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WEYL GROUPS OF FINE GRADINGS ON SIMPLE LIE ALGEBRAS OF TYPES *A*, *B*, *C* AND *D*

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Dedicated to Professor Yuri Bahturin on the occasion of his sixty fifth birthday.

ABSTRACT. Given a grading $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ on a nonassociative algebra \mathcal{L} by an abelian group G , we have two subgroups of $\text{Aut}(\mathcal{L})$: the automorphisms that stabilize each component \mathcal{L}_g (as a subspace) and the automorphisms that permute the components. By the Weyl group of Γ we mean the quotient of the latter subgroup by the former. In the case of a Cartan decomposition of a semisimple complex Lie algebra, this is the automorphism group of the root system, i.e., the so-called extended Weyl group. A grading is called fine if it cannot be refined. We compute the Weyl groups of all fine gradings on simple Lie algebras of types *A*, *B*, *C* and *D* (except D_4) over an algebraically closed field of characteristic different from 2.

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1. Introduction. In [4], we computed the Weyl groups of all fine gradings on matrix algebras, the Cayley algebra \mathcal{C} and the Albert algebra \mathcal{A} over an algebraically closed field \mathbb{F} ($\text{char } \mathbb{F} \neq 2$ in the case of the Albert algebra). It is well known that $\text{Der}(\mathcal{C})$ is a simple Lie algebra of type G_2 ($\text{char } \mathbb{F} \neq 2, 3$) and $\text{Der}(\mathcal{A})$ is a simple Lie algebra of type F_4 ($\text{char } \mathbb{F} \neq 2$). Since the automorphism group schemes of \mathcal{C} and $\text{Der}(\mathcal{C})$, respectively \mathcal{A} and $\text{Der}(\mathcal{A})$, are isomorphic, the classification of fine gradings on $\text{Der}(\mathcal{C})$, respectively $\text{Der}(\mathcal{A})$, is the same as that on \mathcal{C} , respectively \mathcal{A} [3] and, moreover, the Weyl groups of the corresponding fine gradings are isomorphic. The situation with fine gradings on the simple Lie algebras belonging to series A , B , C and D is more complicated, because the fine gradings on matrix algebras yield only a part of the fine gradings on the simple Lie algebras of series A (so-called Type I gradings). In order to obtain the fine gradings for series B , C and D and the remaining (Type II) fine gradings for series A , one has to consider fine φ -gradings on matrix algebras, which were introduced and classified in [2].

The purpose of this paper is to compute the Weyl groups of all fine gradings on the simple Lie algebras of series A , B , C and D , with the sole exception of type D_4 (which differs from the other types due to the triality phenomenon), over an algebraically closed field \mathbb{F} of characteristic different from 2. To achieve this, we first determine the automorphisms of each fine φ -grading on the matrix algebra $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 3$, and then use the transfer technique of [1] to obtain the Weyl group of the corresponding fine grading on the simple Lie algebra $\mathcal{L} = [\mathcal{R}, \mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}])$ or $\mathcal{K}(\mathcal{R}, \varphi)$, where in the second case φ is an involution on \mathcal{R} and $\mathcal{K}(\mathcal{R}, \varphi)$ stands for the set of skew-symmetric elements with respect to φ .

We adopt the terminology and notation of [4], which is recalled in Section 2 for convenience of the reader. In Section 3, we restate the classification of fine φ -gradings on matrix algebras [2] in more explicit terms and determine the relevant automorphism groups of each fine φ -grading (Theorem 3.12). In Section 4, we deal with the simple Lie algebras of series A (Theorems 4.6 and 4.7) and, in Section 5, with those of series B , C and D (Theorems 5.6 and 5.7).

2. Generalities on gradings. Let \mathcal{A} be an algebra (not necessarily associative) over a field \mathbb{F} and let G be a group (written multiplicatively).

Definition 2.1. *A G -grading on \mathcal{A} is a vector space decomposition*

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

such that

$$\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh} \quad \text{for all } g, h \in G.$$

If such a decomposition is fixed, we will refer to \mathcal{A} as a G -graded algebra. The nonzero elements $a \in \mathcal{A}_g$ are said to be homogeneous of degree g ; we will write $\circ a = g$. The support of Γ is the set $\text{Supp } \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$.

There are two natural ways to define equivalence relation on graded algebras. We will use the term ‘‘isomorphism’’ for the case when the grading group is a part of the definition and ‘‘equivalence’’ for the case when the grading group plays a secondary role. Let

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \text{and} \quad \Gamma' : \mathcal{B} = \bigoplus_{h \in H} \mathcal{B}_h$$

be two gradings on algebras, with supports S and T , respectively.

Definition 2.2. We say that Γ and Γ' are equivalent if there exists an isomorphism of algebras $\psi : \mathcal{A} \rightarrow \mathcal{B}$ and a bijection $\alpha : S \rightarrow T$ such that $\psi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$. Any such ψ will be called an equivalence of Γ and Γ' (or of \mathcal{A} and \mathcal{B} if the gradings are clear from the context).

The algebras graded by a fixed group G form a category where the morphisms are the homomorphisms of G -graded algebras, i.e., algebra homomorphisms $\psi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\psi(\mathcal{A}_g) \subset \mathcal{B}_g$ for all $g \in G$.

Definition 2.3. In the case $G = H$, we say that Γ and Γ' are isomorphic if \mathcal{A} and \mathcal{B} are isomorphic as G -graded algebras, i.e., there exists an isomorphism of algebras $\psi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\psi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

It is known that if Γ is a grading on a simple Lie algebra, then $\text{Supp } \Gamma$ generates an abelian group (see e.g. [6, Proposition 3.3]). From now on, we will assume that our grading groups are abelian. Given a group grading Γ on an algebra \mathcal{A} , there are many groups G such that Γ can be realized as a G -grading, but there is one distinguished group among them [7].

Definition 2.4. Suppose that Γ admits a realization as a G_0 -grading for some group G_0 . We will say that G_0 is a universal group of Γ if, for any other realization of Γ as a G -grading, there exists a unique homomorphism $G_0 \rightarrow G$ that restricts to identity on $\text{Supp } \Gamma$.

One shows that the universal group, which we denote by $U(\Gamma)$, exists and depends only on the equivalence class of Γ . Indeed, $U(\Gamma)$ is generated by $S = \text{Supp } \Gamma$ with defining relations $s_1 s_2 = s_3$ whenever $0 \neq \mathcal{A}_{s_1} \mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$ ($s_i \in S$).

As in [7], we associate to Γ three subgroups of the automorphism group $\text{Aut}(\mathcal{A})$ as follows.

Definition 2.5. *The automorphism group of Γ , denoted $\text{Aut}(\Gamma)$, consists of all automorphisms of \mathcal{A} that permute the components of Γ . Each $\psi \in \text{Aut}(\Gamma)$ determines a self-bijection $\alpha = \alpha(\psi)$ of the support S such that $\psi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$ for all $s \in S$. The stabilizer of Γ , denoted $\text{Stab}(\Gamma)$, is the kernel of the homomorphism $\text{Aut}(\Gamma) \rightarrow \text{Sym}(S)$ given by $\psi \mapsto \alpha(\psi)$. Finally, the diagonal group of Γ , denoted $\text{Diag}(\Gamma)$, is the subgroup of the stabilizer consisting of all automorphisms ψ such that the restriction of ψ to any homogeneous component of Γ is the multiplication by a (nonzero) scalar.*

Thus $\text{Aut}(\Gamma)$ is the group of self-equivalences of the graded algebra \mathcal{A} and $\text{Stab}(\Gamma)$ is the group of automorphisms of the graded algebra \mathcal{A} . Also, $\text{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$ via the usual action of characters on \mathcal{A} : if Γ is a G -grading (in particular, we may take $G = U(\Gamma)$), then any character $\chi \in \widehat{G}$ acts as an automorphism of \mathcal{A} by setting $\chi * a = \chi(g)a$ for all $a \in \mathcal{A}_g$ and $g \in G$. If $\dim \mathcal{A} < \infty$, then $\text{Diag}(\Gamma)$ is a diagonalizable algebraic group (quasitorus). If, in addition, \mathbb{F} is algebraically closed and $\text{char } \mathbb{F} = 0$, then Γ is the eigenspace decomposition of \mathcal{A} relative to $\text{Diag}(\Gamma)$ (see e.g. [6]), the group $\text{Stab}(\Gamma)$ is the centralizer of $\text{Diag}(\Gamma)$, in $\text{Aut}(\mathcal{A})$ and $\text{Aut}(\Gamma)$ is its normalizer. If we want to work over an arbitrary field \mathbb{F} , we can define the subgroup scheme $\mathbf{Diag}(\Gamma)$ of the automorphism group scheme $\mathbf{Aut}(\mathcal{A})$ as follows:

$$\mathbf{Diag}(\Gamma)(\mathcal{S}) := \{f \in \text{Aut}_{\mathcal{S}}(\mathcal{A} \otimes \mathcal{S}) \mid f|_{\mathcal{A}_g \otimes \mathcal{S}} \in \mathcal{S}^{\times} \text{id}_{\mathcal{A}_g \otimes \mathcal{S}} \text{ for all } g \in G\}$$

for any unital commutative associative algebra \mathcal{S} over \mathbb{F} . Thus $\text{Diag}(\Gamma)$ is the group of \mathbb{F} -points of $\mathbf{Diag}(\Gamma)$. One checks that $\mathbf{Diag}(\Gamma) = U(\Gamma)^D$, the Cartier dual of $U(\Gamma)$, also $\text{Stab}(\Gamma)$ is the centralizer of $\mathbf{Diag}(\Gamma)$ and $\text{Aut}(\Gamma)$ is its normalizer with respect to the action of $\text{Aut}(\mathcal{A})$ on $\mathbf{Aut}(\mathcal{A})$ by conjugation (see e.g. [3, §2.2]).

Definition 2.6. *The quotient group $\text{Aut}(\Gamma)/\text{Stab}(\Gamma)$, which is a subgroup of $\text{Sym}(S)$, will be called the Weyl group of Γ and denoted by $W(\Gamma)$.*

It follows from the universal property of $U(\Gamma)$ that, for any $\psi \in \text{Aut}(\Gamma)$, the bijection $\alpha(\psi): \text{Supp } \Gamma \rightarrow \text{Supp } \Gamma$ extends to a unique automorphism of $U(\Gamma)$. This gives an action of $\text{Aut}(\Gamma)$ by automorphisms of $U(\Gamma)$. Since the kernel of this action is $\text{Stab}(\Gamma)$, we may regard $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ as a subgroup of $\text{Aut}(U(\Gamma))$. Given a G -grading $\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and a group homomorphism $\alpha: G \rightarrow H$, we obtain the induced H -grading ${}^{\alpha}\Gamma: \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$ by setting $\mathcal{A}'_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$. Clearly, an automorphism α of $U(\Gamma)$ belongs to $W(\Gamma)$ if and only if the $U(\Gamma)$ -gradings ${}^{\alpha}\Gamma$ and Γ are isomorphic.

Given gradings $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$, we say that Γ' is a *coarsening* of Γ , or that Γ is a *refinement* of Γ' , if for any $g \in G$ there exists $h \in H$ such that $\mathcal{A}_g \subset \mathcal{A}'_h$. The coarsening (or refinement) is said to be *proper* if the inclusion is proper for some g . (In particular, ${}^\alpha\Gamma$ is a coarsening of Γ , which is not necessarily proper.) A grading Γ is said to be *fine* if it does not admit a proper refinement in the class of (abelian) group gradings. Any G -grading on a finite-dimensional algebra \mathcal{A} is induced from some fine grading Γ by a homomorphism $\alpha : U(\Gamma) \rightarrow G$. The classification of fine gradings on \mathcal{A} up to equivalence is the same as the classification of maximal diagonalizable subgroup schemes of $\mathbf{Aut}(\mathcal{A})$ up to conjugation by $\mathbf{Aut}(\mathcal{A})$ (see e.g. [3, §2.2]). Fine gradings on simple Lie algebras belonging to the series A , B , C and D (including D_4) were classified in [2] assuming \mathbb{F} algebraically closed of characteristic 0. If we replace automorphism groups by automorphism group schemes, as was done in [1], then the arguments of [2] for all cases except D_4 (which required a completely different method) work under the much weaker assumption — which we adopt from now on — that \mathbb{F} is *algebraically closed of characteristic different from 2*.

3. Fine φ -gradings on matrix algebras. The goal of this section is to determine certain automorphism groups of fine φ -gradings on matrix algebras. These groups will be used in the next two sections to compute the Weyl groups of fine gradings on simple Lie algebras of series A , B , C and D .

3.1. Classification of fine φ -gradings on matrix algebras. Here we present the results of [2, §3] in a more explicit form. We also introduce certain objects that will appear throughout the paper.

