Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **37** (2011), 353–364

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON STRONGLY REGULAR GRAPHS WITH $m_2 = qm_3$ AND $m_3 = qm_2$

Mirko Lepović

Communicated by V. Drensky

Abstract. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. Let $\lambda_1 = r$, λ_2 and λ_3 be the distinct eigenvalues of a connected strongly regular graph. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2 and λ_3, respectively. We here describe the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$.

1. Introduction. Let G be a simple graph of order n. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that G is integral if its spectrum $\sigma(G)$ consists of integral values. Further, we say that a regular graph G of order n and degree

2010 *Mathematics Subject Classification:* 05C50.
Key words: Strongly regular graph, conference graph, integral graph.

$r \geq 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. For a background on strongly regular graphs see e.g. the books [2, ch. 10] or [6, ch. 21]. We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues. Let $\lambda_1 > \lambda_2 > \lambda_3$ denote the distinct eigenvalues of G and let m_1, m_2 and m_3 denote their multiplicities, respectively. It is known that $\lambda_1 = r$ and $m_1 = 1$.

Theorem 1 (Lepović [3]). *Let G be a connected strongly regular graph of order n and degree r. Then $m_2 m_3 \delta^2 = nr\overline{r}$ where $\delta = \lambda_2 - \lambda_3$ and $\overline{r} = (n-1) - r$.*

Remark 1. Let $\overline{r} = (n-1) - r$, $\overline{\lambda}_2 = -\lambda_3 - 1$ and $\overline{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of G. Then $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$ where $\overline{\tau} = \tau(\overline{G})$ and $\overline{\theta} = \theta(\overline{G})$.

Remark 2. (i) a strongly regular graph G of order $4k+1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then G is an integral graph.

Remark 3. (i) if G is a disconnected strongly regular graph of degree r then $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H; (ii) G is a disconnected strongly regular graph if and only if $\theta = 0$.

Using Theorem 1 we have described the parameters n, r, τ and θ for strongly regular graphs of order $2(2p+1)$, $3(2p+1)$ and $4(2p+1)$, where $2p+1$ is a prime number [3], [4]. Besides [5], we have described the parameters n, r, τ and θ for strongly regular graphs with $|m_2 - m_3| \leq 3$. We now proceed to establish the parameters of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$, as follows. First,

Proposition 1 (Elzinga [1]). *Let G be a connected or disconnected strongly regular graph of order n and degree r. Then*

$$r^2 - (\tau - \theta + 1)r - (n-1)\theta = 0\,. \tag{1}$$

Proposition 2 (Elzinga [1]). *Let G be a connected strongly regular graph of order n and degree r. Then*

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0\,, \tag{2}$$

where $\delta = \lambda_2 - \lambda_3$.

2. Main results.

Remark 4. Since $m_2(\overline{G}) = m_3(G)$ and $m_3(\overline{G}) = m_2(G)$ we note that if $m_2(G) = qm_3(G)$ then $m_3(\overline{G}) = qm_2(\overline{G})$.

Remark 5. In Theorems 2, 3 and 4 the complements of strongly regular graphs appear in pairs in (k^0) and $(\overline{k}^0)$ classes, where k denotes the corresponding number of a class.

Theorem 2. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 2m_3$ or $m_3 = 2m_2$. Then G is one of the following strongly regular graphs:*

