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PELL FORM AND PELL EQUATION VIA OBLONG NUMBERS

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Abstract. Let $k \geq 0$ be an integer. Then it is known that oblong (pronic) numbers are numbers of the form $k(k+1)$ which is denoted by O_k. In this work, we derive some algebraic relations on the Pell form $F_{\Delta_k}(x,y) = x^2 - O_k y^2$ of discriminant $\Delta_k = 4O_k$ including cycle, proper cycle, reduction and proper automorphism of it. Also we determine the integer solutions of the Pell equation $F_{\Delta_k}(x,y) = 1$ via oblong numbers O_k.

1. Preliminaries. A real binary quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x,y) = ax^2 + bxy + cy^2 \tag{1.1}$$

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with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by Δ. A quadratic form F of discriminant Δ is called indefinite if $\Delta > 0$, and is called integral if and only if $a, b, c \in \mathbb{Z}$. An indefinite quadratic form $F = (a, b, c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.2}$$

Most properties of quadratic forms can be giving by the aid of extended modular group $\overline{\Gamma}$ (see [14]). Gauss defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x,y) &= \left(ar^2 + brs + cs^2\right)x^2 + (2art + bru + bts + 2csu)\,xy \\ &\quad + \left(at^2 + btu + cu^2\right)y^2 \end{aligned} \tag{1.3}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$, that is, gF is gotten from F by making the substitution $x \to rx + tu$ and $y \to sx + uy$. Moreover $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$. Let F and G be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent and if $\det g = -1$, then F and G are called improperly equivalent. A form F is called ambiguous if it is improperly equivalent to itself. An element $g \in \overline{\Gamma}$ is called an automorphism of F if $gF = F$. If $\det g = 1$, then g is called a proper automorphism of F and if $\det g = -1$, then g is called an improper automorphism of F. Let $Aut(F)^+$ denote the set of proper automorphisms of F and let $Aut(F)^-$ denote the set of improper automorphisms of F (for further details on binary quadratic forms see [3, 4, 7, 11]).

Let $\rho(F)$ denotes the normalization (it means that replacing F by its normalization) of $(c, -b, a)$. To be more explicit, we set

$$\rho^{i+1}(F) = (c_i, -b_i + 2c_i r_i, c_i r_i^2 - b_i r_i + a_i), \tag{1.4}$$

where

$$r_i = \begin{cases} \operatorname{sign}(c_i)\left\lfloor \dfrac{b_i}{2|c_i|} \right\rfloor & \text{for } |c_i| \geq \sqrt{\Delta} \\[2ex] \operatorname{sign}(c_i)\left\lfloor \dfrac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor & \text{for } |c_i| < \sqrt{\Delta} \end{cases} \tag{1.5}$$

for $i \geq 0$. The number r is called the reducing number and the form $\rho^{i+1}(F)$ is called the reduction of F. Further if F is reduced, then so is $\rho^{i+1}(F)$. In fact, ρ is a permutation of the set of all reduced indefinite forms. Let $\tau(F) = \tau(a,b,c) = (-a,b,-c)$. Then the cycle of F is the sequence $((\tau\rho)^i(G))$ for $i \in \mathbb{Z}$, where $G = (A,B,C)$ is a reduced form with $A > 0$ which is equivalent to F. The cycle and proper cycle of F is given by the following theorem.

Theorem 1.1. *Let $F = (a,b,c)$ be reduced indefinite quadratic form of discriminant Δ. Then the cycle of F is a sequence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor \tag{1.6}$$

and

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = \big(|c_i|,\ -b_i + 2s_i|c_i|,\ -(a_i + b_i s_i + c_i s_i^2)\big) \tag{1.7}$$

for $1 \leq i \leq l-2$. If l is odd, then the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim$$
$$\tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $2l$. In this case the equivalence class of F is equal to the proper equivalence class of F, and if l is even, then the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length l. In this case the equivalence class of F is the disjoint union of the proper equivalence class of F and the proper equivalence class of $\tau(F)$ [3].