Definition 3.1. *Let \mathcal{A} be an algebra and let φ be an anti-automorphism of \mathcal{A} . A G -grading $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is said to be a φ -grading if $\varphi(\mathcal{A}_g) = \mathcal{A}_g$ for all $g \in G$ (i.e., φ is an anti-automorphism of the G -graded algebra \mathcal{A}) and $\varphi^2 \in \text{Diag}(\Gamma)$. The universal group of a φ -grading is defined disregarding φ .*

We have natural concepts of isomorphism and equivalence for φ -gradings. In addition, we will need another relation, which is weaker than equivalence.

Definition 3.2. *If Γ_1 is a φ_1 -grading on \mathcal{A} and Γ_2 is a φ_2 -gradings on \mathcal{B} , we will say that (Γ_1, φ_1) is isomorphic (respectively, equivalent) to (Γ_2, φ_2) if there exists an isomorphism (respectively, equivalence) of graded algebras $\psi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi_2 = \psi\varphi_1\psi^{-1}$. In the special case $\mathcal{A} = \mathcal{B}$ and $\varphi_1 = \varphi_2$, we will simply say that Γ_1 is isomorphic (respectively, equivalent) to Γ_2 . We will say that (Γ_1, φ_1) is weakly equivalent to (Γ_2, φ_2) if there exists an equivalence of graded algebras $\psi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\xi\varphi_2 = \psi\varphi_1\psi^{-1}$ for some $\xi \in \text{Diag}(\Gamma_2)$.*

Note that if φ is an involution, then the condition $\varphi^2 \in \text{Diag}(\Gamma)$ is satisfied for any Γ . Also, any φ -grading Γ on \mathcal{A} restricts to the space of skew-symmetric elements $\mathcal{K}(\mathcal{A}, \varphi)$.

Suppose \mathcal{R} is a matrix algebra equipped with a G -grading Γ . Then \mathcal{R} is isomorphic to $\text{End}_{\mathcal{D}}(V)$ where \mathcal{D} is a matrix algebra with a division grading (i.e., a grading that makes it a graded division algebra) and V is a graded right \mathcal{D} -module (which is necessarily free of finite rank). Let $T \subset G$ be the support of \mathcal{D} . Then T is a group and \mathcal{D} can be identified with a twisted group algebra $\mathbb{F}^{\sigma}T$ for some 2-cocycle $\sigma: T \times T \rightarrow \mathbb{F}^{\times}$, i.e., \mathcal{D} has a basis X_t , $t \in T$, such that $X_u X_v = \sigma(u, v) X_{uv}$ for all $u, v \in T$ (we may assume $X_e = I$, the identity element of \mathcal{D}). Let $\beta(u, v) = \frac{\sigma(u, v)}{\sigma(v, u)}$, so

$$X_u X_v = \beta(u, v) X_v X_u.$$

Then $\beta: T \times T \rightarrow \mathbb{F}^{\times}$ is a nondegenerate alternating bicharacter — see e.g. [1, §2]. A division grading on a matrix algebra with a given support T and bicharacter β can be constructed as follows. Since β is nondegenerate and alternating, T admits a “symplectic basis”, i.e., there exists a decomposition of T into the direct product of cyclic subgroups:

$$T = (H'_1 \times H''_1) \times \cdots \times (H'_r \times H''_r)$$

such that $H'_i \times H''_i$ and $H'_j \times H''_j$ are β -orthogonal for $i \neq j$, and H'_i and H''_i are in duality by β . Denote by ℓ_i the order of H'_i and H''_i . (We may assume without loss of generality that ℓ_i are prime powers.) If we pick generators a_i and b_i for H'_i and H''_i , respectively, then $\varepsilon_i := \beta(a_i, b_i) = \beta(b_i, a_i)^{-1}$ is a primitive ℓ_i -th root of unity, and all other values of β on the elements $a_1, b_1, \dots, a_r, b_r$ are 1. Define the following elements of the algebra $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$:

$$X_{a_i} = I \otimes \cdots \otimes I \otimes X_i \otimes I \otimes \cdots \otimes I \quad \text{and} \quad X_{b_i} = I \otimes \cdots \otimes I \otimes Y_i \otimes I \otimes \cdots \otimes I,$$

where

$$X_i = \begin{bmatrix} \varepsilon_i^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_i^{n-2} & 0 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & \varepsilon_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are the generalized Pauli matrices in the i -th factor, $M_{\ell_i}(\mathbb{F})$. Finally, set

$$X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \cdots X_{a_r}^{i_r} X_{b_r}^{j_r}.$$

Identify $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ with $M_\ell(\mathbb{F})$, $\ell = \ell_1 \cdots \ell_r = \sqrt{|T|}$, via Kronecker product. Then

$$M_\ell(\mathbb{F}) = \bigoplus_{t \in T} \mathbb{F} X_t$$

is a division grading with support T and bicharacter β .

Let φ be an anti-automorphism of \mathcal{R} such that Γ is a φ -grading. It is shown in [2, §3] that there exists an involution φ_0 of the graded algebra \mathcal{D} and a φ_0 -sesquilinear form $B: V \times V \rightarrow \mathcal{D}$, which is nondegenerate, homogeneous and balanced, such that, for all $r \in \mathcal{R}$, $\varphi(r)$ is the adjoint of r with respect to B , i.e., $B(x, \varphi(r)y) = B(rx, y)$ for all $x, y \in V$ and $r \in \mathcal{R}$. By φ_0 -sesquilinear we mean that B is \mathbb{F} -bilinear and, for all $x, y \in V$ and $d \in \mathcal{D}$, we have $B(xd, y) = \varphi_0(d)B(x, y)$ and $B(x, yd) = B(x, y)d$; by *balanced* we mean that, for all homogeneous $x, y \in V$, $B(x, y) = 0$ is equivalent to $B(y, x) = 0$. Moreover, the existence of φ_0 forces T to be an elementary 2-group. The pair (φ_0, B) is uniquely determined by φ up to the following transformations: for any nonzero homogeneous $d \in \mathcal{D}$, we may simultaneously replace φ_0 by $\varphi'_0: a \mapsto d\varphi_0(a)d^{-1}$ and B by $B' = dB$. Using Pauli matrices (of order 2) as above to construct a realization of \mathcal{D} , we see that matrix transpose $X \mapsto {}^tX$ preserves the grading: for any $u \in T$, the transpose of X_u equals $\pm X_u$. It follows from [1, Proposition 2.3] that (φ_0, B) can be adjusted so that φ_0 coincides with the matrix transpose. We will always assume that (φ_0, B) is adjusted in this way, which makes B unique up to a scalar in \mathbb{F} . Also, we may write

$$\varphi_0(X_u) = \beta(u)X_u$$

where $\beta(u) \in \{\pm 1\}$ for all $u \in T$. If we regard T as a vector space over the field of two elements, then the function $\beta: T \rightarrow \{\pm 1\}$ is a quadratic form whose polar form is the bicharacter $\beta: T \times T \rightarrow \{\pm 1\}$.

We will say that a φ -grading is *fine* if it is not a proper coarsening of another φ -grading. The following construction of fine φ -gradings on matrix algebras was given in [2] starting from \mathcal{D} . We start from T , an elementary 2-group of even dimension, i.e., $T = \mathbb{Z}_2^{\dim T}$, which we continue to write multiplicatively. Let β be a nondegenerate alternating bicharacter on T . Fix a realization, \mathcal{D} , of the matrix algebra endowed with a division grading with support T and bicharacter β , and let φ_0 be the matrix transpose on \mathcal{D} . Let $q \geq 0$ and $s \geq 0$ be two integers. Let

$$(1) \quad \tau = (t_1, \dots, t_q), \quad t_i \in T.$$

Denote by $\tilde{G} = \tilde{G}(T, q, s, \tau)$ the abelian group generated by T and the symbols $\tilde{g}_1, \dots, \tilde{g}_{q+2s}$ with defining relations

$$(2) \quad \tilde{g}_1^2 t_1 = \dots = \tilde{g}_q^2 t_q = \tilde{g}_{q+1} \tilde{g}_{q+2} = \dots = \tilde{g}_{q+2s-1} \tilde{g}_{q+2s}.$$

Definition 3.3. Let $\mathcal{M}(\mathcal{D}, q, s, \tau)$ be the \tilde{G} -graded algebra $\text{End}_{\mathcal{D}}(V)$ where V has a \mathcal{D} -basis $\{v_1, \dots, v_{q+2s}\}$ with ${}^\circ v_i = \tilde{g}_i$. Let $n = (q + 2s)2^{\frac{1}{2} \dim T}$ and $\mathcal{R} = M_n(\mathbb{F})$. The grading Γ on \mathcal{R} obtained by identifying \mathcal{R} with $\mathcal{M}(\mathcal{D}, q, s, \tau)$ will be denoted by $\Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$. In other words, we define this grading by identifying $\mathcal{R} = M_{q+2s}(\mathcal{D})$ and setting ${}^\circ(E_{ij} \otimes X_t) := \tilde{g}_i t \tilde{g}_j^{-1}$. By abuse of notation, we will also write $\Gamma_{\mathcal{M}}(T, q, s, \tau)$.

Let \tilde{G}^0 be the subgroup of \tilde{G} generated by $\text{Supp } \Gamma$, which consists of the elements $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$, $t \in T$ (so $z_{i,i,t} = t$ for all $t \in T$). Set $z_i := z_{i,i+1,e}$ for $i = 1, \dots, q$ ($i \neq q$ if $s = 0$), $z_{q+i} = z_{q+2i-1, q+2i+1, e}$ for $i = 1, \dots, s-1$, and $z_{q+s} = z_{q+1, q+2, e}$ (if $s > 0$). If $s = 0$, then \tilde{G}^0 is generated by T and the elements z_1, \dots, z_{q-1} . If $s = 1$, then \tilde{G}^0 is generated by T and z_1, \dots, z_{q+1} . If $s > 1$, then relations (2) imply that $z_{q+2i, q+2i+2, e} = z_{q+i}^{-1}$ for $i = 1, \dots, s-1$, hence \tilde{G}^0 is generated by T and z_1, \dots, z_{q+s} . Moreover, relations (2) are equivalent to the following:

$$z_i^2 = t_i t_{i+1} \quad (1 \leq i < q), \quad z_q^2 z_{q+s} = t_q \quad (\text{if } q > 0 \text{ and } s > 0).$$

Let \tilde{G}^1 be the subgroup generated by T and z_1, \dots, z_{q-1} . Let \tilde{G}^2 be the subgroup generated by z_1, \dots, z_s if $q = 0$ and by z_q, \dots, z_{q+s-1} if $q > 0$. Then it is clear from the above relations that $\tilde{G}^0 = \tilde{G}^1 \times \tilde{G}^2$, $\tilde{G}^2 \cong \mathbb{Z}^s$, while $\tilde{G}^1 = T$ if $q = 0$ and $\tilde{G}^1 \cong \mathbb{Z}_2^{\dim T + q - 1 - 2 \dim T_0} \times \mathbb{Z}_4^{\dim T_0}$ if $q > 0$, where T_0 is the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \dots, q-1$. To summarize:

$$(3) \quad \tilde{G}^0 \cong \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s.$$

Note that relations (2) are also equivalent to the following:

$$\begin{aligned} z_{i,j,t} &= z_{j,i,t}, & i, j \leq q, \quad t \in T; \\ z_{i,q+2j-1,t} &= z_{q+2j,i,t}, \quad z_{i,q+2j,t} = z_{q+2j-1,i,t}, & i \leq q, \quad j \leq s, \quad t \in T; \\ z_{q+2i-1,q+2j-1,t} &= z_{q+2j,q+2i,t}, & i, j \leq s, \quad t \in T; \\ z_{q+2i-1,q+2j,t} &= z_{q+2j-1,q+2i,t}, & i, j \leq s, \quad i \neq j, \quad t \in T. \end{aligned}$$

One verifies that, apart from the above equalities and $z_{i,i,t} = t$, the elements $z_{i,j,t}$

are distinct, so the support of $\Gamma = \Gamma_{\mathcal{M}}(\tilde{G}, \mathcal{D}, \kappa, \tilde{\gamma})$ is given by

$$\begin{aligned} \text{Supp } \Gamma &= \{z_{i,j,t} \mid i < j \leq q, t \in T\} \cup \{z_{i,q+j,t} \mid i \leq q, j \leq 2s, t \in T\} \\ &\cup \{z_{q+2i-1,q+2j-1,t} \mid i < j \leq s, t \in T\} \cup \{z_{q+2i,q+2j,t} \mid i < j \leq s, t \in T\} \\ &\cup \{z_{q+2i-1,q+2j,t} \mid i, j \leq s, i \neq j, t \in T\} \\ &\cup \{z_{q+2i-1,q+2i,t} \mid i \leq s, t \in T\} \cup \{z_{q+2i,q+2i-1,t} \mid i \leq s, t \in T\} \cup T, \end{aligned}$$

where the union is disjoint and all homogeneous components except those that appear in the last line have dimension 2, the components of degrees $z_{q+2i-1,q+2i,t}$ and $z_{q+2i,q+2i-1,t}$ have dimension 1, and the components of degree t have dimension $q + 2s$.