- (1^0) *G is the complete bipartite graph $K_{2,2}$ of order $n = 4$ and degree $r = 2$ with $\tau = 0$ and $\theta = 2$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2$ with $m_2 = 2$ and $m_3 = 1$;*
- (2^0) *G is a strongly regular graph of order $n = (3k+1)^2$ and degree $r = k(3k+2)$ with $\tau = (k-1)(k+1)$ and $\theta = k(k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(2k+1)$ with $m_2 = 2k(3k+2)$ and $m_3 = k(3k+2)$;*
- $(\overline{2}^0)$ *G is a strongly regular graph of order $n = (3k+1)^2$ and degree $r = 2k(3k+2)$ with $\tau = 4k^2 + 3k - 1$ and $\theta = 2k(2k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 2k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(3k+2)$ and $m_3 = 2k(3k+2)$;*
- (3^0) *G is a strongly regular graph of order $n = (3k+2)^2$ and degree $r = (k+1)(3k+1)$ with $\tau = k(3k+2)$ and $\theta = k(3k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 2k+1$ and $\lambda_3 = -(k+1)$ with $m_2 = (k+1)(3k+1)$ and $m_3 = 2(k+1)(3k+1)$;*
- $(\overline{3}^0)$ *G is a strongly regular graph of order $n = (3k+2)^2$ and degree $r = 2(k+1)(3k+1)$ with $\tau = k(4k+5)$ and $\theta = 2(k+1)(2k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(2k+2)$ with $m_2 = 2(k+1)(3k+1)$ and $m_3 = (k+1)(3k+1)$.*

Proposition 3. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 2m_3$. Then G belongs to the class (2^0) or $(\overline{3}^0)$ represented in Theorem 2.*

Proof. Let $m_3 = p$ and $m_2 = 2p$ where $p \in \mathbb{N}$. Since $m_2 + m_3 = n-1$ we obtain $n = 3p+1$. Since $\tau - \theta = \lambda_2 + \lambda_3$ and $\delta = \lambda_2 - \lambda_3$ we can easily see that (2) is reduced to $r = p(|\lambda_3| - 2\lambda_2)$. Let $|\lambda_3| - 2\lambda_2 = t$ where $t \in \mathbb{N}$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(2k+t)$; (ii) $\tau - \theta = -(k+t)$; (iii) $\delta = 3k+t$ and (iv) $r = pt$. Since $\delta^2 = (\tau-\theta)^2 + 4(r-\theta)$ (see [1]) we obtain (v) $\theta = pt - 2k^2 - kt$. Using (ii), (iv) and (v) it is not difficult to see that (1) is transformed into

$$(p+1)t^2 - (3p+1)t + 6k^2 + 4kt = 0. \tag{3}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+1)$, $\tau - \theta = -(k+1)$, $\delta = 3k+1$, $r = p$ and $\theta = p - 2k^2 - k$. Using (3) we find that $p = k(3k+2)$. So we obtain that G is a strongly regular graph of order $(3k+1)^2$ and degree $r = k(3k+2)$ with $\tau = (k-1)(k+1)$ and $\theta = k(k+1)$.
Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(2k+2)$, $\tau - \theta = -(k+2)$, $\delta = 3k+2$, $r = 2p$ and $\theta = 2p - 2k^2 - 2k$. Using (3) we find that $p = (k+1)(3k+1)$. So we obtain that G is a strongly regular graph of order $(3k+2)^2$ and degree $r = 2(k+1)(3k+1)$ with $\tau = k(4k+5)$ and $\theta = 2(k+1)(2k+1)$.
Case 3. ($t \geq 3$). Using (iv) we obtain $r = n-1$ if $t = 3$, a contradiction. Using (iv) we obtain $r \geq n$ if $t \geq 4$, a contradiction. □

Proposition 4. *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 2m_2$. Then G belongs to the class $(\overline{2}^0)$ or (3^0) represented in Theorem 2.*

Proof. Let $m_2 = p$, $m_3 = 2p$ and $n = 3p+1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(2|\lambda_3| - \lambda_2)$. Let $2|\lambda_3| - \lambda_2 = t$ where $t = 1, 2$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = 2k - t$; (ii) $\tau - \theta = k - t$; (iii) $\delta = 3k - t$; (iv) $r = pt$ and (v) $\theta = pt - 2k^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - (3p+1)t + 6k^2 - 4kt = 0\,. \tag{4}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-1$ and $\lambda_3 = -k$, $\tau - \theta = k-1$, $\delta = 3k-1$, $r = p$ and $\theta = p - 2k^2 + k$. Using (4) we find that $p = k(3k-2)$. Replacing k with $k+1$ we arrive at $p = (k+1)(3k+1)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(3k+2)^2$ and degree $r = (k+1)(3k+1)$ with $\tau = k(3k+2)$ and $\theta = k(3k+1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k-2$ and $\lambda_3 = -k$, $\tau - \theta = k-2$, $\delta = 3k-2$, $r = 2p$ and $\theta = 2p - 2k^2 + 2k$. Using (4) we find that $p = (k-1)(3k-1)$. Replacing k with $k+1$ we arrive at $p = k(3k+2)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(3k+1)^2$ and degree $r = 2k(3k+2)$ with $\tau = 4k^2+3k-1$ and $\theta = 2k(2k+1)$. □