Let Δ be a non-square discriminant. Then the Pell form F_Δ is defined to be

$$F_\Delta(x,y) = \begin{cases} x^2 - \frac{\Delta}{4}y^2 & \text{if } \Delta \equiv 0 (\mathrm{mod}\ 4) \\ x^2 + xy - \frac{1-\Delta}{4}y^2 & \text{if } \Delta \equiv 1 (\mathrm{mod}\ 4). \end{cases} \tag{1.8}$$

So the Pell equation is the equation $F_\Delta(x,y) = 1$ and the negative Pell equation is the equation $F_\Delta(x,y) = -1$.

Let $\Delta \equiv 0 (\mathrm{mod}\ 4)$, say $\Delta = 4d$ for a positive non–square integer d and let N be any fixed integer. Then the equation

$$x^2 - dy^2 = \pm N \tag{1.9}$$

is known as Pell–type equation and is named after John Pell (1611–1685), a mathematician who searched for integer solutions to equations of this type in the seventeenth century. Ironically, Pell was not the first to work on this problem, nor did he contribute to our knowledge for solving it. Euler (1707–1783), who brought us the ψ-function, accidentally named the equation after Pell, and the name stuck.

For $N = 1$, the equation

$$x^2 - dy^2 = \pm 1$$

is known the classical Pell equation. The Pell equation $x^2 - dy^2 = 1$ was first studied by Brahmagupta (598–670) and Bhaskara (1114–1185). Its complete theory was worked out by Lagrange (1736–1813), not Pell. It is often said that Euler (1707–1783) mistakenly attributed Brouncker's (1620–1684) work on this equation to Pell. However the equation appears in a book by Rahn (1622–1676) which was certainly written with Pell's help: some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation. In 1657, Fermat stated (without giving proof) that the positive Pell equation $x^2 - dy^2 = 1$ has an infinite number of solutions. If (m, n) is a solution, that is, $m^2 - dn^2 = 1$, then $(m^2 + dn^2, 2mn)$ is also a solution since $(m^2 + dn^2)^2 - d(2mn)^2 = (m^2 - dn^2)^2 = 1$. So the Pell equation $x^2 - dy^2 = 1$ has infinitely many. Later, in 1766 Lagrange proved that the Pell equation $x^2 - dy^2 = 1$ has an infinite number of solutions if d is positive and non–square. The first non–trivial solution $(x_1, y_1) \neq (\pm 1, 0)$ of this equation is called the fundamental solution from which all others are easily computed since

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

for $n \geq 1$, can be found using, e.g., the cyclic method [6], known in India in the 12^{th} century, or using the slightly less efficient but more regular English method [6] (17^{th} century). There are other methods to compute this so–called fundamental solution, some of which are based on a continued fraction expansion of the square root of d which given as follows: Let $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_l}]$ denote the continued fraction expansion of period length l. Set $A_{-2} = 0, A_{-1} = 1$, $A_k = a_k A_{k-1} + A_{k-2}$ and $B_{-2} = 1, B_{-1} = 0, B_k = a_k B_{k-1} + B_{k-2}$ for nonnegative integer k. Then $C_k = \dfrac{A_k}{B_k}$ is the $k-$th convergent of $\sqrt{d}$, and the fundamental solution of $x^2 - dy^2 = 1$ is $(x_1, y_1) = (A_{l-1}, B_{l-1})$ if l is even or (A_{2l-1}, B_{2l-1}) if l is odd. Also if l is odd, then the the fundamental solution of $x^2 - dy^2 = -1$ is $(x_1, y_1) = (A_{l-1}, B_{l-1})$ (for further details on Pell equations see [1, 8, 9, 10, 12]).

2. Pell form and Pell equation via oblong numbers. In [15–18], Tekcan (also Bizim and Bayraktar) considered some specific Pell equations and their integer solutions. In [17], Tekcan considered the integer solutions of the Pell equation $x^2 - Dy^2 = \pm 4$ and derived some nice results including two conjectures related to integer solutions of it. Later, these two conjectures were proved by Shabani in [13]. In [5], Chandoul extended the integer solutions of $x^2 - Dy^2 = \pm 4$ to $x^2 - Dy^2 = \pm k^2$ for an integer $k \geq 2$ with the same argument Tekcan used in his work.