Proposition 3.4. *Let $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$. Then $\tilde{G}^0 = \tilde{G}^0(T, q, s, \tau)$ is the universal group of Γ , and $\text{Diag}(\Gamma)$ consists of all automorphisms of the form $X \mapsto DXD^{-1}$, $X \in \mathcal{R}$, where*

$$(4) \quad D = \text{diag}(\lambda_1, \dots, \lambda_{q+2s}) \otimes X_t, \quad \lambda_i \in \mathbb{F}^\times, t \in T,$$

satisfying the relation

$$(5) \quad \lambda_1^2 \beta(t, t_1) = \dots = \lambda_q^2 \beta(t, t_q) = \lambda_{q+1} \lambda_{q+2} = \dots = \lambda_{q+2s-1} \lambda_{q+2s}.$$

Proof. The relations $z_{i,\ell,u} z_{\ell,j,v} = z_{i,j,uv}$, $u, v \in T$, can be rewritten in terms of the elements of $\text{Supp } \Gamma$, producing a set of defining relations for \tilde{G}^0 . It follows that \tilde{G}^0 is the universal group of Γ .

Since \tilde{G}^0 is the universal group of Γ , $\text{Diag}(\Gamma)$ consists of all automorphisms of the form $X \mapsto \chi * X$ where χ is a character of \tilde{G}^0 . Since \mathbb{F}^\times is a divisible group, we can assume that χ is a character of \tilde{G} . Let $\lambda_i = \chi(\tilde{g}_i)$, $i = 1, \dots, q + 2s$. Let t be the element of T such that $\chi(u) = \beta(t, u)$ for all $u \in T$. Looking at relations (2), we see that (5) must hold. Conversely, any $t \in T$ and a set of $\lambda_i \in \mathbb{F}^\times$ satisfying (5) will determine a character χ of \tilde{G} . It remains to observe that the action of χ on \mathcal{R} coincides with the conjugation by D as in (4). \square

The following is Proposition 3.3 from [2].

Theorem 3.5. *Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ on $\mathcal{R} = M_{q+2s}(\mathcal{D})$ by $\tilde{G}^0 = \tilde{G}^0(T, q, s, \tau)$ where τ is given by (1). Let $\mu = (\mu_1, \dots, \mu_s)$ where μ_i are scalars in \mathbb{F}^\times . Let $\varphi = \varphi_{\tau, \mu}$ be the anti-automorphism of \mathcal{R} defined by $\varphi(X) = \Phi^{-1}({}^t X)\Phi$, $X \in \mathcal{R}$, where Φ is the block-diagonal matrix given by*

$$(6) \quad \Phi = \text{diag} \left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \mu_1 I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \mu_s I & 0 \end{bmatrix} \right)$$

and I is the identity element of \mathcal{D} . Then Γ is a fine φ -grading unless $q = 2$, $s = 0$ and $t_1 = t_2$. In the latter case, Γ can be refined to a φ -grading that makes \mathcal{R} a graded division algebra.

This result and the discussion preceding Proposition 3.8 in [2] yield

Theorem 3.6. *Let Γ be a fine φ -grading on the matrix algebra $\mathcal{R} = M_n(\mathbb{F})$ over an algebraically closed field \mathbb{F} , $\text{char } \mathbb{F} \neq 2$. Then (Γ, φ) is equivalent to some $(\Gamma_{\mathcal{M}}(T, q, s, \tau), \varphi_{\tau, \mu})$ as in Theorem 3.5 where $(q + 2s)2^{\frac{1}{2} \dim T} = n$.*

In [2], in order to obtain the classification of fine gradings on simple Lie algebras of series A , one classifies, *up to weak equivalence*, all pairs (Γ, φ) where Γ is a fine φ -grading on a matrix algebra. At the same time, for series B , C and D , one classifies, *up to equivalence*, such pairs where φ is an *involution* of appropriate type: orthogonal for series B and D (we write $\text{sgn}(\varphi) = 1$) and symplectic for series C (we write $\text{sgn}(\varphi) = -1$). The classifications involve equivalences $\mathcal{D} \rightarrow \mathcal{D}'$ satisfying certain conditions, where \mathcal{D} and \mathcal{D}' are matrix algebras with division gradings. If T is the support of \mathcal{D} and T' is the support of \mathcal{D}' , then the graded algebras \mathcal{D} and \mathcal{D}' are equivalent if and only if the groups T and T' are isomorphic. Identifying T and T' , we may assume that $\mathcal{D} = \mathcal{D}'$ and look at self-equivalences of \mathcal{D} , i.e., the elements of $\text{Aut}(\Gamma_0)$ where Γ_0 is the grading on \mathcal{D} . By [4, Proposition 2.7], the Weyl group $W(\Gamma_0)$ is isomorphic to $\text{Aut}(T, \beta)$, the group of automorphisms of T that preserve the bicharacter β . Explicitly, if $\psi_0 \in \text{Aut}(\Gamma_0)$, then $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$, for all $t \in T$, where $\alpha \in \text{Aut}(T, \beta)$, and the mapping $\psi_0 \mapsto \alpha$ yields an isomorphism $\text{Aut}(\Gamma_0)/\text{Stab}(\Gamma_0) \rightarrow \text{Aut}(T, \beta)$. Hence the conditions in [2] can be rewritten in terms of the group T rather than the graded division algebra \mathcal{D} . Note that $\text{Aut}(T, \beta)$ can be regarded as a sort of symplectic group; in particular, if T is an elementary 2-group, then $\text{Aut}(T, \beta) \cong \text{Sp}_m(2)$ where $m = \dim T$.

Definition 3.7. *Given τ as in (1), we will denote by $\Sigma(\tau)$ the multiset in T determined by τ , i.e., the underlying set of $\Sigma(\tau)$ consists of the elements that occur in (t_1, \dots, t_q) , and the multiplicity of each element is the number of times it occurs there.*

The group $\text{Aut}(T, \beta)$ acts naturally on T , so we can form the semidirect product $T \rtimes \text{Aut}(T, \beta)$, which also acts on T : a pair (u, α) sends $t \in T$ to $\alpha(t)u$. Clearly, if $\dim T = 2r$, then $T \rtimes \text{Aut}(T, \beta)$ is isomorphic to $\text{ASp}_{2r}(2)$, the affine symplectic group of order $2r$ over the field of two elements (“rigid motions” of the symplectic space of dimension $2r$).

Using this notation, Theorem 3.17 of [2] can be recast as follows:

Theorem 3.8. *Consider two pairs, (Γ, φ) and (Γ', φ') , as in Theorem*

3.5, namely, $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$, $\varphi = \varphi_{\tau, \mu}$ and $\Gamma' = \Gamma_{\mathcal{M}}(T', q', s', \tau')$, $\varphi' = \varphi_{\tau', \mu'}$, where $T = \mathbb{Z}_2^{2r}$ and $T' = \mathbb{Z}_2^{2r'}$. Then (Γ, φ) and (Γ', φ') are weakly equivalent if and only if $r = r'$, $q = q'$, $s = s'$, and $\Sigma(\tau)$ is conjugate to $\Sigma(\tau')$ by the natural action of $T \rtimes \text{Aut}(T, \beta) \cong \text{ASp}_{2r}(2)$.

Let $\psi_0: \mathcal{D} \rightarrow \mathcal{D}$ be an equivalence. Then the map $\psi_0^{-1}\varphi_0\psi_0$ is an involution of the graded algebra \mathcal{D} , which has the same type as φ_0 (orthogonal). Hence there exists a nonzero homogeneous element $d_0 \in \mathcal{D}$ such that

$$(7) \quad d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi_0\psi_0)(d) \quad \text{for all } d \in \mathcal{D}.$$

Note that d_0 is determined up to a scalar in \mathbb{F} . Moreover, d_0 is symmetric with respect to φ_0 . By a similar argument, $\psi_0(d_0)$ is also symmetric. Let α be the element of $\text{Aut}(T, \beta)$ corresponding to ψ_0 and let t_0 be the degree of d_0 . Then (7) is equivalent to the following:

$$(8) \quad \beta(t_0, t)\beta(t) = \beta(\alpha(t)) \quad \text{for all } t \in T,$$

so t_0 depends only on α . Moreover, $\beta(t_0) = \beta(\alpha(t_0)) = 1$.

Definition 3.9. For any $\alpha \in \text{Aut}(T, \beta)$, the map $t \mapsto \beta(\alpha^{-1}(t))\beta(t)$ is a character of T , so there exists a unique element $t_\alpha \in T$ such that $\beta(t_\alpha, t) = \beta(\alpha^{-1}(t))\beta(t)$ for all $t \in T$. We define a new action of the group $\text{Aut}(T, \beta)$ on T by setting

$$\alpha \cdot t := \alpha(t)t_\alpha \quad \text{for all } \alpha \in \text{Aut}(T, \beta) \text{ and } t \in T.$$

In other words, $\text{Aut}(T, \beta)$ acts through the (injective) homomorphism to $T \rtimes \text{Aut}(T, \beta)$, $\alpha \mapsto (t_\alpha, \alpha)$, and the natural action of $T \rtimes \text{Aut}(T, \beta)$ on T .

Comparing this definition with equation (8), which defines the element t_0 associated to α , we see that $t_\alpha = \alpha(t_0)$. In particular, $\beta(t_\alpha) = 1$. This implies that $\beta(\alpha \cdot t) = \beta(t)$ for all $t \in T$, so the sets

$$T_+ := \{t \in T \mid \beta(t) = 1\} \quad \text{and} \quad T_- := \{t \in T \mid \beta(t) = -1\},$$

which correspond, respectively, to symmetric and skew-symmetric homogeneous components of \mathcal{D} (relative to φ_0), are invariant under the twisted action of $\text{Aut}(T, \beta)$.

Now Proposition 3.8(2) and Theorem 3.22 of [2] can be recast as follows:

Theorem 3.10. Let $\varphi = \varphi_{\tau, \mu}$ be as in Theorem 3.5. Then φ is an involution with $\text{sgn}(\varphi) = \delta$ if and only if

$$\delta = \beta(t_1) = \dots = \beta(t_q) = \mu_1 = \dots = \mu_s.$$

For gradings $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ with $T = \mathbb{Z}_2^{2r}$ and $\Gamma' = \Gamma_{\mathcal{M}}(T', q', s', \tau')$ with $T' = \mathbb{Z}_2^{2r'}$ and for involutions $\varphi = \varphi_{\tau, \mu}$ and $\varphi' = \varphi_{\tau', \mu'}$, the pairs (Γ, φ) and (Γ', φ') are equivalent if and only if $r = r'$, $q = q'$, $s = s'$, $\text{sgn}(\varphi) = \text{sgn}(\varphi')$, and $\Sigma(\tau)$ is conjugate to $\Sigma(\tau')$ by the twisted action of $\text{Aut}(T, \beta) \cong \text{Sp}_{2r}(2)$ as in Definition 3.9.

3.2. Automorphism groups of fine φ -gradings on matrix algebras. We are now going to study automorphisms of the fine φ -gradings $\Gamma_{\mathcal{M}}(T, q, s, \tau)$. We begin with some general observations. Let \mathcal{D} and \mathcal{D}' be graded division algebras, with the same grading group G . Let V be a graded right \mathcal{D} -module and V' a graded right \mathcal{D}' -module, both of nonzero finite rank. By an *isomorphism from (\mathcal{D}, V) to (\mathcal{D}', V')* we mean a pair (ψ_0, ψ_1) where $\psi_0: \mathcal{D} \rightarrow \mathcal{D}'$ is an isomorphism of graded algebras, $\psi_1: V \rightarrow V'$ is an isomorphism of graded vector spaces over \mathbb{F} , and $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in \mathcal{D}$.

Let $\mathcal{R} = \text{End}_{\mathcal{D}}(V)$ and $\mathcal{R}' = \text{End}_{\mathcal{D}'}(V')$. If $\psi: \mathcal{R} \rightarrow \mathcal{R}'$ is an isomorphism of graded algebras, then there exist $g \in G$ and an isomorphism (ψ_0, ψ_1) from $(\mathcal{D}, V^{[g]})$ to (\mathcal{D}', V') such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$ (see e.g. [2, Proposition 2.5]). Here $V^{[g]}$ denotes a shift of grading: the $(\mathcal{R}, \mathcal{D})$ -bimodule structure of $V^{[g]}$ is the same as that of V , but we set $V_h^{[g]} = V_{hg^{-1}}$ for all $h \in G$. Conversely, given an isomorphism (ψ_0, ψ_1) of the above pairs, there exists a unique isomorphism $\psi: \mathcal{R} \rightarrow \mathcal{R}'$ of graded algebras such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$. Two isomorphisms (ψ_0, ψ_1) and (ψ'_0, ψ'_1) determine the same isomorphism $\mathcal{R} \rightarrow \mathcal{R}'$ if and only if there exists a nonzero homogeneous $d \in \mathcal{D}'$ such that $\psi'_0(x) = d^{-1}\psi_0(x)d$ and $\psi'_1(v) = \psi_1(v)d$ for all $x \in \mathcal{D}$ and $v \in V$.

