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TAIL BEHAVIOR OF HÖLDER NORMS OF STOCHASTIC PROCESSES AND WEAK CONVERGENCE OF MAXIMA IN HÖLDER SPACES

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Abstract. Our main goal was to establish functional limit theorems for component–wise maxima of iid processes taking values in Hölder spaces. Given that the finite–dimensional distributions converge, the key technical challenge is to establish tightness. The classical tightness conditions of Lamperti apply, provided that one can control the tail–behavior of Hölder norms. We do so, by using a powerful isomorphism theorem due to Ciesielski, which relates Hölder norms to superma of sequences. As a consequence, we obtain estimates for the tail probabilities of Hölder norms for light, moderate and heavy–tailed stochastic processes. The established inequalities are of independent interest since they provide explicit bounds on the tail probabilities of Hölder norms and suprema of stochastic processes under simple conditions on the bivariate distributions.

We illustrate the results with sufficient conditions for the Hölder regularity of several classes of doubly stochastic max–stable processes of Schlather and Brown–Resnick types. Some extensions to the case of weak convergence in Besov spaces are also considered.

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1. Introduction. The problem of weak convergence for stochastic processes in Hölder spaces has already been studied in many contexts. The seminal work [13] provides very useful tightness criteria and several Donsker–type functional limit theorems. In a series of works Suquet, Račkauskas, and collaborators – [8, 17, 18, 19, 20], to name a few of their works, have obtained a number of theoretical results and many applications of limit theorems in Hölder spaces. [27] offers an illuminating perspective and a general approach to characterizing tightness in Banach spaces with Schauder bases. Thanks to an isomorphism theorem of [5], Suquet's results apply, in particular, to Hölder spaces.

In this note, our focus is on establishing functional limit theorems for maxima in Hölder spaces, which to the best of our knowledge, have not been thoroughly explored. More precisely, consider independent copies $\xi_i = \{\xi_i(t)\}_{t\in[0,1]}$, $i = 1, 2, \ldots$ of a stochastic process ξ with $\gamma-$Hölder continuous paths $\gamma \in (0, 1)$. Our goal is to establish conditions on ξ, which imply the weak convergence

$$M_n(\cdot) := \frac{1}{a_n} \bigvee_{1\le i\le n} \xi_i(\cdot) - b_n \Longrightarrow \eta(\cdot), \quad \text{as } n \to \infty, \tag{1}$$

with suitable normalization $a_n > 0$ and centering $b_n \in \mathbb{R}$ constants. Here '$\Rightarrow$' denotes weak convergence in the Hölder pace $(\mathcal{H}, \|\cdot\|_{\mathcal{H}_\gamma})$ (see Section 2, below).

The limit process in (1) is necessarily max–stable. In particular, its marginals are extreme value distributions (Fréchet, Gumbel or reversed Weibul). For simplicity, we focus here on the Fréchet domain of attraction case when the ξ_i's have heavy–tailed marginals, that is,

$$\mathbb{P}\{\xi(t) > x\} \sim L(x)x^{-\alpha}, \quad \text{as } x \to \infty, \tag{2}$$

for some $\alpha > 0$ and a slowly varying function L. The resulting limit η in (1) is then an $\alpha-$Fréchet max–stable process and the constants a_n and b_n can be taken to be:

$$a_n := \ell(n)n^{1/\alpha}, \quad \text{and} \quad b_n := 0, \tag{3}$$

with $\ell^\alpha(u) \sim L(\ell(u)u^{1/\alpha})$, as $u \to \infty$.

One motivation to study the weak convergence in Hölder spaces is to establish path–regularity results for the limit. The continuous mapping theorem yields further applications to limit theorems for Hölder–continuous functionals. The idea to proving (1) is as follows. Given that the finite–dimensional distributions converge, it remains to show tightness. It turns out that the maximum operator is Lipschitz with respect to the Hölder norm (Lemma 2.1, below). This fact and a classical tightness criterion due to Lampetri, leads to a simple weak convergence result in Theorem 2.1, below. The key to applying this result in practice, however, is the knowledge of the tail–behavior for the Hölder norms of interesting classes of stochastic processes, which is a non–trivial problem. A powerful result due to Ciesielski (see Theorem 3.1 below) shows that there is an isomorphism between the Hölder spaces and the space of bounded sequences equipped with the uniform norm. This lets us relate the tail behavior of the Hölder norm of a stochastic process to that of the supremum of a sequence of random variables. The latter supremum can be handled easily through Borel–Cantelli. The fact that one deals with sequences allows also for a general treatment that does not involve a detailed knowledge of the dependence structure of the underlying processes. This observation leads to simple and yet general estimates for the tail–behavior of Hölder norms for light–tailed processes, which may be of independent interest (see Theorem 3.2 below). Interestingly, bounds on the tails of suprema can be obtained, which apply to general (not necessarily Gaussian) light–tailed processes, under mild regularity conditions (see Corollary 3.1, below). Theorem 3.3 provides further tail bounds for the Hölder norms of processes with moderate and heavy tails. These results are applied to establish concrete functional limit theorems, which in turn yield the Hölder regularity for many classes of max–stable processes including the Schlather and Brown–Resnick processes.