Proof of Theorem 2. According to Proposition 3 it turns out that G belongs to the class (2^0) or $(\overline{3}^0)$ if $m_2 = 2m_3$. According to Proposition 4 it turns out that G belongs to the class $(\overline{2}^0)$ or (3^0) if $m_3 = 2m_2$. □

Remark 6. We note that the complete bipartite graph $K_{2,2}$ is a strongly regular graph with $m_2 = 2m_3$. It is obtained from the class Theorem 2 $(\overline{3}^0)$ for $k = 0$.

Theorem 3. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 3m_3$ or $m_3 = 3m_2$. Then G is one of the following strongly regular graphs:*

(1^0) *G is the strongly regular graph $\overline{3K_3}$ of order $n = 9$ and degree $r = 6$ with $\tau = 3$ and $\theta = 6$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 6$ and $m_3 = 2$;*

(2^0) *G is a strongly regular graph of order $n = (4k+1)^2$ and degree $r = 2k(2k+1)$ with $\tau = k^2 - k - 1$ and $\theta = k(k+1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(3k+1)$ with $m_2 = 6k(2k+1)$ and $m_3 = 2k(2k+1)$;*

$(\overline{2}^0)$ *G is a strongly regular graph of order $n = (4k+1)^2$ and degree $r = 6k(2k+1)$ with $\tau = 9k^2 + 5k - 1$ and $\theta = 3k(3k+1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(k+1)$ with $m_2 = 2k(2k+1)$ and $m_3 = 6k(2k+1)$;*

(3^0) *G is a strongly regular graph of order $n = (4k+3)^2$ and degree $r = 2(k+1)(2k+1)$ with $\tau = k^2 + 3k + 1$ and $\theta = k(k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k+2$ and $\lambda_3 = -(k+1)$ with $m_2 = 2(k+1)(2k+1)$ and $m_3 = 6(k+1)(2k+1)$;*

($\overline{3}^0$) *G is a strongly regular graph of order* $n = (4k+3)^2$ *and degree* $r = 6(k+1)(2k+1)$ *with* $\tau = 9k^2+13k+3$ *and* $\theta = 3(k+1)(3k+2)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = k$ *and* $\lambda_3 = -(3k+3)$ *with* $m_2 = 6(k+1)(2k+1)$ *and* $m_3 = 2(k+1)(2k+1)$*.*

Proposition 5. *Let G be a connected strongly regular graph of order n and degree r with* $m_2 = 3m_3$*. Then G belongs to the class* (2^0) *or* ($\overline{3}^0$) *represented in Theorem* 3*.*

Proof. Let $m_3 = p$, $m_2 = 3p$ and $n = 4p+1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - 3\lambda_2)$. Let $|\lambda_3| - 3\lambda_2 = t$ where $t = 1, 2, 3$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(3k+t)$; (ii) $\tau - \theta = -(2k+t)$; (iii) $\delta = 4k+t$; (iv) $r = pt$ and (v) $\theta = pt - 3k^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - (4p+1)t + 12k^2 + 6kt = 0\,. \tag{5}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+1)$, $\tau - \theta = -(2k+1)$, $\delta = 4k+1$, $r = p$ and $\theta = p - 3k^2 - k$. Using (5) we find that $p = 2k(2k+1)$. So we obtain that G is a strongly regular graph of order $(4k+1)^2$ and degree $r = 2k(2k+1)$ with $\tau = k^2 - k - 1$ and $\theta = k(k+1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+2)$, $\tau - \theta = -(2k+2)$, $\delta = 4k+2$, $r = 2p$ and $\theta = 2p - 3k^2 - 2k$. Using (5) we find that $2p - 1 = 6k(k+1)$, a contradiction because $2 \nmid (2p-1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+3)$, $\tau - \theta = -(2k+3)$, $\delta = 4k+3$, $r = 3p$ and $\theta = 3p - 3k^2 - 3k$. Using (5) we find that $p = 2(k+1)(2k+1)$. So we obtain that G is a strongly regular graph of order $(4k+3)^2$ and degree $r = 6(k+1)(2k+1)$ with $\tau = 9k^2+13k+3$ and $\theta = 3(k+1)(3k+2)$. $\square$