Lemma 3.11. *Let $\psi: \mathcal{R} \rightarrow \mathcal{R}'$ be the isomorphism of graded algebras determined by an isomorphism (ψ_0, ψ_1) from $(\mathcal{D}, V^{[g]})$ to (\mathcal{D}', V') . Suppose that the graded algebras \mathcal{R} and \mathcal{R}' admit anti-automorphisms φ and φ' , respectively, determined by a φ_0 -sesquilinear form $B: V \times V \rightarrow \mathcal{D}$ and a φ'_0 -sesquilinear form $B': V' \times V' \rightarrow \mathcal{D}'$. Then $\varphi' = \psi\varphi\psi^{-1}$ if and only if there exists a nonzero homogeneous $d_0 \in \mathcal{D}$ such that*

$$(9) \quad B'(\psi_1(v), \psi_1(w)) = \psi_0(d_0 B(v, w)) \quad \text{for all } v, w \in V.$$

Moreover, $d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi'_0\psi_0)(d)$ for all $d \in \mathcal{D}$.

Proof. Set $\varphi'' := \psi^{-1}\varphi'\psi$ and $B''(v, w) := \psi_0^{-1}(B'(\psi_1(v), \psi_1(w)))$ for all

$v, w \in V$. Then we compute:

$$\begin{aligned}
 B''(v, wd) &= \psi_0^{-1}(B'(\psi_1(v), \psi_1(w)\psi_0(d))) \\
 &= \psi_0^{-1}(B'(\psi_1(v), \psi_1(w))\psi_0(d)) = B''(v, w)d; \\
 B''(vd, w) &= \psi_0^{-1}(B'(\psi_1(v)\psi_0(d), \psi_1(w))) \\
 &= \psi_0^{-1}(\varphi'_0(\psi_0(d))B'(\psi_1(v), \psi_1(w))) = (\psi_0^{-1}\varphi'_0\psi_0)(d)B''(v, w); \\
 B''(v, \varphi''(r)w) &= \psi_0^{-1}(B'(\psi_1(v), \psi(\varphi''(r))\psi_1(w))) \\
 &= \psi_0^{-1}(B'(\psi_1(v), \varphi'(\psi(r))\psi_1(w))) \\
 &= \psi_0^{-1}(B'(\psi(r)\psi_1(v), \psi_1(w))) = B''(rv, w).
 \end{aligned}$$

We have shown that B'' is a $(\psi_0^{-1}\varphi'_0\psi_0)$ -sesquilinear form corresponding to φ'' . Hence $\varphi'' = \varphi$ if and only if there exists a nonzero homogeneous element $d_0 \in \mathcal{D}$ such that $B'' = d_0B$, i.e., equation (9) holds. \square

Now consider $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ and $\varphi = \varphi_{\tau, \mu}$ as in Theorem 3.5. There are two kinds of automorphism groups that we will need. Namely, there is

$$\text{Aut}^*(\Gamma, \varphi) := \{\psi \in \text{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \xi\varphi \text{ for some } \xi \in \text{Diag}(\Gamma)\},$$

which will be relevant to computing the Weyl group of the corresponding fine grading on the simple Lie algebra of type A , and there is

$$\text{Aut}(\Gamma, \varphi) := \{\psi \in \text{Aut}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\},$$

which will be relevant to computing the Weyl groups of fine gradings on the simple Lie algebras of types B , C and D . Hence, we are interested in $\text{Aut}(\Gamma, \varphi)$ only if φ is an involution. Similarly, define

$$\text{Stab}(\Gamma, \varphi) := \{\psi \in \text{Stab}(\Gamma) \mid \psi\varphi\psi^{-1} = \varphi\}.$$

(We could also define $\text{Stab}^*(\Gamma, \varphi)$, but we will not need it.)

Recall that Γ is the grading on $\mathcal{R} = \text{End}_{\mathcal{D}}(V)$ where \mathcal{D} is a matrix algebra equipped with a division grading with support $T = \mathbb{Z}_2^{2r}$ and bicharacter β , and V has a \mathcal{D} -basis $\{v_1, \dots, v_k\}$ with ${}^\circ v_i = \tilde{g}_i$ and $k = q + 2s$. We will use the universal group \tilde{G}^0 for the grading Γ . If $\psi: \mathcal{R} \rightarrow \mathcal{R}$ is an equivalence, then there exists an automorphism α of the group \tilde{G}^0 such that ψ sends ${}^\alpha\Gamma$ to Γ . In other words, $\psi: \mathcal{R}' \rightarrow \mathcal{R}$ is an isomorphism of graded algebras where \mathcal{R}' is \mathcal{R} as an algebra, but equipped with the grading ${}^\alpha\Gamma$. Define \mathcal{D}' similarly to \mathcal{R}' , using the restriction of α to $T \subset \tilde{G}^0$. The support of \mathcal{D}' is $T' = \alpha(T)$. Since $V[\tilde{g}_1^{-1}]$ is \tilde{G}^0 -graded, we

can also define V' so that $\mathcal{R}' = \text{End}_{\mathcal{D}'}(V')$ as a graded algebra. Therefore, ψ is determined by (ψ_0, ψ_1) where $\psi_0: \mathcal{D}' \rightarrow \mathcal{D}$ is an isomorphism of graded algebras and $\psi_1: V' \rightarrow V$ is an isomorphism up to a shift of grading. Hence $T' = T$ and $\psi_0 \in \text{Aut}(\Gamma_0)$, so $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$, for all $t \in T$, and the map $\alpha: T \rightarrow T$ belongs to $\text{Aut}(T, \beta) \cong \text{Sp}_{2r}(2)$. Also, if Ψ is the matrix of ψ_1 relative to $\{v_1, \dots, v_k\}$, we have

$$\psi(X) = \Psi\psi_0(X)\Psi^{-1} \quad \text{for all } X \in \mathcal{R}.$$

Since all \tilde{g}_i are distinct modulo T , matrix Ψ necessarily has the form $\Psi = PD$ where P is a permutation matrix and $D = \text{diag}(d_1, \dots, d_k)$ where d_i are nonzero homogeneous elements of \mathcal{D} . Moreover, the permutation $\pi \in \text{Sym}(k)$ corresponding to P and the coset of ψ_0 modulo $\text{Stab}(\Gamma_0)$ are uniquely determined by ψ . Hence, we have a well-defined homomorphism

$$\text{Aut}(\Gamma) \rightarrow \text{Sym}(k) \times \text{Aut}(T, \beta)$$

that sends ψ to the corresponding (π, α) .

Now we turn to the anti-automorphism $\varphi: \mathcal{R} \rightarrow \mathcal{R}$, which is given by the adjoint with respect to a φ_0 -sesquilinear form B on V where $\varphi_0: \mathcal{D} \rightarrow \mathcal{D}$ is given by matrix transpose, $X_t \mapsto \beta(t)X_t$ for all $t \in T$. Recall that such B is determined up to a scalar in \mathbb{F} . We can take for B the φ_0 -sesquilinear form whose matrix with respect to $\{v_1, \dots, v_k\}$ is Φ displayed in Theorem 3.5. Pick $\xi \in \text{Diag}(\Gamma)$ and let B' be a φ_0 -sesquilinear form on V corresponding to $\xi\varphi$. By Lemma 3.11, ψ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if condition (9) holds for some nonzero homogeneous $d_0 \in \mathcal{D}$. Clearly, (9) is equivalent to (7) and

$$(10) \quad \widehat{\Phi} = \psi_0(d_0\Phi),$$

where $\widehat{\Phi}$ is the matrix of B' relative to $\{\psi_1(v_1), \dots, \psi_1(v_k)\}$. Recall that (7) is equivalent to condition (8) on $t_0 := {}^\circ d_0$. To summarize, ψ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if

$$(11) \quad \widehat{\Phi} = d_0\psi_0(\Phi)$$

for some $d_0 \in \mathcal{D}$ of degree t_α as in Definition 3.9 (we have replaced $\psi_0(d_0)$ in (10) by d_0 to simplify notation).

The matrix of B' relative to $\{v_1, \dots, v_k\}$ is $\Phi(D')^{-1}$ where $\xi(X) = D'X(D')^{-1}$, for all $X \in \mathcal{R}$, with D' of the form given by Proposition 3.4: $D' = \text{diag}(\nu_1 X_u, \dots, \nu_k X_u)$ for some $u \in T$ and $\nu_i \in \mathbb{F}^\times$ satisfying

$$(12) \quad \nu_1^2\beta(u, t_1) = \dots = \nu_q^2\beta(u, t_q) = \nu_{q+1}\nu_{q+2} = \dots = \nu_{q+2s-1}\nu_{q+2s}.$$

It follows at once that, for $\psi \in \text{Aut}^*(\Gamma, \varphi)$, the permutation π must preserve the set $\{1, \dots, q\}$ and the pairing of $q + 2i - 1$ with $q + 2i$, for $i = 1, \dots, s$. It is convenient to introduce the group $W(s) := \mathbb{Z}_2^s \rtimes \text{Sym}(s)$ (i.e., the wreath product of $\text{Sym}(s)$ and \mathbb{Z}_2), which will be regarded as the group of permutations on $\{q + 1, \dots, q + 2s\}$ that respect the block decomposition $\{q + 1, q + 2\} \cup \dots \cup \{q + 2s - 1, q + 2s\}$. The reason for the notation $W(s)$ is that $\mathbb{Z}_2^s \rtimes \text{Sym}(s)$ is the classical Weyl group of type B_s or C_s (and also the extended Weyl group of type D_s if $s > 4$). By the above discussion, we have a homomorphism:

$$(13) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut}(T, \beta).$$

We need some more notation to state the main result of this section. Let Σ be a multiset of cardinality q and let m_1, \dots, m_ℓ be the multiplicities of the elements of Σ , written in some order. Thus, m_i are positive integers whose sum is q . We will denote by $\text{Sym}\Sigma$ the subgroup $\text{Sym}(m_1) \times \dots \times \text{Sym}(m_\ell)$ of $\text{Sym}(q)$, which may be thought of as “interior symmetries” of Σ . For a multiset Σ in T , let $\text{Aut}^*\Sigma$ be the stabilizer of Σ under the natural action of $T \rtimes \text{Aut}(T, \beta)$ on T , i.e., $\text{Aut}^*\Sigma$ is the set of “rigid motions” of the symplectic space T that permute the elements of Σ preserving multiplicity. These are “exterior symmetries” of Σ . Note that each bijection $\theta: T \rightarrow T$ that stabilizes Σ determines an element of $\text{Sym}(q)$ that permutes the blocks of sizes m_1, \dots, m_ℓ in the same way θ permutes the elements of Σ (thus, only blocks of equal size may be permuted) and preserves the order within each block; we will call this permutation the *restriction of θ to Σ* . Hence, we obtain a restriction homomorphism $\text{Aut}^*\Sigma \rightarrow \text{Sym}(q)$. In particular, $\text{Aut}^*\Sigma$ acts naturally on $\text{Sym}\Sigma$ by permuting factors (of equal order). Finally, let $\text{Aut}\Sigma$ be the stabilizer of Σ under the twisted action of $\text{Aut}(T, \beta)$ on T as in Definition 3.9. Note that $\text{Aut}\Sigma$ may be regarded as a subgroup of $\text{Aut}^*\Sigma$.

Theorem 3.12. *Let $\Gamma = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ and let φ be as in Theorem 3.5 such that Γ is a fine φ -grading. Let $\Sigma = \Sigma(\tau)$, so $|\Sigma| = q$.*

1) $\text{Stab}(\Gamma, \varphi) = \text{Diag}(\Gamma)$.