In the present paper, we aim to consider the same problem by considering the oblong (pronic, rectangular or heteromecic) numbers which are the product of two consecutive integers, that is, $k(k+1)$ for and integer $k \geq 0$, and is denoted by O_k. So

$$O_k = k(k+1). \tag{2.1}$$

The first few oblongs numbers are $0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, \ldots$ (sequence A002378 in OEIS). So the k^{th} oblong number represents the number of points in a rectangular array having k columns and $k+1$ rows. (see [19]). Further the product of two oblong numbers O_{k-1} and O_k is another oblong number O_{k^2-1} (see [2]), that is,

$$O_{k-1}O_k = [(k-1)k][k(k+1)] = (k^2-1)k^2 = O_{k^2-1}.$$

Also the half of O_k is a triangular number denoted by T_k, that is, $T_k = \dfrac{O_k}{2}$ (sequence A000217 in OEIS).

Now we set $\Delta_k = 4O_k$. Then by (1.8), we get the Pell form

$$F_{\Delta_k}(x,y) = x^2 - O_k y^2 \tag{2.2}$$

and so the Pell equation

$$F_{\Delta_k}(x,y) = x^2 - O_k y^2 = 1. \tag{2.3}$$

Then we can give the following results.

Theorem 2.1. *Let O_k denote the k^{th} oblong number. Then for the Pell equation $F_{\Delta_k}(x,y) = 1$ we have,*

(1) *The continued fraction expansion of* $\sqrt{O_k}$ *is*

$$\sqrt{O_k} = \begin{cases} [1, \overline{O_1}] & \text{for } k = 1 \\ \left[\frac{\sqrt{4O_k+1}-1}{2}; \overline{O_1, \sqrt{4O_k+1}-1}\right] & \text{for } k > 1. \end{cases}$$

(2) $(x_1, y_1) = (\sqrt{4O_k+1}, O_1)$ *is the fundamental solution and the other solutions are* $(x_n,\ y_n)$, *where*

$$\frac{x_n}{y_n} = \left[\frac{\sqrt{4O_k+1}-1}{2}; \underbrace{O_1, \sqrt{4O_k+1}-1}_{n-1 \text{ times}}, O_1\right]$$

for $n \geq 2$.

(3) *The* n^{th} *integer solution* (x_n, y_n) *can be given as a linear combination of* O_1, O_k *and* $\sqrt{4O_k+1}$, *namely,*

$$\begin{aligned} x_n &= \sqrt{4O_k+1}x_{n-1} + O_1O_ky_{n-1} \\ y_n &= O_1x_{n-1} + \sqrt{4O_k+1}y_{n-1} \end{aligned}$$

for $n \geq 2$ *and also satisfy the recurrence relation*

$$\begin{aligned} x_n &= (2\sqrt{4O_k+1}-1)\,(x_{n-1}+x_{n-2}) - x_{n-3} \\ y_n &= (2\sqrt{4O_k+1}-1)\,(y_{n-1}+y_{n-2}) - y_{n-3} \end{aligned}$$

for $n \geq 4$.

Proof. (1) Let $k = 1$. Then $O_1 = 2$ and hence $\sqrt{Q_1} = [1; \overline{2}]$. Now let $k > 1$. Then since

$$\sqrt{O_k} = \frac{\sqrt{4O_k+1}-1}{2} + \left(\sqrt{O_k} - \frac{\sqrt{4O_k+1}-1}{2}\right)$$

we deduce that

$$\sqrt{O_k} = \frac{\sqrt{4O_k+1}-1}{2} + \cfrac{1}{O_1 + \cfrac{1}{\sqrt{4O_k+1}-1+\left(\sqrt{O_k}-\frac{\sqrt{4O_k+1}-1}{2}\right)}}.$$

Therefore $\sqrt{O_k} = \left[\frac{\sqrt{4O_k+1}-1}{2}; \overline{O_1, \sqrt{4O_k+1}-1}\right]$.

(2) We derive the fundamental solution of it by using the continued fraction expansion of $\sqrt{O_k}$. Since $\sqrt{O_k} = \left[\frac{\sqrt{4O_k+1}-1}{2}; \overline{O_1, \sqrt{4O_k+1}-1}\right]$ of period length 2, we deduce that $A_0 = \frac{\sqrt{4O_k+1}-1}{2}$, $A_1 = \sqrt{4O_k+1}$, $B_0 = 1$ and $B_1 = O_1$. Therefore $(x_1, y_1) = (A_1, B_1) = (\sqrt{4O_k+1}, O_1)$ is the fundamental solution.