Proposition 6. *Let G be a connected strongly regular graph of order n and degree r with* $m_3 = 3m_2$*. Then G belongs to the class* ($\overline{2}^0$) *or* (3^0) *represented in Theorem* 3*.*

Proof. Let $m_2 = p$, $m_3 = 3p$ and $n = 4p+1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(3|\lambda_3| - \lambda_2)$. Let $3|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, 3$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = 3k - t$; (ii) $\tau - \theta = 2k - t$; (iii) $\delta = 4k - t$;

(iv) $r = pt$ and (v) $\theta = pt - 3k^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - (4p+1)t + 12k^2 - 6kt = 0\,. \tag{6}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 3k-1$ and $\lambda_3 = -k$, $\tau - \theta = 2k-1$, $\delta = 4k-1$, $r = p$ and $\theta = p - 3k^2 + k$. Using (6) we find that $p = 2k(2k-1)$. Replacing k with $k+1$ we arrive at $p = 2(k+1)(2k+1)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(4k+3)^2$ and degree $r = 2(k+1)(2k+1)$ with $\tau = k^2 + 3k + 1$ and $\theta = k(k+1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 3k-2$ and $\lambda_3 = -k$, $\tau - \theta = 2k-2$, $\delta = 4k-2$, $r = 2p$ and $\theta = 2p - 3k^2 + 2k$. Using (6) we find that $2p - 1 = 6k(k-1)$, a contradiction because $2 \nmid (2p-1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 3k-3$ and $\lambda_3 = -k$, $\tau - \theta = 2k-3$, $\delta = 4k-3$, $r = 3p$ and $\theta = 3p - 3k^2 + 3k$. Using (6) we find that $p = 2(k-1)(2k-1)$. Replacing k with $k+1$ we arrive at $p = 2k(2k+1)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(4k+1)^2$ and degree $r = 6k(2k+1)$ with $\tau = 9k^2 + 5k - 1$ and $\theta = 3k(3k+1)$. □

Proof of Theorem 3. According to Proposition 5 it turns out that G belongs to the class (2^0) or $(\overline{3}^0)$ if $m_2 = 3m_3$. According to Proposition 6 it turns out that G belongs to the class $(\overline{2}^0)$ or (3^0) if $m_3 = 3m_2$. □

Remark 7. We note that $\overline{3K_3}$ is a strongly regular graph with $m_2 = 3m_3$. It is obtained from the class Theorem 3 $(\overline{3}^0)$ for $k = 0$.

Theorem 4. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 4m_3$ or $m_3 = 4m_2$. Then G is one of the following strongly regular graphs:*

(1^0) *G is the complete bipartite $K_{3,3}$ of order $n = 6$ and degree $r = 3$ with $\tau = 0$ and $\theta = 3$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 4$ and $m_3 = 1$;*

(2^0) *G is the strongly regular graph $\overline{4K_4}$ of order $n = 16$ and degree $r = 12$ with $\tau = 8$ and $\theta = 12$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -4$ with $m_2 = 12$ and $m_3 = 3$;*

(3^0) *G is a strongly regular graph of order* $n=(5k+1)^2$ *and degree* $r=k(5k+2)$ *with* $\tau=k^2-2k-1$ *and* $\theta=k(k+1)$*, where* $k\geq 3$*. Its eigenvalues are* $\lambda_2=k$ *and* $\lambda_3=-(4k+1)$ *with* $m_2=4k(5k+2)$ *and* $m_3=k(5k+2)$*;*