2) $\text{Aut}^*(\Gamma, \varphi)/\text{Stab}(\Gamma, \varphi)$ is isomorphic to an extension of the group $((T^{q+s-1} \times \mathbb{Z}_2^s) \times (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^*\Sigma$ by \mathbb{Z}_2^{q+s-1} , with the following actions: T^{q+s-1} is identified with T^{q+s}/T and \mathbb{Z}_2^{q+s-1} is identified with $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$, where T and \mathbb{Z}_2 are imbedded diagonally, then

- $\text{Sym}\Sigma \subset \text{Sym}(q)$ acts on T^{q+s}/T and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the first q components and trivially on \mathbb{Z}_2^s ;

- $\text{Sym}(s)$ acts on T^{q+s}/T and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the last s components and naturally on \mathbb{Z}_2^s ;
 - $\text{Aut}^* \Sigma$ acts on $\text{Sym} \Sigma$ and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ through the restriction homomorphism $\text{Aut}^* \Sigma \rightarrow \text{Sym}(q)$, trivially on $\text{Sym}(s)$, and as follows on $(T^{q+s}/T) \times \mathbb{Z}_2^s$: an element $(u, \alpha) \in \text{Aut}^* \Sigma \subset T \rtimes \text{Aut}(T, \beta)$ sends a pair $((u_1, \dots, u_q, u_{q+1}, \dots, u_{q+s})T, \underline{x}) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$ to $((\alpha(u_{\pi^{-1}(1)}), \dots, \alpha(u_{\pi^{-1}(q)}), \alpha(u_{q+1})u^{x_1}, \dots, \alpha(u_{q+s})u^{x_s})T, \underline{x})$, where π is the image of (u, α) under the restriction homomorphism;
 - $T^{q+s-1} \times \mathbb{Z}_2^s$ acts trivially on \mathbb{Z}_2^{q+s-1} .
- 3) If φ is an involution, then $\text{Aut}(\Gamma, \varphi)/\text{Stab}(\Gamma, \varphi)$ is isomorphic to $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym} \Sigma \times \text{Sym}(s))) \rtimes \text{Aut} \Sigma$, with the following actions: T^{q+s-1} is identified with T^{q+s}/T , where T is imbedded diagonally, then
- $\text{Sym} \Sigma \subset \text{Sym}(q)$ acts on T^{q+s}/T by permuting the first q components and trivially on \mathbb{Z}_2^s ;
 - $\text{Sym}(s)$ acts on T^{q+s}/T by permuting the last s components and naturally on \mathbb{Z}_2^s ;
 - $\text{Aut} \Sigma$ acts on $\text{Sym} \Sigma$ as a subgroup of $\text{Aut}^* \Sigma$, i.e., through the twisted action on T (Definition 3.9) and restriction to Σ , trivially on $\text{Sym}(s)$, and as follows on $(T^{q+s}/T) \times \mathbb{Z}_2^s$: an element $\alpha \in \text{Aut} \Sigma \subset \text{Aut}(T, \beta)$ sends a pair $((u_1, \dots, u_q, u_{q+1}, \dots, u_{q+s})T, \underline{x}) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$ to $((\alpha(u_{\pi^{-1}(1)}), \dots, \alpha(u_{\pi^{-1}(q)}), \alpha(u_{q+1})t_\alpha^{x_1}, \dots, \alpha(u_{q+s})t_\alpha^{x_s})T, \underline{x})$, where π is the image of (t_α, α) under the restriction to Σ .

Proof. 1) If $\psi \in \text{Stab}(\Gamma, \varphi)$, then $\Psi = PD$ where P corresponds to $\pi \in \text{Sym}(q) \times W(s)$, and $\psi_0 \in \text{Stab}(\Gamma_0)$. Adjusting D if necessary, we may assume $\psi_0 = \text{id}$. We claim that π is the trivial permutation. Since ψ does not permute the homogeneous components of Γ , π must act trivially on \tilde{G}^0/T . So, we consider the action of $\text{Sym}(q) \times W(s)$ on \tilde{G}^0/T in terms of the generators z_i ($i = 1, \dots, q-1$ if $s = 0$ and $i = 1, \dots, q+s$ if $s > 0$) that were introduced after Definition 3.3.

$\text{Sym}(q)$ acts trivially on the subgroup $\langle z_{q+1}, \dots, z_{q+s} \rangle$ and via the action of the classical Weyl group of type A_{q-1} , taken modulo 2, on the subgroup $\langle z_1, \dots, z_{q-1} \rangle \cong \mathbb{Z}_2^{q-1}$ where z_i is identified with the element $\varepsilon_i - \varepsilon_{i+1}$, with $\{\varepsilon_1, \dots, \varepsilon_q\}$ being the standard basis of \mathbb{Z}_2^q , on which $\text{Sym}(q)$ acts naturally.

$W(s)$ acts trivially on the subgroup $\langle z_1, \dots, z_{q-1} \rangle$ and via the action of the classical Weyl group of type B_s or C_s on the subgroup $\langle z_{q+1}, \dots, z_{q+s} \rangle \cong \mathbb{Z}^s$

where z_{q+i} is identified with the element $\varepsilon_i - \varepsilon_{i+1}$ for $i \neq s$ and z_{q+s} is identified with the element $2\varepsilon_1$, with $\{\varepsilon_1, \dots, \varepsilon_s\}$ being the standard basis of \mathbb{Z}^s . The easiest way to see this is to extend \widetilde{G} by adding a new element \widehat{g}_0 satisfying $(\widehat{g}_0)^{-2} = \widetilde{g}_1 \widetilde{g}_2$ and set $\widehat{g}_i = \widetilde{g}_i \widehat{g}_0$. The elements of the subgroup \widetilde{G}^0 are not affected if we replace \widetilde{g}_i by \widehat{g}_i , but then we have $\widehat{g}_{q+2j} = \widehat{g}_{q+2j-1}^{-1}$ for $j = 1, \dots, s$, so we can map \widehat{g}_{q+2j-1} to ε_j and \widehat{g}_{q+2j} to $-\varepsilon_j$.

Note that the action of $W(s)$ on $\langle z_{q+1}, \dots, z_{q+s} \rangle$ is always faithful, while the action of $\text{Sym}(q)$ on $\langle z_1, \dots, z_{q-1} \rangle$ is faithful unless $q = 2$. If $q > 0$ and $s > 0$, then we also have the generator z_q , on which $\pi \in \text{Sym}(q) \times W(s)$ acts in this way (note that $\pi(q) \leq q$ and $\pi(q+1) > q$):

$$z_q \mapsto \begin{cases} z_{\pi(q)} \cdots z_q z_{q+1} \cdots z_{q+j} & \text{if } \pi(q+1) = q+2j+1; \\ z_{\pi(q)} \cdots z_q z_{q+1}^{-1} \cdots z_{q+j}^{-1} z_{q+s} & \text{if } \pi(q+1) = q+2j+2. \end{cases}$$

If π acts trivially on $\langle z_{q+1}, \dots, z_{q+s} \rangle$, then $\pi(q+1) = q+1$. Hence, if π also acts trivially on z_q , then $\pi(q) = q$. It follows that the action of $\text{Sym}(q) \times W(s)$ on \widetilde{G}^0/T is faithful unless $q = 2$ and $s = 0$. In this remaining case, we have $\tau = (t_1, t_2)$ where $t_1 \neq t_2$ (otherwise Γ is not a fine φ -grading). If ψ_1 yields $\pi = (12)$, then $\psi_1(v_1) = v_2 d_1$ and $\psi(v_2) = v_1 d_2$ for some nonzero homogeneous $d_1, d_2 \in \mathcal{D}$, but then $B(\psi_1(v_1), \psi_1(v_1))$ has degree t_2 , while $B(v_1, v_1)$ has degree t_1 . This contradicts (11), because here we have $\psi_0 = \text{id}$, $d_0 \in \mathbb{F}^\times$ and $B' = B$. The proof of the claim is complete.

Since $P = I$, we have $\Psi = \text{diag}(d_1, \dots, d_k)$, where the d_i must necessarily have the same degree, say, t , so $\Psi = \text{diag}(\lambda_1, \dots, \lambda_k) \otimes X_t$, but then (11) implies that (5) must hold, hence $\psi \in \text{Diag}(\Gamma)$. We have proved that $\text{Stab}(\Gamma, \varphi) \subset \text{Diag}(\Gamma)$. The opposite inclusion is obvious.

2) We can extract more information about an element $\psi \in \text{Aut}^*(\Gamma, \varphi)$ than given by its image under the homomorphism (13) if we look at the action of ψ on φ . Write $\psi \varphi \psi^{-1} = \xi_\psi \varphi$ where ξ_ψ is a uniquely determined element of $\text{Diag}(\Gamma)$. Clearly, we have $\xi_{\psi \psi'} = \xi_\psi (\psi \xi_{\psi'} \psi^{-1})$. Since ξ_ψ is the conjugation by $\text{diag}(\nu_1, \dots, \nu_k) \otimes X_{u_\psi}$, for a uniquely determined $u_\psi \in T$, we obtain $u_{\psi \psi'} = u_\psi \alpha_\psi(u_{\psi'})$ where α_ψ is the element of $\text{Aut}(T, \beta)$ corresponding to ψ under (13). Hence, we can construct a homomorphism

$$(14) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times (T \rtimes \text{Aut}(T, \beta)),$$

where the first two components are as in (13) and the third is $\psi \mapsto (u_\psi, \alpha_\psi)$.

Theorem 3.8 implies that we may assume without loss of generality that

$$\Phi = \text{diag} \left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right).$$

(In other words, the scalars μ_i are all equal to 1.) Then, for ψ given by $\Psi = PD$ and $\psi_0 \in \text{Aut}(\Gamma_0)$, with P corresponding to $\pi \in \text{Sym}(q) \times W(s)$, condition (11) is equivalent to the following, with $u = u_\psi$:

$$(15) \quad \varphi_0(d_i)X_{t_{\pi(i)}}\nu_{\pi(i)}^{-1}X_u^{-1}d_i = d_0\psi_0(X_{t_i}), \quad i = 1, \dots, q,$$

and, for each $j = 1, \dots, s$, one of the following depending on whether $\pi(q + 2j - 1) < \pi(q + 2j)$ or $\pi(q + 2j - 1) > \pi(q + 2j)$:

$$(16) \quad \varphi_0(d_{q+2j-1})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j-1} = d_0$$

in the first case, and

$$(17) \quad \varphi_0(d_{q+2j-1})\nu_{\pi(q+2j-1)}^{-1}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu_{\pi(q+2j)}^{-1}X_u^{-1}d_{q+2j-1} = d_0$$

in the second case.

If $\psi \in \text{Aut}^*(\Gamma, \varphi)$, then, looking at the degrees in (15), we obtain

$$(18) \quad t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}u_\psi, \quad i = 1, \dots, q,$$

which implies that $(t_{\alpha_\psi}u_\psi, \alpha_\psi)$ belongs to $\text{Aut}^*\Sigma$. Composing the third component of the homomorphism (14) with the automorphism $(u, \alpha) \mapsto (t_\alpha u, \alpha)$ of the group $T \rtimes \text{Aut}(T, \beta)$, we obtain a homomorphism

$$(19) \quad \text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut}^*\Sigma.$$

For any element $(t_\alpha u, \alpha) \in \text{Aut}^*\Sigma$, let $\pi_{u, \alpha} \in \text{Sym}(q)$ be its restriction to Σ . Then (18) implies that the permutation $\pi_{u_\psi, \alpha_\psi}^{-1}$ does not move the elements of the underlying set of Σ , so it belongs to $\text{Sym}\Sigma$. It follows that (19) can be rearranged as follows:

$$f: \text{Aut}^*(\Gamma, \varphi) \rightarrow W(s) \times (\text{Sym}\Sigma \rtimes \text{Aut}^*\Sigma).$$

We claim that f is surjective. We will construct representatives in $\text{Aut}^*(\Gamma, \varphi)$ for the elements of each of the subgroups $W(s)$, $\text{Sym}\Sigma$ and $\text{Aut}^*\Sigma$.

For any $\pi \in W(s)$, let P be the corresponding permutation matrix and let ψ_π be given by $\Psi = P$ and $\psi_0 = \text{id}$. Let α be the automorphism of \tilde{G} that restricts to identity on T and sends \tilde{g}_i to $\tilde{g}_{\pi(i)}$ (in particular, \tilde{g}_i are fixed for $i = 1, \dots, q$). Then ψ_π sends ${}^\alpha\Gamma$ to Γ , so $\psi_\pi \in \text{Aut}(\Gamma)$. Also, conditions (15) through (17) are satisfied with $d_0 = I$, $u = e$ and $\nu_i = 1$, so $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$.

For any $\pi \in \text{Sym}\Sigma$, let P be the corresponding permutation matrix and let ψ_π be given by $\Psi = P$ and $\psi_0 = \text{id}$. Since we have $t_{\pi(i)} = t_i$ for all $i = 1, \dots, q$,

we can define the automorphism α of \tilde{G} in the same way as above (this time, \tilde{g}_i are fixed for $i = q + 1, \dots, q + 2s$). Then ψ_π sends ${}^\alpha\Gamma$ to Γ , so $\psi_\pi \in \text{Aut}(\Gamma)$. Also, conditions (15) and (16) are satisfied with $d_0 = I$, $u = e$ and $\nu_i = 1$, so $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$.

Now, for any $(t_\alpha u, \alpha) \in \text{Aut}^* \Sigma$, let $\pi = \pi_{u, \alpha}$. Then $t_{\pi(i)} = \alpha(t_i)t_\alpha u$ for $i = 1, \dots, q$ and hence we can extend $\alpha: T \rightarrow T$ to an automorphism of \tilde{G} by setting $\alpha(\tilde{g}_i) = \tilde{g}_{\pi(i)}$ for $i = 1, \dots, q$, $\alpha(\tilde{g}_{q+2j-1}) = \tilde{g}_{q+2j-1}$ and $\alpha(\tilde{g}_{q+2j}) = \tilde{g}_{q+2j}t_\alpha u$ for $j = 1, \dots, s$. Choose $\nu_i \in \mathbb{F}^\times$ such that $\nu_i^2 = \beta(u, t_i)\beta(u)$, $i = 1, \dots, q$, and set $\nu_{q+2j} = 1$ and $\nu_{q+2j-1} = \beta(u)$, $j = 1, \dots, s$. Then (12) holds, so the conjugation by $\text{diag}(\nu_1 X_u, \dots, \nu_k X_u)$ is an element $\xi \in \text{Diag}(\Gamma)$. Choose ψ_0 such that $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$. Let P be the permutation matrix corresponding to π and let

$$D = \text{diag}(\lambda_1 I, \dots, \lambda_q I, I, X_u X_{t_\alpha}, \dots, I, X_u X_{t_\alpha}),$$

where $\lambda_i \in \mathbb{F}^\times$ are selected in such a way that condition (15) holds with $d_0 = X_{t_\alpha}$ (the degrees of both sides match, so it is indeed possible to find such λ_i). Since $\beta(t_\alpha) = 1$, condition (16) also holds. Finally, let $\psi_{u, \alpha}$ be given by $\Psi = PD$ and ψ_0 . Then $\psi_{u, \alpha}$ sends ${}^\alpha\Gamma$ to Γ and φ to $\xi\varphi$, with α and ξ indicated above. Therefore, $\psi_{u, \alpha}$ belongs to $\text{Aut}^*(\Gamma, \varphi)$.