The paper is structured as follows. In Section 2, we review sufficient conditions for tightness and establish a general weak convergence result for maxima in Hölder spaces. In Section 3, we provide upper bounds on the tail–probabilities for the Hölder norms of various stochastic processes. Concrete functional limit theorems in Hölder and Besov spaces are presented in Section 4. We conclude with some applications in Section 5.

2. Preliminaries and tightness in Hölder spaces. Let $\mathcal{H}_\gamma$, $\gamma \in (0,1)$ be the space of all γ–Hölder continuous functions, defined on the interval $[0,1]$. Namely, all $f : [0,1] \to \mathbb{R}$, with finite norm

$$\|f\|_{\mathcal{H}_\gamma} := \|f\|_\infty \vee |f|_\gamma \equiv \sup_{t\in[0,1]} |f(t)| \vee \sup_{t,s\in[0,1]} \frac{|f(t)-f(s)|}{|t-s|^\gamma}.$$

Recall that our goal here is to establish functional convergence results as in (1). The Hölder spaces are particularly natural in this context because the maximum operator is Lipschitz in the Hölder norm $\|\cdot\|_{\mathcal{H}_\gamma}$. More precisely, we have the following result.

Lemma 2.1. *For any $f, g \in \mathcal{H}_\gamma$, $0 < \gamma \le 1$, we have $\|f \vee g\|_{\mathcal{H}_\gamma} \le \|f\|_{\mathcal{H}_\gamma} \vee \|g\|_{\mathcal{H}_\gamma}$. In particular, $\mathcal{H}_\gamma$ is closed with respect to the point–wise maxima operator $(f, g) \mapsto f \vee g$.*

Proof. Observe first that for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$, we have

$$|a_1 \vee a_2 - b_1 \vee b_2| \le |a_1 - b_1| \vee |a_2 - b_2|. \tag{4}$$

Indeed, without loss of generality, it is enough to show the inequality in the case $a_1 \ge a_2$ and $a_1 \ge b_1 \vee b_2$. Then, (4) follows by considering the cases $a_1 \ge b_1 > b_2$ and $a_1 \ge b_2 \ge b_1$. In the first scenario $|a_1 \vee a_2 - b_1 \vee b_2| = a_1 - b_1$ and in the second, $|a_1 \vee a_2 - b_1 \vee b_2| = a_1 - b_2 \le a_1 - b_1$. Both imply (4).

Now, by applying (4) to $a_1 := f(t), a_2 := g(t),\ b_1 := f(s)$ and $b_2 := g(s)$, we obtain

$$|f(t) \vee g(t) - f(s) \vee g(s)| \le |f(t) - f(s)| \vee |g(t) - g(s)|, \quad \text{for all } t, s \in [0, 1].$$

This implies that $|f \vee g|_\gamma \le |f|_\gamma \vee |g|_\gamma$. Since $|f(t) \vee g(t)| \le |f(t)| \vee |g(t)|$, we also obtain $\|f \vee g\|_\infty \le \|f\|_\infty \vee \|g\|_\infty$. Consequently,

$$\|f \vee g\|_{\mathcal{H}_\gamma} \le \|f\|_\infty \vee \|g\|_\infty \vee |f|_\gamma \vee |g|_\gamma = \|f\|_{\mathcal{H}_\gamma} \vee \|g\|_{\mathcal{H}_\gamma},$$

which completes the proof. □

By Prokhorov's well known theorem (see e.g. [2]), the convergence in (1) follows provided that: (i) the finite–dimensional distributions converge and (ii) the laws of $M_n,\ n \in \mathbb{N}$ are tight. The following tightness criterion due to Lamperti is particularly useful.