Now we assume that (x_{n-1}, y_{n-1}) is a solution, that is, $x_{n-1}^2 - O_k y_{n-1}^2 = 1$. Then we find that

$$\begin{aligned}
\frac{x_n}{y_n} &= \frac{\sqrt{4O_k+1}-1}{2} + \cfrac{1}{O_1 + \cfrac{1}{\sqrt{4O_k+1}-1 + \cfrac{1}{O_1 + \cfrac{1}{\begin{array}{c}+\cdots\\ +\sqrt{4O_k+1}-1+\frac{1}{O_1}\end{array}}}}} \\
&= \frac{\sqrt{4O_k+1}-1}{2} \\
&\quad + \cfrac{1}{O_1 + \cfrac{1}{\frac{\sqrt{4O_k+1}-1}{2} + \frac{\sqrt{4O_k+1}-1}{2} + \cfrac{1}{O_1 + \cfrac{1}{\begin{array}{c}+\cdots\\ +\sqrt{4O_k+1}-1+\frac{1}{O_1}\end{array}}}}} \\
&= \frac{\sqrt{4O_k+1}-1}{2} + \cfrac{1}{O_1 + \cfrac{1}{\frac{\sqrt{4O_k+1}-1}{2} + \frac{x_{n-1}}{y_{n-1}}}} \\
(2.4) &= \frac{\sqrt{4O_k+1}\,x_{n-1} + O_1 O_k y_{n-1}}{O_1 x_{n-1} + \sqrt{4O_k+1}\,y_{n-1}}.
\end{aligned}$$

So

$$x_n^2 - O_k y_n^2 = \left[\sqrt{4O_k+1}\,x_{n-1} + O_1 O_k y_{n-1}\right]^2 - O_k\left[O_1 x_{n-1} + \sqrt{4O_k+1}\,y_{n-1}\right]^2$$

$$
\begin{aligned}
&= (4O_k+1)x_{n-1}^2 + 4O_k\sqrt{4O_k+1}x_{n-1}y_{n-1} + 4O_k^2y_{n-1}^2 \\
&\quad -O_k[O_1^2x_n^2 + 2O_1\sqrt{4O_k+1}x_{n-1}y_{n-1} + (4O_k+1)]y_{n-1}^2 \\
&= x_{n-1}^2 - O_ky_{n-1}^2 \\
&= 1.
\end{aligned}
$$

Therefore (x_n, y_n) is also a solution.

(3) The first assertion is easily seen from (2.4). The second assertion can be proved by induction on n. □

Let O_k denote the k^{th} oblong number. We set the matrix M as

$$
M = \begin{pmatrix} \sqrt{4O_k+1} & O_1O_k \\ O_1 & \sqrt{4O_k+1} \end{pmatrix}. \tag{2.5}
$$

In the following theorem, we able to determine the n^{th} power of M which we use it later.

Theorem 2.2. *For the matrix M, we let*

$$
M^n = \begin{pmatrix} M_{11}^n & M_{12}^n \\ M_{21}^n & M_{22}^n \end{pmatrix}.
$$

(1) *If n is even, then*

$$
\begin{aligned}
M_{11}^n &= \sum_{i=0}^{\frac{n}{2}} C(n,2i)(\sqrt{4O_k+1})^{n-2i}O_1^{2i}O_k^i = M_{22}^n \\
M_{12}^n &= \sum_{i=0}^{\frac{n-2}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i}O_1^{2i+1}O_k^{i+1} \\
M_{21}^n &= \sum_{i=0}^{\frac{n-2}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i}O_1^{2i+1}O_k^{i}.
\end{aligned}
$$

(2) *If n is odd, then*

$$
M_{11}^n = \sum_{i=0}^{\frac{n-1}{2}} C(n,2i)(\sqrt{4O_k+1})^{n-2i}O_1^{2i}O_k^i = M_{22}^n
$$

$$
\begin{aligned}
M_{12}^{n} &= \sum_{i=0}^{\frac{n-1}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i} O_1^{2i+1} O_k^{i+1} \\
M_{21}^{n} &= \sum_{i=0}^{\frac{n-1}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i} O_1^{2i+1} O_k^{i}
\end{aligned}
$$

for $n \geq 2$ *(here* $C(n,i)$ *denotes the binomial coefficient).*

P r o o f. (1) Let $n = 2$. Then

$$
M^2 = \begin{pmatrix} 4O_k + 1 + O_1^2 O_k & 2O_1 O_k \sqrt{4O_k+1} \\ 2O_1\sqrt{4O_k+1} & 4O_k + 1 + O_1^2 O_k \end{pmatrix}.
$$