($\overline{3}^0$) *G is a strongly regular graph of order* $n=(5k+1)^2$ *and degree* $r=4k(5k+2)$ *with* $\tau=16k^2+7k-1$ *and* $\theta=4k(4k+1)$*, where* $k\geq 3$*. Its eigenvalues are* $\lambda_2=4k$ *and* $\lambda_3=-(k+1)$ *with* $m_2=k(5k+2)$ *and* $m_3=4k(5k+2)$*;*

(4^0) *G is a strongly regular graph of order* $n=(5k+4)^2$ *and degree* $r=(k+1)(5k+3)$ *with* $\tau=k^2+4k+2$ *and* $\theta=k(k+1)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=4k+3$ *and* $\lambda_3=-(k+1)$ *with* $m_2=(k+1)(5k+3)$ *and* $m_3=4(k+1)(5k+3)$*;*

($\overline{4}^0$) *G is a strongly regular graph of order* $n=(5k+4)^2$ *and degree* $r=4(k+1)(5k+3)$ *with* $\tau=16k^2+25k+8$ *and* $\theta=4(k+1)(4k+3)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=k$ *and* $\lambda_3=-(4k+4)$ *with* $m_2=4(k+1)(5k+3)$ *and* $m_3=(k+1)(5k+3)$*;*

(5^0) *G is a strongly regular graph of order* $n=6(5k-1)^2$ *and degree* $r=2(30k^2-12k+1)$ *with* $\tau=24k^2-15k+1$ *and* $\theta=6k(4k-1)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=3k-1$ *and* $\lambda_3=-(12k-2)$ *with* $m_2=4(30k^2-12k+1)$ *and* $m_3=30k^2-12k+1$*;*

($\overline{5}^0$) *G is a strongly regular graph of order* $n=6(5k-1)^2$ *and degree* $r=3(30k^2-12k+1)$ *with* $\tau=18k(3k-1)$ *and* $\theta=3(3k-1)(6k-1)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=12k-3$ *and* $\lambda_3=-3k$ *with* $m_2=30k^2-12k+1$ *and* $m_3=4(30k^2-12k+1)$*;*

(6^0) *G is a strongly regular graph of order* $n=6(5k+1)^2$ *and degree* $r=2(30k^2+12k+1)$ *with* $\tau=24k^2+15k+1$ *and* $\theta=6k(4k+1)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=12k+2$ *and* $\lambda_3=-(3k+1)$ *with* $m_2=30k^2+12k+1$ *and* $m_4=4(30k^2+12k+1)$*;*

($\overline{6}^0$) *G is a strongly regular graph of order* $n=6(5k+1)^2$ *and degree* $r=3(30k^2+12k+1)$ *with* $\tau=18k(3k+1)$ *and* $\theta=3(3k+1)(6k+1)$*, where* $k\in\mathbb{N}$*. Its eigenvalues are* $\lambda_2=3k$ *and* $\lambda_3=-(12k+3)$ *with* $m_2=4(30k^2+12k+1)$ *and* $m_3=30k^2+12k+1$.

Proposition 7. *Let G be a connected strongly regular graph of order n* *and degree r with $m_2 = 4m_3$. Then G belongs to the class* (3^0) *or* $(\overline{4}^0)$ *or* (5^0) *or* $(\overline{6}^0)$ *represented in Theorem* 4.