Lemma 2.2 (Lemma 2, [13]). *If for all $\epsilon > 0$, there exist $\delta > 0$ and $B > 0$, such that*

$$\inf_{n \in \mathbb{N}} \mathbb{P}\{\|M_n\|_{\mathcal{H}_{\gamma+\delta}} \le B\} \ge 1 - \epsilon,$$

then the laws of $M_n,\ n \in \mathbb{N}$ are tight in $(\mathcal{H}_\gamma, \|\cdot\|_{\mathcal{H}_\gamma})$.

Let $\xi = \{\xi(t)\}_{t \in [0,1]}$ be a process with paths in $(\mathcal{H}_\gamma, \|\cdot\|_{\mathcal{H}_\gamma})$ and heavy–tailed marginal distributions as in (2). The following theorem provides a simple general functional limit theorem for this class of heavy–tailed processes.

Theorem 2.1. *Let $0 < \gamma < 1$ and ξ be as in (2) and suppose that (1) holds in the sense of the finite–dimensional distributions, with a_n and $b_n = 0$ as in (3).*

If for some $\delta > 0$ such that $\gamma + \delta \le 1$, we have

$$\mathbb{P}\{\|\xi\|_{\mathcal{H}_{\gamma+\delta}} > x\} \propto L(x)x^{-\alpha}, \quad \text{as } x \to \infty, \tag{5}$$

then (1) *holds also in the sense of weak convergence of distributions in* $(\mathcal{H}_\gamma, \|\cdot\|_{\mathcal{H}_\gamma})$.

Proof. With this choice of a_n and b_n, by Lemma 2.1, we obtain

$$\|M_n\|_{\mathcal{H}_{\gamma+\delta}} \le \frac{1}{a_n} \bigvee_{1\le i\le n} \|\xi_i\|_{\mathcal{H}_{\gamma+\delta}}. \tag{6}$$

Focus on the right–hand side of the last expression. It consists of maxima of independent and identically distributed random variables $X_i := \|\xi_i\|_{\mathcal{H}_{\gamma+\delta}}$. By (5) $\mathbb{P}\{X_i > x\} \propto L(x)x^{-\alpha},\ x \to \infty$, which implies that the right–hand side of (6) converges in distribution to either 0 or to a finite, $\alpha-$Fréchet random variable Z. Therefore,

$$\mathbb{P}\{\|M_n\|_{\mathcal{H}_{\gamma+\delta}} > B\} \le \mathbb{P}\Big\{\frac{1}{a_n} \bigvee_{1\le i\le n} X_i > B\Big\} \longrightarrow \mathbb{P}\{Z > B\},$$

as $n \to \infty$, which by Lemma 2.2 readily yields the tightness of $\{M_n,\ n \in \mathbb{N}\}$ in $\mathcal{H}_\gamma$ and completes the proof of the theorem. □

Remark 2.1. Lemma 2.2 follows essentially from the Arcella–Ascoli theorem, which yields the curious fact that the closed unit ball in $(\mathcal{H}_{\gamma+\delta}, \|\cdot\|_{\mathcal{H}_{\gamma+\delta}})$, $\delta > 0$ is a *compact* set in $(\mathcal{H}_\gamma, \|\cdot\|_{\mathcal{H}_\gamma})$. This is rather helpful in applications since establishing tightness in $\mathcal{H}_\gamma$ amounts to proving boundedness in $\mathcal{H}_{\gamma+\delta}$.

In more abstract terms, this result can be read as the fact that the Banach space $(\mathcal{H}_{\gamma+\delta}, \|\cdot\|_{\mathcal{H}_{\gamma+\delta}})$, $\delta > 0$ is embedded compactly in $(\mathcal{H}_\gamma, \|\cdot\|_{\mathcal{H}_\gamma})$. Namely, the identity operator $\mathrm{id} : \mathcal{H}_{\gamma+\delta} \to \mathcal{H}_\gamma$ is compact, i.e. maps bounded and closed sets to compacts. Many other function spaces have this interesting feature. Most notably the Besov spaces $B^{\gamma+\delta}_{p,q}(D)$, $\delta > 0$ on a *bounded domain* $D \subset \mathbb{R}^d$ embed compactly into $B^\gamma_{p,q}(D)$ (see e.g. Proposition 4.6, p. 197 in [28]).

Remark 2.2. The two key elements in the proof of Theorem 2.1 are that: (i) $\|f \vee g\|_{\mathcal{H}_{\gamma+\delta}} \le \|f\|_{\mathcal{H}_{\gamma+\delta}} \vee \|g\|_{\mathcal{H}_{\gamma+\delta}}$ (Lemma 2.1) and (ii) the space $\mathcal{H}_{\gamma+\delta}$ embeds *compactly* into $\mathcal{H}_\gamma$. Therefore, the result of Theorem 2.1 holds for any other pair of function spaces for which (i) the maximum operation satisfies an analog of Lemma 2.1 and (ii) we have compact embedding of one space into the other. Proposition 4.2 below, for example, establishes a weak convergence result in Besov spaces.