Also

$$
\begin{aligned}
M_{11}^{2} &= \sum_{i=0}^{1} C(2,2i)(\sqrt{4O_k+1})^{2-2i} O_1^{2i} O_k^{i} = 4O_k + 1 + O_1^2 O_k = M_{22}^2 \\
M_{12}^{2} &= \sum_{i=0}^{0} C(2,2i+1)(\sqrt{4O_k+1})^{1-2i} O_1^{2i+1} O_k^{i+1} = 2O_1 O_k \sqrt{4O_k+1} \\
M_{21}^{2} &= \sum_{i=0}^{0} C(2,2i+1)(\sqrt{4O_k+1})^{1-2i} O_1^{2i+1} O_k^{i} = 2O_1 \sqrt{4O_k+1}.
\end{aligned}
$$

So it is true for $n = 2$. Let us assume that it is satisfied for $n-2$, that is,

$$
M^{n-2} = \begin{pmatrix} M_{11}^{n-2} & M_{12}^{n-2} \\ M_{21}^{n-2} & M_{22}^{n-2} \end{pmatrix},
$$

where

$$
\begin{aligned}
M_{11}^{n-2} &= \sum_{i=0}^{\frac{n-2}{2}} C(n-2,2i)(\sqrt{4O_k+1})^{n-2-2i} O_1^{2i} O_k^{i} = M_{22}^{n-2} \\
M_{12}^{n-2} &= \sum_{i=0}^{\frac{n-4}{2}} C(n-2,2i+1)(\sqrt{4O_k+1})^{n-3-2i} O_1^{2i+1} O_k^{i+1}
\end{aligned}
$$

$$M_{21}^{n-2} = \sum_{i=0}^{\frac{n-4}{2}} C(n-2,2i+1)(\sqrt{4O_k+1})^{n-3-2i} O_1^{2i+1} O_k^i.$$

Then since $M^{n-2} \cdot M^2$, we get

$$
\begin{aligned}
& M_{11}^{n-2}[4O_k+1+O_1^2O_k] + M_{12}^{n-2}[2O_1\sqrt{4O_k+1}] \\
&= \left[\begin{array}{c} (\sqrt{4O_k+1})^{n-2} \\ +C(n-2,2)(\sqrt{4O_k+1})^{n-4}O_1^2O_k + \cdots \\ +C(n-2,n-4)(\sqrt{4O_k+1})^2 O_1^{n-4} {O_k}^{\frac{n-4}{2}} \\ +O_1^{n-2} {O_k}^{\frac{n-2}{2}} \end{array}\right] [4O_k+1+2O_1O_k] \\
&+ \left[\begin{array}{c} C(n-2,1)(\sqrt{4O_k+1})^{n-3}O_1O_k \\ +C(n-2,3)(\sqrt{4O_k+1})^{n-5}O_1^3{O_k}^2 + \cdots \\ +C(n-2,n-5)(\sqrt{4O_k+1})^3 O_1^{n-5} {O_k}^{\frac{n-4}{2}} \\ +C(n-2,n-3)(\sqrt{4O_k+1}) O_1^{n-3} {O_k}^{\frac{n-2}{2}} \end{array}\right] [2O_1\sqrt{4O_k+1}] \\
&= (\sqrt{4O_k+1})^n + [C(n-2,2)+1+2C(n-2,1)]\,(\sqrt{4O_k+1})^{n-2}O_1^2O_k \\
&\quad + [C(n-2,4)+C(n-2,2)+2C(n-2,3)]\,(\sqrt{4O_k+1})^{n-4}O_1^4O_k^2 \\
&\quad + \cdots \\
&\quad + \left[\begin{array}{c} C(n-2,n-4)+C(n-2,n-6) \\ +2C(n-2,n-5) \end{array}\right] (\sqrt{4O_k+1})^4 O_1^{n-4} O_k^{\frac{n-4}{2}} \\
&\quad + [1+C(n-2,n-4)+2C(n-2,n-3)]\,(\sqrt{4O_k+1})^2 O_1^{n-2} O_k^{\frac{n-2}{2}} \\
&\quad + O_1^n O_k^{\frac{n}{2}} \\
&= (\sqrt{4O_k+1})^n + C(n,2)(\sqrt{4O_k+1})^{n-2}O_1^2O_k + C(n,4)(\sqrt{4O_k+1})^{n-4}O_1^4O_k^2 \\
&\quad + \cdots + C(n,n-2)(\sqrt{4O_k+1})^2 O_1^{n-2} O_k^{\frac{n-2}{2}} + O_1^n O_k^{\frac{n}{2}}
\end{aligned}
$$