We have proved that the homomorphism f is surjective. Let K be the kernel of f . It consists of the conjugations by matrices of the form $D = \text{diag}(d_1, \dots, d_k)$ such that (15) and (16) are satisfied with $\pi = \text{id}$, $\psi_0 = \text{id}$, $d_0 \in \mathbb{F}^\times$ and $u = e$. Hence ${}^\circ d_{q+2j-1} = {}^\circ d_{q+2j}$ for all $j = 1, \dots, s$. Conversely, given $(u_1, \dots, u_k) \in T^k$ with $u_{q+2j-1} = u_{q+2j}$ for $j = 1, \dots, s$, we can find elements d_i with ${}^\circ d_i = u_i$ such that the conjugation by D belongs to $\text{Aut}(\Gamma, \varphi)$.

According to 1), the subgroup

$$N = \{\psi \in K \mid {}^\circ d_1 = \dots = {}^\circ d_k\}$$

contains $\text{Stab}(\Gamma, \varphi)$. Clearly, N is normal in $\text{Aut}^*(\Gamma, \varphi)$. From the previous paragraph it follows that $K/N \cong T^{q+s}/T$ where T is imbedded into T^{q+s} diagonally. The representatives ψ_π that we constructed above for $\pi \in W(s)$ and for $\pi \in \text{Sym}\Sigma$ form subgroups of $\text{Aut}(\Gamma, \varphi)$ that commute with one another. But observe also that the representatives $\psi_{u, \alpha}$ for $(t_\alpha u, \alpha) \in \text{Aut}^* \Sigma$ form a subgroup modulo N . Moreover, for $\pi \in \text{Sym}(s) \subset W(s)$ the elements $\psi_{u, \alpha}$ and ψ_π commute modulo N , while for $\pi \in \text{Sym}\Sigma$ we have $\psi_{u, \alpha}\psi_\pi\psi_{u, \alpha}^{-1} \in \psi_{\pi u, \alpha\pi\pi u, \alpha}^{-1}N$. Finally, for the transposition $\pi = (q + 2j - 1, q + 2j)$, we have $\psi_\pi\psi_{u, \alpha}\psi_\pi\psi_{u, \alpha}^{-1} \in \psi N$ where ψ is the conjugation by $\text{diag}(d_1, \dots, d_k)$ with $d_{q+2j-1} = d_{q+2j} = X_{t_\alpha u}$ and all other $d_i = I$. It follows that $\text{Aut}^*(\Gamma, \varphi)/N$ is isomorphic to $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^* \Sigma$, with the stated actions.

It remains to compute the quotient $N/\text{Stab}(\Gamma, \varphi)$. Since any element $\psi \in N$ belongs to $\text{Stab}(\Gamma)$, the mapping $\psi \mapsto \xi_\psi$ is a homomorphism $N \rightarrow \text{Diag}(\Gamma)$ whose kernel is exactly $\text{Stab}(\Gamma, \varphi)$. Hence, it suffices to compute the image. Since here $u = e$ and ${}^\circ d_{q+2j-1} = {}^\circ d_{q+2j}$, condition (16) implies that $\nu_{q+2j-1} = \nu_{q+2j}$ for $j = 1, \dots, s$. But then (12) implies that all ν_i^2 are equal to each other. Since multiplying all ν_i by the same scalar in \mathbb{F}^\times does not change ξ , we may assume that $\nu_i \in \{\pm 1\}$. In fact, for $D = \text{diag}(\lambda_1 I, \dots, \lambda_k I)$, conditions (15) and (16) reduce to the following: up to a common scalar multiple, $\nu_i = \lambda_i^2$ for $i = 1, \dots, q$, and $\nu_{q+2j-1} = \nu_{q+2j} = \lambda_{q+2j-1} \lambda_{q+2j}$ for $j = 1, \dots, s$. Hence every (ν_1, \dots, ν_k) with $\nu_i \in \{\pm 1\}$ and $\nu_{q+2j-1} = \nu_{q+2j}$ indeed appears in ξ_ψ for some $\psi \in N$. Therefore, the quotient $N/\text{Stab}(\Gamma, \varphi)$ is isomorphic to $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ where \mathbb{Z}_2 is imbedded into \mathbb{Z}_2^{q+s} diagonally.

3) The proof is similar to 2), so we will merely point out the differences. According to Theorem 3.10, here we have

$$\Phi = \text{diag} \left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix} \right),$$

where $\delta = \text{sgn}(\varphi)$ and $\beta(t_i) = \delta$ for $i = 1, \dots, q$. Also, B' equals B and hence, for ψ given by $\Psi = PD$ and $\psi_0 \in \text{Aut}(\Gamma_0)$, with P corresponding to $\pi \in \text{Sym}(q) \times W(s)$, condition (11) is equivalent to the following:

$$(20) \quad \varphi_0(d_i) X_{t_{\pi(i)}} d_i = d_0 \psi_0(X_{t_i}), \quad i = 1, \dots, q,$$

and, for each $j = 1, \dots, s$, one of the following depending on whether $\pi(q+2j-1) < \pi(q+2j)$ or $\pi(q+2j-1) > \pi(q+2j)$:

$$(21) \quad \varphi_0(d_{q+2j-1}) d_{q+2j} = d_0$$

in the first case, and

$$(22) \quad \varphi_0(d_{q+2j-1}) d_{q+2j} = \delta d_0$$

in the second case. Here we took into account that, since $\varphi_0(d_0) = d_0$, either (21) or (22) implies $\varphi_0(d_{q+2j-1}) d_{q+2j} = \varphi_0(d_{q+2j}) d_{q+2j-1}$.

If $\psi \in \text{Aut}(\Gamma, \varphi)$, then, looking at the degrees in (20), we obtain

$$(23) \quad t_{\pi(i)} = \alpha_\psi(t_i) t_{\alpha_\psi}, \quad i = 1, \dots, q,$$

which implies that $(t_{\alpha_\psi}, \alpha_\psi)$ stabilizes Σ , i.e., α_ψ belongs to $\text{Aut} \Sigma$. Hence we obtain a homomorphism

$$(24) \quad \text{Aut}(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times \text{Aut} \Sigma.$$

For any element $\alpha \in \text{Aut } \Sigma$, let $\pi_\alpha \in \text{Sym}(q)$ be the restriction of its twisted action to Σ . Then (23) implies that the permutation $\pi\pi_{\alpha\psi}^{-1}$ does not move the elements of the underlying set of Σ , so it belongs to $\text{Sym}\Sigma$. It follows that (24) can be rearranged as follows:

$$f: \text{Aut}(\Gamma, \varphi) \rightarrow W(s) \times (\text{Sym}\Sigma \rtimes \text{Aut } \Sigma).$$

To prove that f is surjective, we construct representatives in $\text{Aut}(\Gamma, \varphi)$ for the elements of each of the subgroups $W(s)$, $\text{Sym}\Sigma$ and $\text{Aut } \Sigma$.

For π in $\text{Sym}\Sigma$ or in $\text{Sym}(s) \subset W(s)$, we take the same representatives as in the proof of 2). For $\pi = (q + 2j - 1, q + 2j) \in W(s)$, a slight modification is needed: we take $\Psi = PD$ rather than just P , where $d_{q+2j} = \delta I$ and all other $d_i = I$. For any $\alpha \in \text{Aut } \Sigma$, let $\pi = \pi_\alpha$. Then $t_{\pi(i)} = \alpha(t_i)t_\alpha$ for $i = 1, \dots, q$ and hence we can extend $\alpha: T \rightarrow T$ to an automorphism of \tilde{G} by setting $\alpha(\tilde{g}_i) = \tilde{g}_{\pi(i)}$ for $i = 1, \dots, q$, $\alpha(\tilde{g}_{q+2j-1}) = \tilde{g}_{q+2j-1}$ and $\alpha(\tilde{g}_{q+2j}) = \tilde{g}_{q+2j}t_\alpha$ for $j = 1, \dots, s$. Choose ψ_0 such that $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$. Let P be the permutation matrix corresponding to π and let

$$D = \text{diag}(\lambda_1 I, \dots, \lambda_q I, I, X_{t_\alpha}, \dots, I, X_{t_\alpha}),$$

where $\lambda_i \in \mathbb{F}^\times$ are selected in such a way that condition (20) holds with $d_0 = X_{t_\alpha}$. Clearly, condition (21) also holds. Finally, let ψ_α be given by $\Psi = PD$ and ψ_0 . Then ψ_α sends ${}^\alpha\Gamma$ to Γ and fixes φ , so ψ_α belongs to $\text{Aut}(\Gamma, \varphi)$.

Let K be the kernel of f and let

$$N = \{\psi \in K \mid {}^\circ d_1 = \dots = {}^\circ d_k\}.$$

The same arguments as in 2) show that $K/N \cong T^{q+s}/T$ and $\text{Aut}(\Gamma, \varphi)/N$ is isomorphic to $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut } \Sigma$, with the stated actions. But here we have $N = \text{Stab}(\Gamma, \varphi)$, which completes the proof. \square

4. Series A. In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series A . Thus, we take $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 2$, and $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{R}, \mathcal{R}]/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}])$. First we review the classification of fine gradings on \mathcal{L} from [2] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for \mathcal{L} from what we already know about automorphisms of fine gradings ([4]) and fine φ -gradings (Section 3) on \mathcal{R} .

4.1. Classification of fine gradings. The case $n = 2$ is easy, because the restriction from \mathcal{R} to \mathcal{L} yields an isomorphism $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$. It follows

that the classification of fine gradings on \mathcal{L} is the same as that on \mathcal{R} . Namely, there are two fine gradings on $\mathfrak{sl}_2(\mathbb{F})$, up to equivalence: the Cartan grading, whose universal group is \mathbb{Z} , and the Pauli grading, whose universal group is \mathbb{Z}_2^2 .

Now assume $n \geq 3$. Then the restriction and passing modulo the center yields a closed imbedding $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$, which is not an isomorphism. To rectify this, one introduces the affine group scheme $\overline{\mathbf{Aut}}(\mathcal{R})$ corresponding to the algebraic group of automorphisms and anti-automorphisms of \mathcal{R} (see [1, §3]). Unless $n = \text{char } \mathbb{F} = 3$, we obtain an isomorphism $\overline{\mathbf{Aut}}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$. It is convenient to divide gradings on \mathcal{L} into two types: for Type I the corresponding diagonalizable subgroup scheme of $\mathbf{Aut}(\mathcal{L})$ is contained in the image of the closed imbedding $\mathbf{Aut}(\mathcal{R}) \rightarrow \mathbf{Aut}(\mathcal{L})$, while for Type II it is not. In other words, a grading on \mathcal{L} is of Type I if and only if it is induced from a (unique) grading on \mathcal{R} by restriction and passing modulo the center.

In [1], the *distinguished element* of a Type II grading Γ is introduced. It can be characterized as the unique element h of order 2 in the grading group G such that the coarsening $\overline{\Gamma}$ induced from Γ by the quotient map $G \rightarrow \overline{G} := G/\langle h \rangle$ is a Type I grading. The original grading Γ can be recovered from $\overline{\Gamma}$ if we know the action of some character χ of G with $\chi(h) = -1$. Indeed, we just have to split each component of $\overline{\Gamma}$ into eigenspaces with respect to the action of χ . We can transfer this procedure to \mathcal{R} in the following way. The action of χ on \mathcal{L} is induced by $-\varphi$ where φ is an anti-automorphism of \mathcal{R} . The Type I grading $\overline{\Gamma}$ on \mathcal{L} comes from a grading $\overline{\Gamma}'$ on \mathcal{R} . Since $-\varphi$ is an automorphism of $\mathcal{R}^{(-)}$ (the Lie algebra \mathcal{R} under commutator) and φ^2 acts as a scalar on each component of $\overline{\Gamma}'$, we can refine the \overline{G} -grading $\overline{\Gamma}' : \mathcal{R} = \bigoplus_{\overline{g} \in \overline{G}} \mathcal{R}_{\overline{g}}$ to a G -grading $\Gamma' : \mathcal{R}^{(-)} = \bigoplus_{g \in G} \mathcal{R}_g$ by splitting each component $\mathcal{R}_{\overline{g}}$ into eigenspaces of φ . In detail, φ^2 acts on $\mathcal{R}_{\overline{g}}$ as multiplication by $\chi^2(\overline{g})$ (where we regard χ^2 as a character of \overline{G} , since $\chi^2(h) = 1$), so we set

$$(25) \quad \mathcal{R}_g = \{X \in \mathcal{R}_{\overline{g}} \mid \varphi(X) = -\chi(g)X\} = \{\varphi(X) - \chi(g)X \mid X \in \mathcal{R}_{\overline{g}}\}.$$

Then Γ' induces the original Type II grading Γ on \mathcal{L} by restriction and passing modulo the center.