Condition (5) in Theorem 2.1 may be difficult to check in practice. In the following section, we obtain inequalities that can be used to establish the tail–behavior of the Hölder norm in many cases. Theorem 2.1 will then be applied in Section 4 to derive concrete limit theorems and thus to establish the path properties of several max–stable processes in Section 5.

3. Tail–behavior of Hölder norms. The classical result of [5] provides a powerful way to deal with the Hölder norms. It turns out that there is a Banach space isomorphism between the Hölder spaces and the space ℓ_∞ of bounded infinite sequences equipped with the sup–norm. This isomorphism allows one to compute (equivalent) Hölder norms by taking suprema of sequences. More precisely, following [5] let $\mathcal{H}_\gamma^0$ be the set of all continuous functions defined on $[0,1]$, vanishing at 0, and such that

$$|x|_\gamma \equiv \sup_{t,s\in[0,1]} \frac{|x(t)-x(s)|}{|t-s|^\gamma} < \infty.$$

Then $(\mathcal{H}_\gamma^0, |\cdot|_\gamma)$, $0<\gamma<1$ becomes a Banach space. Also, let $\|\xi\|_\infty := \sup_{n\ge 1}|\xi_n| < \infty$, for $\xi = \{\xi_n\}_{n\ge1} \in \ell_\infty$.

Now, we introduce certain basis functions, used to express the Ciesielski's isomporphisms. Consider the Haar mother wavelet function $\psi(t) = 1_{[0,1/2)}(t) - 1_{(1/2,1]}(t)$. For all $0<\gamma<1$, let

$$\chi_1(t) \equiv \chi_1^{(\gamma)}(t) = 1, \quad \text{and} \quad \varphi_1(t) \equiv \varphi_1^{(\gamma)}(t) = t,\ t\in[0,1].$$

Also, for all $2 \le n \equiv 2^j + k$, $(1\le k\le 2^j,\ j\ge 0,\ j,k\in\mathbb{N}\cup\{0\})$, let

$$\varphi_n(t) := \int_0^t \chi_n(\tau)d\tau, \quad \text{where} \quad \chi_n(t) \equiv \chi_{2^j+k}(t) = 2^{j/2}\psi(2^jt-k+1).$$

Finally, introduce the functions:

$$\chi_n^{(\gamma)}(t) \equiv \frac{2^{(j+1)\gamma}}{2^{j/2+1}}\chi_{2^j+k}(t) \quad \text{and} \quad \varphi_n^{(\gamma)}(t) \equiv \varphi_{2^j+k}^{(\gamma)}(t) = \frac{2^{j/2+1}}{2^{(j+1)\gamma}}\varphi_{2^j+k}(t),\ t\in[0,1].$$

For all $x\in C[0,1]$, and $n = 2^j+k \ge 2$, we write

$$(7)\qquad \xi_n = \int_0^1 \chi_{2^j+k}^{(\gamma)}(\tau)dx(\tau) := \frac{2^{(j+1)\gamma}}{2}\Big(2x\Big(\frac{2k-1}{2^{j+1}}\Big) - x\Big(\frac{k}{2^j}\Big) - x\Big(\frac{k-1}{2^j}\Big)\Big),$$

for all $1 \le k \le 2^j$, $j \ge 0$ and $\xi_1 = \int_0^1 \chi_1^{(\gamma)}(\tau)dx(\tau) := x(1)-x(0)$. For convenience of the reader, we state next the Cisielski's isomorphism theorem.

Theorem 3.1 (Theorem 1 in [5]). *Let* $0 < \gamma < 1$. *The spaces* $(\mathcal{H}_\gamma^0, |\cdot|_\gamma)$ *and* $(\ell_\infty, \|\cdot\|_\infty)$ *are isomorphic Banach spaces. The isomorphism* T_γ *from* ℓ_∞ *onto* $\mathcal{H}_\gamma^0$ *is given by*

$$x(\cdot) = \sum_{n=1}^{\infty} \xi_n \varphi_n^{(\gamma)}(\cdot), \tag{8}$$

where the series converges with respect to the sup–norm $\|\cdot\|_\infty$ *in* $C[0,1]$, *and* T_γ^{-1} (*the isomorphism from* $\mathcal{H}_\gamma^0$ *onto* ℓ_∞) *is given by* (*recall* (7)),

$$\xi_n = \int_0^1 \chi_n^{(\gamma)}(\tau)dx(\tau), \quad n = 1, 2, \dots. \tag{9}$$