$$= \sum_{i=0}^{\frac{n}{2}} C(n,2i)(\sqrt{4O_k+1})^{n-2i} O_1^{2i} O_k^i$$

$$= M_{11}^n.$$

(Here we note that $C(n,2i) = C(n-2,2i) + C(n-2,2i-2) + 2C(n-2,2i-1)$ for $i = 1,2,\dots,\frac{n-2}{2}$). Similarly it can be shown that

$$\begin{aligned} M_{11}^{n-2}[2O_1O_k\sqrt{4O_k+1}] + M_{12}^{n-2}[4O_k+1+O_1^2O_k] &= M_{12}^n \\ M_{21}^{n-2}[4O_k+1+O_1^2O_k] + M_{22}^{n-2}[2O_1\sqrt{4O_k+1}] &= M_{21}^n \\ M_{21}^{n-2}[2O_1O_k\sqrt{4O_k+1}] + M_{22}^{n-2}[4O_k+1+O_1^2O_k] &= M_{22}^n. \end{aligned}$$

So

$$M^n = \begin{pmatrix} M_{11}^n & M_{12}^n \\ M_{21}^n & M_{22}^n \end{pmatrix}$$

as we claimed. The other case can be proved similarly. $\square$

In the following theorem, we will show that the n^{th} integer solution (x_n, y_n) of $F_{\Delta_k}(x,y) = 1$ can be deduce via oblong numbers O_k.

Theorem 2.3. *The n^{th} integer solution of $F_{\Delta_k}(x,y) = 1$ is (x_n, y_n), where*

$$x_n = \begin{cases} \sum_{i=0}^{\frac{n}{2}} C(n,2i)(\sqrt{4O_k+1})^{n-2i} O_1^{2i} O_k^i & \text{if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n,2i)(\sqrt{4O_k+1})^{n-2i} O_1^{2i} O_k^i & \text{if } n \text{ is odd} \end{cases}$$

$$y_n = \begin{cases} \sum_{i=0}^{\frac{n-2}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i} O_1^{2i+1} O_k^i & \text{if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n,2i+1)(\sqrt{4O_k+1})^{n-1-2i} O_1^{2i+1} O_k^i & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 2$.

Proof. It can be proved as in the same way that Theorem 2.2 was proved. □

Now we can consider the Pell form F_{Δ_k}. Note that this form is not reduced since $|\sqrt{4O_k}-1|>0$ which is contradiction to (1.2). So we can give the following theorem related to reduction of F_{Δ_k}.

Theorem 2.4. *The reduction of F_{Δ_k} is*

$$\rho^2(F_{\Delta_k})=\left(1,\sqrt{4O_k+1}-1,\frac{1-\sqrt{4O_k+1}}{2}\right).$$

Proof. Let $F_{\Delta_k}=F_{\Delta_{k,0}}=(1,0,-O_k)$. Then from (1.5), we get $r_0=0$ and hence from (1.4)

$$\rho^1(F_{\Delta_k})=(-O_k,0,1)$$

which is not reduced. If we apply the reduction algorithm to $\rho^1(F_{\Delta_k})$ again, then we find that $r_1=\dfrac{\sqrt{4O_k+1}-1}{2}$ and so

$$\rho^2(F_{\Delta_k})=\left(1,\sqrt{4O_k+1}-1,\frac{1-\sqrt{4O_k+1}}{2}\right)$$

which is reduced. So the reduction of F_{Δ_k} is $\rho^2(F_{\Delta_k})$. □

Now we can consider the cycle and proper cycle of $\rho^2(F_{\Delta_k})$.