Proof. Let $m_3 = p$, $m_2 = 4p$ and $n = 5p+1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - 4\lambda_2)$. Let $|\lambda_3| - 4\lambda_2 = t$ where $t = 1,2,3,4$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(4k+t)$; (ii) $\tau - \theta = -(3k+t)$; (iii) $\delta = 5k+t$; (iv) $r = pt$ and (v) $\theta = pt - 4k^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - (5p+1)t + 20k^2 + 8kt = 0\,. \tag{7}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(4k+1)$, $\tau - \theta = -(3k+1)$, $\delta = 5k+1$, $r = p$ and $\theta = p - 4k^2 - k$. Using (7) we find that $p = k(5k+2)$. So we obtain that G is a strongly regular graph of order $(5k+1)^2$ and degree $r = k(5k+2)$ with $\tau = k^2 - 2k - 1$ and $\theta = k(k+1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(4k+2)$, $\tau - \theta = -(3k+2)$, $\delta = 5k+2$, $r = 2p$ and $\theta = 2p - 4k^2 - 2k$. Using (7) we find that $3p - 1 = 2k(5k+4)$. Replacing k with $3k-1$ we arrive at $p = 30k^2 - 12k + 1$, where k is positive integer. So we obtain that G is a strongly regular graph of order $6(5k-1)^2$ and degree $r = 2(30k^2 - 12k + 1)$ with $\tau = 24k^2 - 15k + 1$ and $\theta = 6k(4k-1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(4k+3)$, $\tau - \theta = -(3k+3)$, $\delta = 5k+3$, $r = 3p$ and $\theta = 3p - 4k^2 - 3k$. Using (7) we find that $3(p-1) = 2k(5k+6)$. Replacing k with $3k$ we arrive at $p = 30k^2 + 12k + 1$, where k is positive integer. So we obtain that G is a strongly regular graph of order $6(5k+1)^2$ and degree $r = 3(30k^2 + 12k + 1)$ with $\tau = 18k(3k+1)$ and $\theta = 3(3k+1)(6k+1)$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(4k+4)$, $\tau - \theta = -(3k+4)$, $\delta = 5k+4$, $r = 4p$ and $\theta = 4p - 4k^2 - 4k$. Using (7) we find that $p = (k+1)(5k+3)$. So we obtain that G is a strongly regular graph of order $(5k+4)^2$ and degree $r = 4(k+1)(5k+3)$ with $\tau = 16k^2 + 25k + 8$ and $\theta = 4(k+1)(4k+3)$. □

Proposition 8. *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 4m_2$. Then G belongs to the class* $(\overline{3}^0)$ *or* (4^0) *or* $(\overline{5}^0)$ *or* (6^0) *represented in Theorem* 3.

Proof. Let $m_2 = p$, $m_3 = 4p$ and $n = 5p+1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(4|\lambda_3| - \lambda_2)$. Let $4|\lambda_3| - \lambda_2 = t$ where $t = 1,2,3,4$. Let $\lambda_3 = -k$

where k is a positive integer. Then (i) $\lambda_2 = 4k - t$; (ii) $\tau - \theta = 3k - t$; (iii) $\delta = 5k - t$; (iv) $r = pt$ and (v) $\theta = pt - 4k^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - (5p+1)t + 20k^2 - 8kt = 0\,. \tag{8}$$

Case 1. $(t = 1)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 4k-1$ and $\lambda_3 = -k$, $\tau - \theta = 3k-1$, $\delta = 5k-1$, $r = p$ and $\theta = p-4k^2+k$. Using (8) we find that $p = k(5k-2)$. Replacing k with $k+1$ we arrive at $p = (k+1)(5k+3)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(5k+4)^2$ and degree $r = (k+1)(5k+3)$ with $\tau = k^2+4k+2$ and $\theta = k(k+1)$.

Case 2. $(t = 2)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 4k-2$ and $\lambda_3 = -k$, $\tau - \theta = 3k-2$, $\delta = 5k-2$, $r = 2p$ and $\theta = 2p-4k^2+2k$. Using (8) we find that $3p - 1 = 2k(5k-4)$. Replacing k with $3k+1$ we arrive at $p = 30k^2+12k+1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $6(5k+1)^2$ and degree $r = 2(30k^2+12k+1)$ with $\tau = 24k^2+15k+1$ and $\theta = 6k(4k+1)$.