Moreover, for the operator norms of these linear mappings, we have

$$\frac{2}{3(2^\gamma - 1)(2^{1-\gamma} - 1)} \le \|T_\gamma\| \le \frac{2}{(2^\gamma - 1)(2^{1-\gamma} - 1)}, \qquad and \qquad \|T_\gamma^{-1}\| = 1.$$

Remark 3.1. The Banach space $(\mathcal{H}_\gamma^0, |\cdot|_\gamma)$ is not separable since, in fact, it is isomorphic to the sequence space $(\ell_\infty, \|\cdot\|_\infty)$. Therefore, the functions $\varphi_n^{(\gamma)}(\cdot)$, $n \ge 1$ do not provide a Schauder basis for $\mathcal{H}_\gamma^0$ (since there is none). [5] (Theorem 2) has shown that the $\varphi_n^{(\gamma)}$'s provide a Schauder basis of the separable spaces $\mathcal{H}_{\gamma,0}^0 := \overline{\cup_{\delta>0}\mathcal{H}_{\gamma+\delta}^0}^{|\cdot|_\gamma}$, which consist of all functions $f \in \mathcal{H}_\gamma^0$ with

$$\lim_{\epsilon \to 0} \sup_{|t-s|\le\epsilon,\ t,s\in[0,1]} \frac{|f(t) - f(s)|}{|t-s|^\gamma} = 0.$$

[27] provides an excellent treatment of tightness in Schauder–decomposable spaces. Since in this section we are concerned with the behavior of the Hölder norms, we did not need to focus on the separable Schauder–decomposable spaces $\mathcal{H}_{\gamma,0}^0$ and did not use directly the characterizations of Suquet.

Remark 3.2. Relation (8) resembles a wavelet decomposition of the function x. The rate of decay of the coefficients ξ_j essentially determines the Hölder regularity of the function x. The modern theory of multi–resolution analysis (wavelet decompositions) provides far–reaching extensions of this type of characterizations to the case of Hölder, Besov, and other function spaces see e.g. [14].

Theorem 3.2. *Consider a stochastic process* $X = \{X(t)\}_{t\in[0,1]}$, $X(0) = 0$, *such that for some* $H \in (0,1]$, $C > 0, d > 0$ *and* $\delta > 0$, *we have*

$$\sup_{t,s\in[0,1]} \mathbb{P}\Big\{\frac{|X(t)-X(s)|}{|t-s|^H} > u\Big\} \le Ce^{-du^\delta}, \qquad (u > 0). \tag{10}$$

Then, X *has a modification* $\widetilde{X}$ *with continuous paths. Also, for all* $\gamma \in (0, H)$, *we have* $\widetilde{X} \in \mathcal{H}_\gamma^0$ *and*

$$\mathbb{P}\{|\widetilde{X}|_\gamma > u\} \le 6C\frac{e^{-cdu^\delta}}{cdu^\delta - 1}, \qquad (u > 1/(cd)^{1/\delta}), \tag{11}$$

for some $c = c_{\delta(H-\gamma)} > 0$.

Remark 3.3. The fact that X has a version $\widetilde{X}$ with $\gamma-$Hölder paths for all $\gamma \in (0,H)$ is an easy consequence of the Kolmogorov–Chentzov criterion. The proof of the more delicate result in (11) on the tail behavior of the norm $|\widetilde{X}|_\gamma$, relies on Ciesielski's isomorphism (see Theorem 3.1 above).

Remark 3.4. If X is a Gaussian process with $\gamma-$Hölder–continuous paths, then the Gaussian process $G(t,s) = (X(t)-X(s))/|t-s|^\gamma$, $(t,s) \in [0,1]^2$ is bounded with probability one. Therefore, the tail behavior of the supremum of G (over $[0,1]^2$) follows immediately from well–known results such as the Borel–TIS inequality (see, e.g. Theorem 2.1.2 in [1]).

Theorem 3.2, however, applies also to non–Gaussian processes and relies only on information about the bivariate marginal distributions of the process X. Therefore, Corollary 3.1 below can be used to establish the tail behavior of suprema, where the Borel–TIS inequality does not apply.