Theorem 2.5. *The cycle of $\rho^2(F_{\Delta_k})$ is*

$$\left(1,\sqrt{4O_k+1}-1,\frac{1-\sqrt{4O_k+1}}{2}\right)\sim\left(\frac{\sqrt{4O_k+1}-1}{2},\sqrt{4O_k+1}-1,-1\right)$$

of length 2, and the proper cycle of $\rho^2(F_{\Delta_k})$ is

$$\left(1,\sqrt{4O_k+1}-1,\frac{1-\sqrt{4O_k+1}}{2}\right)\sim\left(\frac{1-\sqrt{4O_k+1}}{2},\sqrt{4O_k+1}-1,1\right)$$

of length 2.

Proof. Let $\rho^2(F_{\Delta_k})=\rho^2(F_{\Delta_{k,0}})=\left(1,\sqrt{4O_k+1}-1,\dfrac{1-\sqrt{4O_k+1}}{2}\right)$. Then from (1.6), we get $s_0=O_1$ and hence

$$\rho^2(F_{\Delta_{k,1}})=\left(\frac{\sqrt{4O_k+1}-1}{2},\sqrt{4O_k+1}-1,-1\right).$$

$s_1 = \sqrt{4O_k+1} - 1$ and hence

$$\rho^2(F_{\Delta_{k,2}}) = \left(1, \sqrt{4O_k+1} - 1, \frac{1-\sqrt{4O_k+1}}{2}\right) = \rho^2(F_{\Delta_{k,0}}).$$

So the cycle of $\rho^2(F_{\Delta_k})$ is $\rho^2(F_{\Delta_{k,0}}) \sim \rho^2(F_{\Delta_{k,1}})$. Note that $l = 2$. Therefore from Theorem 1.1, the proper cycle of $\rho^2(F_{\Delta_k})$ is $\left(1, \sqrt{4O_k+1} - 1, \frac{1-\sqrt{4O_k+1}}{2}\right) \sim \left(\frac{1-\sqrt{4O_k+1}}{2}, \sqrt{4O_k+1} - 1, 1\right)$ of length 2. □

Now we consider the proper automorphisms of F_{Δ_k}. To get this we first set

$$g_{F_{\Delta_k}} = \begin{pmatrix} \sqrt{4O_k+1} & O_1 \\ O_1 O_k & \sqrt{4O_k+1} \end{pmatrix}. \tag{2.6}$$

Then we can give the following theorem which can be proved as in the same way that Theorem 2.2 was proved.

Theorem 2.6. *Let O_k denote the k^{th} oblong number. Then*

(1) *The set of proper automorphisms of F_{Δ_k} is*

$$Aut^+(F_{\Delta_k}) = \{\pm g^n_{F_{\Delta_k}} : n \in \mathbb{Z}\}.$$

(2) *The integer solutions of $F_{\Delta_k}(x,y) = 1$ are (x_n, y_n), where*

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (g^n_{F_{\Delta_k}})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $n \geq 1$.

Example 2.1. Let $k = 3$. Then $O_3 = 12$ and so $\sqrt{12} = [3; \overline{2, 6}]$. The fundamental solution of $F_{\Delta_3} : x^2 - 12y^2 = 1$ is $(x_1, y_1) = (7, 2)$ and other solutions are

$$\frac{x_2}{y_2} = [3; 2, 6, 2] = \frac{97}{28}$$

$$\frac{x_3}{y_3} = [3; 2, 6, 2, 6, 2] = \frac{1351}{390}$$

$$\frac{x_4}{y_4} = [3;2,6,2,6,2,6,2] = \frac{18817}{5432}$$

$$\frac{x_5}{y_5} = [3;2,6,2,6,2,6,2,6,2] = \frac{262087}{75658}$$

and etc. $x_n = 7x_{n-1} + 24y_{n-1}, y_n = 2x_{n-1} + 7y_{n-1}$ for $n \geq 2$ and $x_n = 13(x_{n-1} + x_{n-2}) - x_{n-3}, y_n = 13(y_{n-1} + y_{n-2}) - y_{n-3}$ for $n \geq 4$.