Now we apply the above to fine gradings on \mathcal{L} . The fine gradings of Type I come from the fine gradings on \mathcal{R} that do not admit an anti-automorphism φ making them φ -gradings. All fine gradings on \mathcal{R} are obtained as follows. We start from T , a finite abelian group that admits a nondegenerate alternating bicharacter β (hence $|T|$ is a square). Fix a realization, \mathcal{D} , of the matrix algebra endowed with a division grading with support T and bicharacter β . Let $k \geq 1$ be an integer. Denote by $\tilde{G} = \tilde{G}(T, k)$ the abelian group freely generated by T

and the symbols $\tilde{g}_1, \dots, \tilde{g}_k$.

Definition 4.1. Let $\mathcal{M}(\mathcal{D}, k)$ be the \tilde{G} -graded algebra $\text{End}_{\mathcal{D}}(V)$ where V has a \mathcal{D} -basis $\{v_1, \dots, v_k\}$ with ${}^\circ v_i = \tilde{g}_i$. Let $n = k\sqrt{|T|}$ and $\mathcal{R} = M_n(\mathbb{F})$. The grading on \mathcal{R} obtained by identifying \mathcal{R} with $\mathcal{M}(\mathcal{D}, k)$ will be denoted by $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$. In other words, we define this grading by identifying $\mathcal{R} = M_k(\mathcal{D})$ and setting ${}^\circ(E_{ij} \otimes X_t) := \tilde{g}_i t \tilde{g}_j^{-1}$. By abuse of notation, we will also write $\Gamma_{\mathcal{M}}(T, k)$.

The universal group of $\Gamma_{\mathcal{M}}(T, k)$ is the subgroup $\tilde{G}^0 = \tilde{G}(T, k)^0$ of \tilde{G} generated by the support, i.e., by the elements $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$, $t \in T$. Clearly, $\tilde{G}^0 \cong T \times \mathbb{Z}^{k-1}$. By [2, Proposition 3.24], $\Gamma_{\mathcal{M}}(T, k)$ is a φ -grading for some φ if and only if T is an elementary 2-group and $k \leq 2$. Two gradings, $\Gamma_{\mathcal{M}}(T, k)$ and $\Gamma_{\mathcal{M}}(T', k')$, are equivalent if and only if $T \cong T'$ and $k = k'$.

Definition 4.2. Consider the grading $\Gamma_{\mathcal{M}}(T, k)$ on \mathcal{R} by the group $\tilde{G}(T, k)^0$ where $k \geq 3$ if T is an elementary 2-group. The $\tilde{G}(T, k)^0$ -grading on \mathcal{L} obtained by restriction and passing modulo the center will be denoted by $\Gamma_A^{(1)}(T, k)$.

The grading $\Gamma_A^{(1)}(T, k)$ is fine, and $\tilde{G}(T, k)^0$ is its universal group. To deal with fine gradings of Type II, we will need the following general observation:

Lemma 4.3. Let $\bar{\Gamma}$ be a φ -grading on an algebra \mathcal{A} and let \bar{G} be its universal group. Then there exist an abelian group G , an element $h \in G$ of order 2, a character χ of G with $\chi(h) = -1$ such that $\bar{G} = G/\langle h \rangle$ and the action of χ^2 on the \bar{G} -graded algebra \mathcal{A} (regarding χ^2 as a character of the group \bar{G}) coincides with φ^2 . The pair (G, h) is determined uniquely up to isomorphism over \bar{G} (i.e., $\langle h \rangle \rightarrow G \rightarrow \bar{G}$ is unique up to equivalence of extensions).

Proof. For each $\bar{g} \in \bar{G}$, φ^2 acts on $\mathcal{A}_{\bar{g}}$ as multiplication by some $\lambda(\bar{g}) \in \mathbb{F}^\times$. Since \bar{G} is the universal group of $\bar{\Gamma}$, $\lambda: \bar{G} \rightarrow \mathbb{F}^\times$ is a homomorphism. For each $\bar{g} \in \bar{G}$, we select $\mu(\bar{g}) \in \mathbb{F}^\times$ such that $\mu(\bar{g})^2 = \lambda(\bar{g})$ (there are two choices). It will be convenient to choose $\mu(\bar{e}) = 1$. It follows that

$$(26) \quad \mu(\bar{x}\bar{y}) = \varepsilon(\bar{x}, \bar{y})\mu(\bar{x})\mu(\bar{y}) \quad \text{for all } \bar{x}, \bar{y} \in \bar{G}$$

where $\varepsilon(\bar{x}, \bar{y}) \in \{\pm 1\}$. One immediately verifies that ε is a symmetric 2-cocycle on \bar{G} with $\varepsilon(\bar{g}, \bar{e}) = 1$ for all $\bar{g} \in \bar{G}$ and, moreover, the class of ε in $H^2(\bar{G}, \mathbb{Z}_2)$ (where we identified $\{\pm 1\}$ with \mathbb{Z}_2) does not depend on the choices of $\mu(\bar{g})$. Let G be the central extension of \bar{G} by \mathbb{Z}_2 determined by ε , i.e., G consists of the pairs (\bar{g}, δ) , $\bar{g} \in \bar{G}$, $\delta \in \{\pm 1\}$, with multiplication given by

$$(27) \quad (\bar{x}, \delta_1)(\bar{y}, \delta_2) = (\bar{x}\bar{y}, \varepsilon(\bar{x}, \bar{y})\delta_1\delta_2) \quad \text{for all } \bar{x}, \bar{y} \in \bar{G} \text{ and } \delta_1, \delta_2 \in \{\pm 1\}.$$

Define $\chi: G \rightarrow \mathbb{F}^\times$ by $(\bar{g}, \delta) \mapsto \mu(\bar{g})\delta$. Comparing (26) and (27), we see that χ is a homomorphism. Set $h = (\bar{e}, -1) \in G$. Then h has order 2 and $\chi(h) = -1$. By construction, the action of χ^2 on \mathcal{A} determined by $\bar{\Gamma}$ coincides with φ^2 . \square

Let T be an elementary 2-group of even dimension. Recall the group $\tilde{G}(T, q, s, \tau)$, which was introduced before Definition 3.3, and its subgroup $\tilde{G}(T, q, s, \tau)^0$.

Definition 4.4. *Consider the grading $\bar{\Gamma} = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ on \mathcal{R} by the group $\bar{G} = \tilde{G}(T, q, s, \tau)^0$ where $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Let Φ be the matrix given by*

$$\Phi = \text{diag} \left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right).$$

Define $\varphi(X) = \Phi^{-1}({}^t X)\Phi$. Let G , h and χ be as in Lemma 4.3, so we obtain a G -grading on $\mathcal{R}^{(-)}$ defined by (25). The G -grading on \mathcal{L} obtained by restriction and passing modulo the center will be denoted by $\Gamma_A^{(\text{II})}(T, q, s, \tau)$.

The grading $\Gamma_A^{(\text{II})}(T, q, s, \tau)$ is fine, and G is its universal group. Note that $\varphi^4 = \text{id}$. It can be shown (cf. [2, Example 3.21]) that the extension $\langle h \rangle \rightarrow G \rightarrow \bar{G}$ is split if and only if there exists $t \in T$ such that $t_i t$ are in T_+ for all i or in T_- for all i . Taking into account (3), we see that G is isomorphic to

$$\begin{cases} \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1) + 1} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s & \text{if } \exists t \in T \quad \beta(t_1 t) = \dots = \beta(t_q t); \\ \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0 + 1} \times \mathbb{Z}^s & \text{otherwise,} \end{cases}$$

where T_0 is the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \dots, q-1$.

Now Theorem 4.2 of [2] can be extended to positive characteristic and recast as follows:

Theorem 4.5. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 3$ if $\text{char } \mathbb{F} \neq 3$ and $n \geq 4$ if $\text{char } \mathbb{F} = 3$. Then any fine grading on $\mathfrak{psl}_n(\mathbb{F})$ is equivalent to one of the following:*

- $\Gamma_A^{(\text{I})}(T, k)$ as in Definition 4.2 with $k\sqrt{|T|} = n$,
- $\Gamma_A^{(\text{II})}(T, q, s, \tau)$ as in Definition 4.4 with $(q + 2s)\sqrt{|T|} = n$.

Gradings belonging to different types listed above are not equivalent. Within each type, we have the following:

- $\Gamma_A^{(\text{I})}(T_1, k_1)$ and $\Gamma_A^{(\text{I})}(T_2, k_2)$ are equivalent if and only if $T_1 \cong T_2$ and $k_1 = k_2$;

- $\Gamma_A^{(\text{II})}(T_1, q_1, s_1, \tau_1)$ and $\Gamma_A^{(\text{II})}(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the natural action of $\text{ASP}_{2r}(2)$.

The missing case $n = \text{char } \mathbb{F} = 3$ can be treated using octonions, because in characteristic 3 the algebra of traceless octonions under commutator is a Lie algebra isomorphic to $\mathfrak{psl}_3(\mathbb{F})$ (cf. [1, Remark 4.11]).

4.2. Weyl groups of fine gradings. By [4, Theorem 2.8], the Weyl group of $\Gamma_{\mathcal{M}}(T, k)$ is isomorphic to $T^{k-1} \rtimes (\text{Sym}(k) \times \text{Aut}(T, \beta))$, with $\text{Sym}(k)$ and $\text{Aut}(T, \beta)$ acting on T^{k-1} through their natural action on T^k and identification of T^{k-1} with T^k/T where T is imbedded into T^k diagonally. Thanks to the isomorphism $\mathbf{Aut}(M_2(\mathbb{F})) \rightarrow \mathbf{Aut}(\mathfrak{sl}_2(\mathbb{F}))$, it follows that the Weyl group of the Cartan grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\text{Sym}(2)$ (the classical Weyl group of type A_1) and the Weyl group of the Pauli grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\text{Sp}_2(2) = \text{GL}_2(2)$ (this is known in the case $\text{char } \mathbb{F} = 0$ — see [5]).

To state our result for $\mathfrak{psl}_n(\mathbb{F})$, $n \geq 3$, it is convenient to introduce the following notation:

$$\overline{\text{Aut}}(T, \beta) := \text{Aut}(T, \beta) \rtimes \langle \sigma \rangle,$$

where σ is an element of order 2 acting as the automorphism of T that sends a_i to a_i^{-1} and b_i to b_i , where a_i and b_i are the generators of T used for the chosen realization of \mathcal{D} (a “symplectic basis” of T with respect to β). We observe that $\beta(\sigma \cdot u, \sigma \cdot v) = \beta(u, v)^{-1}$, for all $u, v \in T$, and hence we obtain an induced action of σ on $\text{Aut}(T, \beta)$ by setting $(\sigma \cdot \alpha)(t) := \sigma \cdot \alpha(\sigma \cdot t)$ for all $\alpha \in \text{Aut}(T, \beta)$ and $t \in T$. The elements of $\overline{\text{Aut}}(T, \beta)$ act as automorphisms of T that send β to $\beta^{\pm 1}$. However, this action is not faithful if T is an elementary 2-group.

Theorem 4.6. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 3$ if $\text{char } \mathbb{F} \neq 3$ and $n \geq 4$ if $\text{char } \mathbb{F} = 3$. Consider the fine grading $\Gamma = \Gamma_A^{(\text{I})}(T, k)$ on $\mathfrak{psl}_n(\mathbb{F})$ as in Definition 4.2, $k\sqrt{|T|} = n$. Then*

$$W(\Gamma) \cong T^{k-1} \rtimes (\text{Sym}(k) \times \overline{\text{Aut}}(T, \beta)),$$

with $\text{Sym}(k)$ and $\overline{\text{Aut}}(T, \beta)$ acting on T^{k-1} through their natural action on T^k and identification of T^{k-1} with T^k/T where T is imbedded into T^k diagonally.