Corollary 3.1. *Let* $Y = \{Y(t)\}_{t\in[0,1]}$ *be such that the process* $X(t) := Y(t) - Y(0)$ *satisfies the assumptions of Theorem* 3.2. *Then,* Y *can be modified to have continuous paths and*

$$\mathbb{P}\Big\{\sup_{t\in[0,1]} |Y(t)| > u\Big\} \le \mathbb{P}\{|Y(0)| > u/2\} + 6C\frac{e^{-cu^\delta}}{cu^\delta - 1}, \qquad (u > 1/c^{1/\delta}),$$

for some $c = c_{\delta,H} > 0$.

P r o o f o f T h e o r e m 3.2. By (10), we have that for all $k > 0$, and $t \neq s \in [0,1]$,

$$\mathbb{E}|X(t) - X(s)|^k = \int_0^\infty \mathbb{P}\Big\{\frac{|X(t) - X(s)|^k}{\Delta^{kH}} > \frac{u}{\Delta^{kH}}\Big\}du \leq C\int_0^\infty e^{-du^{\delta/k}\Delta^{-\delta H}}du,$$

where $\Delta := |t-s|$. By making the change of variables $v := du^{\delta/k}\Delta^{-\delta H}$, we obtain that the last integral equals

$$d^{-k/\delta}(k/\delta)\int_0^\infty v^{k/\delta-1}e^{-v}dv\Delta^{kH} = d^{-k/\delta}\Gamma(k/\delta+1)|t-s|^{kH},$$

where Γ stands for the Gamma function. This implies that for all $k > 0$,

$$\mathbb{E}|X(t) - X(s)|^k \leq d^{-k/\delta}\Gamma(k/\delta+1)|t-s|^{kH}, \quad t, s \in [0,1].$$

Thus, the Kolmogorov–Chentzov continuity criterion (see e.g. p. 53 in [11]) implies that X has a modification $\widetilde{X}$ with $\gamma-$Hölder paths, for all $\gamma \in (0, H - 1/k)$. By taking arbitrarily large k's, we obtain $\widetilde{X} \in \mathcal{H}_\gamma^0$, for all $\gamma \in (0, H)$, with probability one.

Ciesielski's isomorphism (Theorem 3.1, above) implies that with probability one,

$$|\widetilde{X}|_\gamma = |T_\gamma^{-1}(\xi)|_\gamma \leq \|T_\gamma^{-1}\|\|\xi\|_\infty = \|\xi\|_\infty, \tag{12}$$

for all $\gamma \in (0, H)$, where $\|\xi\|_\infty = \sup_{n\geq 1}|\xi_n|$, with

$$\xi_n = \frac{2^{(j+1)\gamma}}{2}\Big(\widetilde{X}((k-1/2)/2^j) - \widetilde{X}(k/2^j) + \widetilde{X}((k-1/2)/2^j) - \widetilde{X}((k-1)/2^j)\Big). \tag{13}$$

Let now $Z_n := \xi_n 2^{(j+1)(H-\gamma)}$, and observe that by (13), with $\Delta := 2^{-(j+1)}$, we have

$$\begin{aligned}\mathbb{P}\{|Z_n| > u\} \quad &\leq \quad \mathbb{P}\Big\{\frac{|\widetilde{X}((k-1/2)/2^j) - \widetilde{X}(k/2^j)|}{\Delta^H} > u\Big\} \\ &+\mathbb{P}\Big\{\frac{|\widetilde{X}((k-1/2)/2^j) - \widetilde{X}((k-1)/2^j)|}{\Delta^H} > u\Big\} \leq 2Ce^{-du^\delta},\end{aligned} \tag{14}$$

where the first inequality follows from the fact that $\mathbb{P}\{|\eta+\zeta|/2 > u) \leq \mathbb{P}\{|\eta| > u\} + \mathbb{P}\{|\zeta| > u\}$, and the second one from the assumption (10).

Note that $Z_n = \xi_n a_n$, where $a_n = 2^{(j+1)(H-\gamma)}$, with $j = [\log_2 n] - 1$. Therefore, for all $\gamma \in (0, H)$, we have

$$a_n^\delta \geq n^{\delta(H-\gamma)} \geq c(\log(n+1)+1),$$

for some $c = c_{\delta(H-\gamma)} > 0$ and for all $n \geq 1$. Relation (14) also implies that (15) holds with C replaced by $2C$. Hence, by applying Lemma 3.1 to the Z_n's and a_n's, we obtain

$$\mathbb{P}\{\sup_{n\geq 1} |\xi_n| > u\} = \mathbb{P}\{\sup_{n\geq 1} |Z_n|/a_n\} \leq 2C\frac{e^{-cdu^\delta}}{cdu^\delta - 1},$$

for all $u^\delta > 1/cd$. The last inequality and Relation (12) imply (11). □

The next lemma was used in the proof of Theorem 3.2.