The reduction of $F_{\Delta_3}(x,y) = x^2 - 12y^2$ is $\rho^2(F_{\Delta_3}) = (1,6,-3)$ and hence the cycle of $\rho^2(F_{\Delta_3})$ is $\rho^2(F_{\Delta_{3,0}}) = (1,6,-3) \sim \rho^2(F_{\Delta_{3,1}}) = (3,6,-1)$ and the proper cycle of $\rho^2(F_{\Delta_3})$ is $\rho^2(F_{\Delta_{3,0}}) = (1,6,-3) \sim \rho^2(F_{\Delta_{3,1}}) = (-3,6,1)$.

The set of proper automorphisms of F_{Δ_3} is $Aut^+(F_{\Delta_3}) = \{\pm g^n_{F_{\Delta_3}} : n \in \mathbb{Z}\}$, where

$$g_{F_{\Delta_3}} = \begin{pmatrix} 7 & 2 \\ 24 & 7 \end{pmatrix}.$$

Further the integer solutions of $F_{\Delta_3} : x^2 - 12y^2 = 1$ are

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (g^2_{F_{\Delta_3}})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 97 \\ 28 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = (g^3_{F_{\Delta_3}})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1351 \\ 390 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = (g^4_{F_{\Delta_3}})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 18817 \\ 5432 \end{pmatrix}$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = (g^5_{F_{\Delta_3}})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 262087 \\ 75658 \end{pmatrix},$$

and etc.

REFERENCES

[1] E. J. Barbeau. Pell's Equation. New York, Springer-Verlag Inc, 2003.

[2] T. Breiteig. When is the product of two oblong numbers another oblong? *Math. Mag.* **73** (2000), 120–129.

[3] J. Buchmann, U. Vollmer. Binary Quadratic Forms: An Algorithmic Approach. Berlin, Heidelberg, Springer-Verlag, 2007.

[4] D. A. Buell. Binary Quadratic Forms, Clasical Theory and Modern Computations. New York, Springer-Verlag, 1989.

[5] A. Chandoul. The Pell Equation $x^2 - Dy^2 = \pm k^2$. *Adv. in Pure Maths.* **1** (2011), 16–22.

[6] H. M. Edward. Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory. Graduate Texts in Mathematics, vol. **50**, Springer-Verlag, 1977.

[7] D. E. Flath. Introduction to Number Theory. New York etc., Wiley-Interscience Publication, 1989.

[8] M. Jacobson, H. Williams. Solving the Pell Equation, CMS Books in Mathematics. New York, Springer, 2009.

[9] H. W. Lenstra. Solving the Pell Equation. *Notices Amer. Math. Soc.* **49** (2) (2002), 182–192.

[10] K. Matthews. The Diophantine Equation $x^2 - Dy^2 = N$, $D > 0$. *Exposition. Math.* **18** (2000), 323–331.

[11] R. A. Mollin. Quadratics. CRS Press, Boca Raton, New York, London, Tokyo, 1996.

[12] R. A. Mollin. Fundamental Number Theory with Applications. Second Edition (Discrete Mathematics and Its Applications) Chapman & Hall/ CRC, Boca Raton, London, New York, 2008.

[13] A. S. Shabani. The Proof of Two Conjectures Related to Pell's Equation $x^2 - Dy^2 = \pm 4$. *Int. J. Comput. Math. Sci.* **2**, *1* (2008), 24–27.

[14] A. Tekcan, O. Bizim. The Connection Between Quadratic Forms and the Extended Modular Group. *Math. Bohem.* **128**, *3* (2003), 225–236.

[15] A. Tekcan. Pell Equation $x^2 - Dy^2 = 2$, II. *Ir. Math. Soc. Bull.* **54** (2004) 73–89.

[16] A. Tekcan, O. Bizim, M. Bayraktar. Solving the Pell Equation using the Fundamental Element of the Field $Q(\sqrt{\Delta})$. *Southeast Asian Bull. Math.* **30**, *2* (2006), 355–366.

[17] *A. Tekcan.* The Pell Equation $x^2 - Dy^2 = \pm 4$. *Appl. Math. Sci., Ruse* **1**, *5–8* (2007), 363–369.

[18] A. Tekcan. The Number of Solutions of Pell Equations $x^2 - ky^2 = N$ and $x^2 + xy - ky^2 = N$ over $\mathbb{F}_p$. *Ars Comb.* **102** (2011), 225–236.

[19] The On–Line Encyclopedia of Integer Sequences, `http://www.oeis.org`.

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