilde{v}(\phi(z_n))} &\leq \frac{w(z_n)|\psi(z_n)||\phi'(z_n)||\phi(z_n)|^n}{(1-|\phi(z_n)|^2)\tilde{v}(\phi(z_n))}\\ &\leq w(z_n)|DC_{\phi,\psi}f_n(z_n)| \leq \|DC_{\phi,\psi}f_n\|_w \to 0.\end{aligned}$$

Since v is essential, the claim follows. It remains to show condition (b). Let $(z_n)_n \subset \mathbb{D}$ be a sequence with $|\phi(z_n)| \to 1$ such that

$$\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\psi'(z)|}{v(\phi(z))} = \lim_{n\to\infty} \frac{w(z_n)|\psi'(z_n)|}{v(\phi(z_n))}.$$

For every $n \in \mathbb{N}$ we consider the following function

$$f_n(z) := 3\frac{v(\phi(z_n))}{\nu(\overline{\phi(z_n)}z)^2} - 2\frac{v(\phi(z_n))^2}{\nu(\overline{\phi(z_n)}z)^3}$$

for every $z \in \mathbb{D}$. Then

$$\|f_n\|_v = \sup_{n\in\mathbb{N}}\sup_{z\in\mathbb{D}} 3\frac{v(\phi(z_n))v(z)}{|\nu(\overline{\phi(z_n)}z)^2|} + \sup_{n\in\mathbb{N}}\sup_{z\in\mathbb{D}} 2\frac{v(\phi(z_n))^2 v(z)}{|\nu(\overline{\phi(z_n)}z)^3|} \leq 5M.$$

Hence $(f_n)_n$ is a bounded sequence in H_v^∞ that tends to zero uniformly on the compact subsets of $\mathbb{D}$. Moreover,

$$f_n'(z) = -6\frac{v(\phi(z_n))\overline{\phi(z_n)}\nu'(\overline{\phi(z_n)}z)}{\nu(\overline{\phi(z_n)}z)^3} + 6\frac{v(\phi(z_n))^2\overline{\phi(z_n)}\nu'(\overline{\phi(z_n)}z)}{v(\overline{\phi(z_n)}z)^4}$$

for every $z \in \mathbb{D}$. Then $f_n(\phi(z_n)) = \dfrac{1}{v(\phi(z_n))}$ and $f_n'(\phi(z_n)) = 0$. Thus, we arrive at

$$\frac{w(z_n)|\psi'(z_n)|}{v(\phi(z_n))} = w(z_n)|(DC_{\phi,\psi}f_n)(z_n)| \le \|(DC_{\phi,\psi})f_n\|_w \to 0$$

if $n \to \infty$ and (b) follows since v is radial.

Conversely, let $(f_n)_n \subset H_v^\infty$ be a bounded sequence which converges to zero uniformly on the compact subsets of $\mathbb{D}$. W.l.o.g. we may assume that $\|f_n\|_v \le 1$ for every $n \in \mathbb{N}$. By Proposition 4 we have to show that

$$\|DC_{\phi,\psi}f_n\|_w \to 0 \text{ if } n \to \infty.$$

Let us fix $\varepsilon > 0$. By hypothesis there is $0 < r < 1$ such that

$$\frac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} < \frac{\varepsilon}{4C_v} \text{ and } \frac{w(z)|\psi'(z)|}{v(\phi(z))} < \frac{\varepsilon}{4} \text{ if } |\phi(z)| > r.$$

Moreover, since $(f_n)_n$ converges to zero uniformly on the compact subsets of $\mathbb{D}$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{|\phi(z)|\le r} w(z)|\psi(z)||\phi'(z)||f_n'(\phi(z))| < \frac{\varepsilon}{4} \text{ and}$$

$$\sup_{|\phi(z)|\le r} w(z)|\psi'(z)||f_n(\phi(z))| < \frac{\varepsilon}{4} \text{ for every } n \ge n_0.$$

Finally, we can conclude

$$\begin{aligned}
\|DC_{\phi,\psi}f_n\|_w &= \sup_{z\in\mathbb{D}} w(z)|\phi'(z)\psi(z)f_n'(\phi(z)) + \psi'(z)f_n(\phi(z))| \\
&\le \sup_{|\phi(z)|\le r} w(z)|\phi'(z)\psi(z)||f_n'(\phi(z))| + \sup_{|\phi(z)|\le r} w(z)|\psi'(z)||f_n(\phi(z))| \\
&+ \sup_{|\phi(z)|>r} w(z)|\phi'(z)\psi(z)||f_n'(\phi(z))| + \sup_{|\phi(z)|>r} w(z)|\psi'(z)||f_n(\phi(z))| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&+ \sup_{|\phi(z)|>r} C_v \frac{w(z)|\phi'(z)\psi(z)|}{\tilde{v}(\phi(z))(1-|\phi(z)|^2)} + \sup_{|\phi(z)|>r} \frac{w(z)|\psi'(z)|}{v(\phi(z))} < \varepsilon
\end{aligned}$$

for every $n \ge n_0$. Hence the claim follows. $\square$

Corollary 6. *Let w be an arbitrary weight and v be a radial weight satisfying conditions (L1), (4.1) and (4.2). Then the operator $C_{\phi,\psi} : H_v^\infty \to B_w$ is compact if and only if the following conditions hold:*

(*a*) $\displaystyle\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\phi'(z)||\psi(z)|}{v(\phi(z))(1-|\phi(z)|^2)} = 0,$

(*b*) $\displaystyle\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{w(z)|\psi'(z)|}{v(\phi(z))} = 0.$

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