Lemma 3.1. *Let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables with arbitrary dependence structure and such that for some $d > 0, \delta > 0$, and $C > 0$,*

$$\sup_{n\geq 1} \mathbb{P}\{|Z_n| > u\} \leq Ce^{-du^\delta}, \quad (u > 0). \tag{15}$$

Then, for all $a_n^\delta \geq c(\log(n+1)+1)$, $n \geq 1$, and $cdu^\delta > 1$,

$$\mathbb{P}\Big\{\sup_{n\geq 1} |Z_n|/a_n > u\Big\} \leq C\frac{e^{-cdu^\delta}}{cdu^\delta - 1}. \tag{16}$$

Proof. By using the union bound and (15), we obtain

$$\mathbb{P}\Big\{\sup_{n\geq 1} |Z_n|/a_n > u\Big\} \leq \sum_{n\geq 1} \mathbb{P}\{|Z_n| > ua_n\} = C\sum_{n\geq 1} e^{-a_n^\delta du^\delta},$$

The last sum is bounded above by

$$\sum_{n\geq 1} e^{-c(\log(n+1)+1)du^\delta} = e^{-cdu^\delta}\sum_{n\geq 1}(n+1)^{-cdu^\delta} \tag{17}$$

$$\leq e^{-cdu^\delta}\int_1^\infty x^{-cdu^\delta}dx = \frac{e^{-cdu^\delta}}{(cdu^\delta - 1)},$$

which implies (16). □

Theorem 3.2 imposes stringent tail decay on the marginal distributions and it does not apply to many interesting classes of processes such as the Geometric Brownian motion, for example. The following result provides less precise but useful tail–bounds on the Hölder norm that apply even to processes with heavy–tailed marginal distributions. Observe that in the heavy–tailed case, however, we pay a price in a restricted range of Hölder exponents.

Theorem 3.3. *Let the stochastic process* $X = \{X(t)\}_{t\in[0,1]}$, $X(0) = 0$ *be such that*

$$\sup_{t,s\in[0,1]} \mathbb{P}\Big\{\frac{|X(t) - X(s)|}{|t - s|^H} > u\Big\} \le \Lambda(u), \quad (u > 0) \tag{18}$$

for some $\Lambda(u) \in (0, 1]$. *Let also*

$$\limsup_{u\to\infty} \frac{\log \Lambda(u)}{\log u} = -c, \tag{19}$$

for some $c \in (1/H, \infty]$.

Then X *has a version* $\widetilde{X}$ *with continuous paths and for all* $\gamma \in (0, H-1/c)$, *we have* $\widetilde{X} \in \mathcal{H}^0_\gamma$. *We have moreover that*

$$\limsup_{u\to\infty} \frac{\log \mathbb{P}\{|\widetilde{X}|_\gamma > u\}}{\log u} \le -c. \tag{20}$$

Proof. As in the proof of Theorem 3.2, we get

$$\mathbb{E}|X(t) - X(s)|^k \le \Big(\int_0^\infty \Lambda(v)kv^{k-1}dv\Big)|t - s|^{kH},$$

for all $t, s \in [0, 1]$ and $k > 0$. Relation (19) implies that $\int_0^\infty \Lambda(v)kv^{k-1}dv < \infty$, for all $k \in (0, c)$. Hence, the Kolmogorov–Chentzov criterion yields the existence of a $\gamma-$Hölder continuous version $\widetilde{X}$ of X, for all $\gamma \in (0, H - 1/c)$.

Proceeding as in the proof of Theorem 3.2, with $Z_n := 2^{(j+1)(H-\gamma)}\xi_n \equiv a_n\xi_n$, where $j = [\log_2 n] - 1$ and ξ_n as in (13), we obtain by (18) that

$$\mathbb{P}\{|Z_n| > u\} \le 2\Lambda(u), \quad (u > 0).$$

Thus, Ciesielski's isomorphism theorem, implies that for all $\gamma \in (0, H - 1/c)$,

$$\mathbb{P}\{|\widetilde{X}|_\gamma > u\} \le \mathbb{P}\{\|\xi\|_\infty > u\} \le \sum_{n=1}^\infty \mathbb{P}\{|Z_n| > a_n u\} \le 2\sum_{n=1}^\infty \Lambda(a_n u).$$

Since $a_n = 2^{[\log_2 n](H-\gamma)} \ge n^{H-\gamma}$, where $H - \gamma > 1/c$, Lemma 3.2 (see below) yields (20). □

The following result was used in the proof of Theorem 3.3.