Proof. The grading Γ on $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F})$ is induced by the grading $\Gamma' = \Gamma_{\mathcal{M}}(T, k)$ on $\mathcal{R} = M_n(\mathbb{F})$. The universal group of both gradings is $G = \tilde{G}(T, k)^0$. Since restriction is a bijection between gradings on \mathcal{R} and Type I gradings on \mathcal{L} , an automorphism ψ' of \mathcal{R} sends ${}^\alpha\Gamma'$ to Γ' , for some automorphism α of G , if and only if the induced automorphism ψ of \mathcal{L} sends ${}^\alpha\Gamma$ to Γ . The automorphism

group of \mathcal{L} is the semidirect product of $\text{Aut}(\mathcal{R})$, in its induced action on \mathcal{L} , and $\langle \sigma \rangle$, where σ is given by the negative of matrix transpose. To compute the action of σ , recall that $(u_1, \dots, u_k)T \in T^k/T$ can be represented by the automorphism $X \mapsto DXD^{-1}$ where $D = \text{diag}(X_{u_1}, \dots, X_{u_k})$, $\pi \in \text{Sym}(k)$ can be represented by $X \mapsto PXP^{-1}$ where P is the permutation matrix corresponding to π , and $\alpha \in \text{Aut}(T, \beta)$ can be represented by $X \mapsto \psi_0(X)$ where ψ_0 is an automorphism of \mathcal{D} such that $\psi_0(X_t) \in \mathbb{F}X_{\alpha(t)}$ for all $t \in T$. The conjugation by σ sends the automorphism $X \mapsto \Psi X \Psi^{-1}$ to the automorphism $X \mapsto ({}^t\Psi^{-1})X({}^t\Psi)$, i.e., replaces Ψ by ${}^t\Psi^{-1}$. Hence, σ commutes with $\text{Sym}(k)$, while the conjugation by σ sends $(u_1, \dots, u_k)T$ to $(\sigma \cdot u_1, \dots, \sigma \cdot u_k)T$, where the action of σ on T is as indicated above. Also, the action of σ on G sends $z_{i,j,t} := \tilde{g}_i t \tilde{g}_j^{-1}$ to $z_{i,j,\sigma \cdot t}^{-1}$, so σ belongs to $\text{Aut}(\Gamma)$. Hence we obtain $\text{Aut}(\Gamma) = \text{Aut}(\Gamma') \rtimes \langle \sigma \rangle$. On the other hand, $\text{Stab}(\Gamma)$ does not contain outer automorphisms because Γ' does not admit an anti-automorphism φ that would make it a φ -grading. Hence $\text{Stab}(\Gamma) = \text{Stab}(\Gamma')$. The result follows. \square

Theorem 4.7. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 3$ if $\text{char } \mathbb{F} \neq 3$ and $n \geq 4$ if $\text{char } \mathbb{F} = 3$. Consider the fine grading $\Gamma = \Gamma_A^{(\text{II})}(T, q, s, \tau)$ on $\mathfrak{psl}_n(\mathbb{F})$ as in Definition 4.4, $(q+2s)\sqrt{|T|} = n$. Let $\Sigma = \Sigma(\tau)$. Then $W(\Gamma)$ contains a normal subgroup N isomorphic to \mathbb{Z}_2^{q+s-1} such that*

$$W(\Gamma)/N \cong ((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^* \Sigma,$$

where the actions are described naturally if we identify T^{q+s-1} with T^{q+s}/T and \mathbb{Z}_2^{q+s-1} with $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ (diagonal imbeddings). Moreover, $W(\Gamma)$ contains a subgroup isomorphic to $((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s))) \rtimes \text{Aut} \Sigma$ that is disjoint from N .

Proof. The grading $\Gamma = \Gamma_A^{(\text{II})}(T, q, s, \tau)$ on $\mathcal{L} = \mathfrak{psl}_n(\mathbb{F})$ is induced by the grading Γ' on $\mathcal{R}^{(-)}$, where $\mathcal{R} = M_n(\mathbb{F})$, obtained from $\overline{\Gamma}' = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ and φ as in Definition 4.4. The universal group of $\overline{\Gamma}'$ is $\overline{G} = \tilde{G}(T, q, s, \tau)^0$, while the universal group of Γ is the extension G of \overline{G} as in Lemma 4.3.

Similarly to Type I, an automorphism ψ' of \mathcal{R} sends ${}^\alpha\Gamma'$ to Γ' , for some automorphism α of G , if and only if the induced automorphism ψ of \mathcal{L} sends ${}^\alpha\Gamma$ to Γ . Note that α fixes the distinguished element $h = (\bar{e}, -1)$ and hence yields an automorphism $\bar{\alpha}$ of \overline{G} . It follows that ψ' sends ${}^{\bar{\alpha}}\overline{\Gamma}'$ to $\overline{\Gamma}'$. For any $g \in G$ and $X \in \mathcal{R}_g$, we have $\varphi(X) = -\chi(g)X$. Since $(\psi')^{-1}(X) \in \mathcal{R}_{\alpha^{-1}(g)}$, we also have $(\varphi(\psi')^{-1})(X) = -\chi(\alpha^{-1}(g))(\psi')^{-1}(X)$. It follows that $\psi' \varphi (\psi')^{-1} = \xi \varphi$ where ξ is the action of the character $(\chi \circ \alpha^{-1})\chi^{-1}$ on \mathcal{R} determined by the G -grading Γ' . Since $\alpha(h) = h$, $(\chi \circ \alpha^{-1})\chi^{-1}$ can be regarded as a character of \overline{G} , hence ξ

belongs to $\text{Diag}(\overline{\Gamma}')$. Conversely, if ψ' sends $\overline{\alpha}\overline{\Gamma}'$ to $\overline{\Gamma}'$ and $\psi'\varphi(\psi')^{-1} = \xi\varphi$ for some $\xi \in \text{Diag}(\overline{\Gamma}')$, then for any $\overline{g} \in \overline{G}$ and $X \in \mathcal{R}_{\overline{g}}$, we have $\psi'(X) \in \mathcal{R}_{\overline{\alpha}(\overline{g})}$ and $\varphi(\psi'(X)) = \nu\psi'(X)$ where $\nu \in \mathbb{F}^\times$ depends only on \overline{g} . It follows that ψ' permutes the components of Γ' and hence sends ${}^\alpha\Gamma'$ to Γ' where α is a lifting of $\overline{\alpha}$. We have proved that an automorphism ψ' of \mathcal{R} belongs to $\text{Aut}^*(\overline{\Gamma}', \varphi)$, respectively $\text{Stab}(\overline{\Gamma}', \varphi)$, if and only if the induced automorphism ψ of \mathcal{L} belongs to $\text{Aut}(\Gamma)$, respectively $\text{Stab}(\Gamma)$. Finally, note that $-\varphi$ induces an automorphism of \mathcal{L} that belongs to $\text{Stab}(\Gamma)$. It follows that the Weyl group of Γ is isomorphic to $\text{Aut}^*(\overline{\Gamma}', \varphi)/\text{Stab}(\overline{\Gamma}', \varphi)$. The latter group was described in Theorem 3.12. \square

If $\text{char } \mathbb{F} = 3$, there are two fine gradings on $\mathfrak{psl}_3(\mathbb{F})$: the Cartan grading, whose universal group is \mathbb{Z}^2 , and the grading induced by the Cayley–Dickson doubling process for octonions, whose universal group is \mathbb{Z}_2^3 . The Weyl groups of these gradings are, respectively, the classical Weyl group of type G_2 [4, Theorem 3.3] and $\text{GL}_3(2)$ [4, Theorem 3.5].

5. Series B , C and D . In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series B , C and D with exception of type D_4 . Thus, we take $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 4$, and $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$ where φ is an involution on \mathcal{R} . If φ is symplectic, then, of course, n has to be even. If φ is orthogonal, we assume $n \geq 5$ and $n \neq 8$. First we review the classification of fine gradings on \mathcal{L} from [2] (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for \mathcal{L} from what we already know about automorphisms of fine φ -gradings (Section 3) on \mathcal{R} .

5.1. Classification of fine gradings. Under the stated assumptions on n , the restriction from \mathcal{R} to \mathcal{L} yields an isomorphism $\mathbf{Aut}(\mathcal{R}, \varphi) \rightarrow \mathbf{Aut}(\mathcal{L})$ (see [1, §3]). It follows that the classification of fine gradings on \mathcal{L} is the same as the classification of fine φ -gradings on \mathcal{R} (here φ is fixed).

The case of series B is quite easy, because n is odd and hence the elementary 2-group T must be trivial. Let $G = \widetilde{G}(\{e\}, q, s, \tau)^0$ where $\tau = (e, \dots, e)$, so $G \cong \mathbb{Z}_2^{q-1} \times \mathbb{Z}^s$.

Definition 5.1. Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\{e\}, q, s, \tau)$ on \mathcal{R} by G . Let Φ be the matrix given by

$$\Phi = \text{diag} \left(\underbrace{1, \dots, 1}_q, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Then Γ is a fine φ -grading for $\varphi(X) = \Phi^{-1}({}^tX)\Phi$ and hence its restriction is a fine grading on $\mathcal{L} \cong \mathfrak{so}_n(\mathbb{F})$. We will denote this grading by $\Gamma_B(q, s)$.

Now we turn to series C and D , where n is even and hence T may be nontrivial. So, let T be an elementary 2-group of even dimension. Choose τ as in (1) with all $t_i \in T_-$ in case of series C and all $t_i \in T_+$ in case of series D . Let $G = \tilde{G}(T, q, s, \tau)^0$, so $G \cong \mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0, q-1)} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s$ where T_0 is the subgroup of T generated by the elements $t_i t_{i+1}$, $i = 1, \dots, q-1$.

Definition 5.2. Consider the grading $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, q, s, \tau)$ on \mathcal{R} by G where $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Let Φ be the matrix given by

$$\Phi = \text{diag} \left(X_{t_1}, \dots, X_{t_q}, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ \delta I & 0 \end{bmatrix} \right),$$

where $\delta = -1$ for series C and $\delta = 1$ for series D . Then Γ is a fine φ -grading for $\varphi(X) = \Phi^{-1}({}^t X)\Phi$ and hence its restriction is a fine grading on $\mathcal{L} \cong \mathfrak{sp}_n(\mathbb{F})$ or $\mathfrak{so}_n(\mathbb{F})$. We will denote this grading by $\Gamma_C(T, q, s, \tau)$ or $\Gamma_D(T, q, s, \tau)$, respectively.

The following three results are Theorem 5.2 of [2], stated separately for series B , C and D (and extended to positive characteristic).

Theorem 5.3. Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 5$ be odd. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_B(q, s)$ where $q + 2s = n$. Also, $\Gamma_B(q_1, s_1)$ and $\Gamma_B(q_2, s_2)$ are equivalent if and only if $q_1 = q_2$ and $s_1 = s_2$.

Theorem 5.4. Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 4$ be even. Then any fine grading on $\mathfrak{sp}_n(\mathbb{F})$ is equivalent to $\Gamma_C(T, q, s, \tau)$ where $(q + 2s)\sqrt{|T|} = n$. Moreover, $\Gamma_C(T_1, q_1, s_1, \tau_1)$ and $\Gamma_C(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\text{Sp}_{2r}(2)$ as in Definition 3.9.

Theorem 5.5. Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 6$ be even. Assume $n \neq 8$. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_D(T, q, s, \tau)$ where $(q + 2s)\sqrt{|T|} = n$. Moreover, $\Gamma_D(T_1, q_1, s_1, \tau_1)$ and $\Gamma_D(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\text{Sp}_{2r}(2)$ as in Definition 3.9.

5.2. Weyl groups of fine gradings. Let $\Gamma = \Gamma_B(q, s)$, $\Gamma_C(T, q, s, \tau)$ or $\Gamma_D(T, q, s, \tau)$, so Γ is the restriction of the grading $\Gamma' = \Gamma_{\mathcal{M}}(T, q, s, \tau)$ on \mathcal{R} to $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$. By arguments similar to the proof of Theorem 4.7, one shows that the Weyl group of Γ is isomorphic to $\text{Aut}(\Gamma', \varphi) / \text{Stab}(\Gamma', \varphi)$, which was described

in Theorem 3.12. For $\Gamma = \Gamma_B(q, s)$, T is trivial and Σ is a singleton of multiplicity q , so we obtain:

Theorem 5.6. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 5$ be odd. Consider the fine grading $\Gamma = \Gamma_B(q, s)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition 5.1, where $q + 2s = n$. Let $\Sigma = \Sigma(\tau)$. Then $W(\Gamma) \cong \text{Sym}(q) \times W(s)$ where $W(s) = \mathbb{Z}_2^s \rtimes \text{Sym}(s)$ (wreath product of $\text{Sym}(s)$ and \mathbb{Z}_2).*

For $\Gamma_C(T, q, s, \tau)$ and $\Gamma_D(T, q, s, \tau)$, T may be nontrivial, so the answer is more complicated:

Theorem 5.7. *Let \mathbb{F} be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 4$ be even. Consider the fine grading $\Gamma = \Gamma_C(T, q, s, \tau)$ on $\mathfrak{sp}_n(\mathbb{F})$ or $\Gamma = \Gamma_D(T, q, s, \tau)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition 5.2, where $(q + 2s)\sqrt{|T|} = n$ and $n \neq 4, 8$ in the case of $\mathfrak{so}_n(\mathbb{F})$. Let $\Sigma = \Sigma(\tau)$. Then*

$$W(\Gamma) \cong ((T^{q+s-1} \times \mathbb{Z}_2^s) \rtimes (\text{Sym } \Sigma \times \text{Sym}(s))) \rtimes \text{Aut } \Sigma,$$

where the actions on T^{q+s-1} are via the identification with T^{q+s}/T (diagonal imbedding of T into T^{q+s}).

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