Case 3. $(t = 3)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 4k-3$ and $\lambda_3 = -k$, $\tau - \theta = 3k-3$, $\delta = 5k-3$, $r = 3p$ and $\theta = 3p-4k^2+3k$. Using (8) we find that $3(p-1) = 2k(5k-6)$. Replacing k with $3k$ we arrive at $p = 30k^2-12k+1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $6(5k-1)^2$ and degree $r = 3(30k^2-12k+1)$ with $\tau = 18k(3k-1)$ and $\theta = 3(3k-1)(6k-1)$.

Case 4. $(t = 4)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 4k-4$ and $\lambda_3 = -k$, $\tau - \theta = 3k-4$, $\delta = 5k-4$, $r = 4p$ and $\theta = 4p-4k^2+4k$. Using (8) we find that $p = (k-1)(5k-3)$. Replacing k with $k+1$ we arrive at $p = k(5k+2)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $(5k+1)^2$ and degree $r = 4k(5k+2)$ with $\tau = 16k^2+7k-1$ and $\theta = 4k(4k+1)$. □

Proof of Theorem 4. According to Proposition 7 it turns out that G belongs to the class (3^0) or $(\overline{4}^0)$ or (5^0) or $(\overline{6}^0)$ if $m_2 = 4m_3$. According to Proposition 8 it turns out that G belongs to the class $(\overline{3}^0)$ or (4^0) or $(\overline{5}^0)$ or (6^0) if $m_3 = 4m_2$. □

Remark 8. We note that the complete bipartite graph $K_{3,3}$ is a strongly regular graph with $m_2 = 4m_3$. It is obtained from the class Theorem 4 $(\overline{6}^0)$ for $k = 0$.

Remark 9. We note that $\overline{4K_4}$ is a strongly regular graph with $m_2 = 4m_3$. It is obtained from the class Theorem 4 ($\overline{4}^0$) for $k = 0$.

3. Concluding remarks. Using the same procedure applied in this work we can establish the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, as follows. First, let $m_3 = p$, $m_2 = qp$ and $n = (q+1)p + 1$ where $q \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - q\lambda_2)$. Let $|\lambda_3| - q\lambda_2 = t$ where $t = 1, 2, \ldots, q$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(qk + t)$; (ii) $\tau - \theta = -((q-1)k + t)$; (iii) $\delta = (q+1)k + t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - \big((q+1)p + 1\big)t + q(q+1)k^2 + 2qkt = 0\,. \tag{9}$$

Second, let $m_2 = p$, $m_3 = qp$ and $n = (q+1)p + 1$ where $q \in \mathbb{N}$. Using (2) we obtain $r = p(q|\lambda_3| - \lambda_2)$. Let $q|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \ldots, q$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = qk - t$; (ii) $\tau - \theta = (q-1)k - t$; (iii) $\delta = (q+1)k - t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - \big((q+1)p + 1\big)t + q(q+1)k^2 - 2qkt = 0\,. \tag{10}$$

Using (9) and (10) we can obtain for $t = 1, 2, \ldots, q$ the corresponding classes of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, respectively.

REFERENCES

[1] R. J. Elzinga. Strongly regular graphs: values of λ and μ for which there are only finitely many feasible (v, k, λ, μ). *Electron. J. Linear Algebra* **10**, 232–239, electronic only (October 2003), ISSN 1081-3810.

[2] C. Godsil, G. Royle. Algebraic Graph Theory. Springer-Verlag, New York, 2001.

[3] M. Lepović. Some characterizations of strongly regular graphs. *J. Appl. Math. Comput.* **29**, *1–2* (2009), 373–381.

[4] M. Lepović. On strongly regular graphs of order $3(2p+1)$ and $4(2p+1)$ where $2p+1$ is a prime number. *Math. J. Okayama Univ.* (submitted).

[5] M. Lepović. On strongly regular graphs with $|m_2 - m_3| \leq 3$. *Global Journal of Pure and Applied Mathematics* **6**, *2* (2010), 125–132.

[6] J. H. van Lint, R. M. Wilson. A Course of Combinatorics. Cambridge University Press, New York, 1992.

Mirko Lepović
Tihomira Vuksanovića 32
34000, Kragujevac, Serbia
e-mail: `lepovic@kg.ac.rs`

Received April 18, 2012