Lemma 3.2. *Let* $\Lambda(u) \in (0,1]$ *be such that*

$$\limsup_{u\to\infty} \frac{\log \Lambda(u)}{\log u} = -c, \tag{21}$$

for some $c \in (0,\infty]$. *If* $a_n \geq Cn^\delta$, *for some* $\delta > 1/c$ *and* $C > 0$, *then*

$$\limsup_{u\to\infty} \frac{\log(\sum_{n=1}^\infty \Lambda(a_n u))}{\log u} = -c. \tag{22}$$

Proof. Since $\Lambda(a_n u) > 0$, the left–hand side in (22) is no smaller than the right–hand side of (21). Therefore, it suffices to show that the left–hand side of (22) is no greater than $-c$.

Let $D \in (0,c)$, be such that $\delta > 1/D > 1/c$. By (21), there exists a constant $\widetilde{C}$, such that $\Lambda(u) \leq \widetilde{C} u^{-D},\ u > 0$ and hence

$$\sum_{n=1}^\infty \Lambda(a_n u) \leq \widetilde{C} \sum_{n=1}^\infty a_n^{-D} u^{-D} \leq \widetilde{C}_1 u^{-D}, \tag{23}$$

where $\widetilde{C}_1 = \widetilde{C} \sum_{n=1}^\infty a_n^{-D} < \infty$ because $a_n^{-D} = \mathcal{O}(n^{-\delta D}),\ n \to \infty$ and $\delta D > 1$. Since $D \in (1/\delta, c)$ was arbitrary, Relation (23) implies (22). □

Remark 3.5. Theorem 3.3 (in the case $c < \infty$) is related in spirit to the classical moment tightness condition in [13] and to a result in [12]. See also Theorem 15 in [27]. Here, we provide explicit bounds on the tails of the Hölder norms, not directly available in these results.

Remark 3.6. The isomorphism theorem of [5] is the key tool used to establish Theorems 3.2 and 3.3. Other function spaces enjoy similar appealing isomorphisms properties. Most notably, the class of Besov spaces $B^\gamma_{p,q}(\mathbb{R})$, which consists of all $L^p(\mathbb{R})$ functions of certain "regularity" γ are isomorphic to weighted sequence spaces. In fact, Hölder spaces are a special case of Besov spaces with $p = q = \infty$. In principle, similar results to Theorems 3.2 and 3.3 on the tail behavior of Besov norms can be established *mutatis mutandis* by using wavelet characterizations (see e.g. [27]). This is beyond the scope of the present work, which focuses on Hölder spaces. Partial results on weak convergence in Besov spaces are given in Section 4, below.

4. Functional limit theorems in Hölder and Besov spaces. We start by introducing some notation and sketching the ideas behind the results in this section.

Let $\xi(t) = X\zeta(t),\ t \in [0,1]$, where X is a positive heavy–tailed random variable, independent of the process $\zeta = \{\zeta(t)\}_{t\in[0,1]}$. Suppose that ζ takes values in $\mathcal{H}_\gamma$. If

$$\mathbb{P}\{X > x\} \sim L(x)x^{-\alpha}, \quad \text{as } x \to \infty, \ \text{ and } \ \mathbb{E}|\zeta(t)|^{\alpha+\delta} < \infty, \tag{24}$$

for some $\delta > 0$, then a well–known result dating back to [3] implies that

$$\mathbb{P}\{\xi(t) > x\} \sim (\mathbb{E}\zeta(t)_+^\alpha)L(x)x^{-\alpha}, \quad \text{as } x \to \infty,$$

where $\zeta(t)_+ := \max\{\zeta(t), 0\}$ (see e.g. Lemma 1.1 in [16]).

Using this fact and a Wold–type argument one can show (see e.g. the proof of Proposition 4.1 below) that

$$\frac{1}{a_n} \bigvee_{1\le i\le n} X_i\zeta_i(\cdot) \overset{f.d.d.}{\Longrightarrow} \eta(\cdot), \quad \text{as } n \to \infty, \tag{25}$$

where $\eta = \{\eta(t)\}_{t\in[0,1]}$ is an $\alpha-$Fréchet max–stable process, where the X_i's and $\zeta_i(\cdot)$'s are independent copies of X and ζ. The finite–dimensional distributions of η are as follows:

$$\mathbb{P}\{\eta(t_j) \le x_j,\ 1 \le j \le m\} = \exp\Big\{ - \mathbb{E}\Big(\max_{1\le j\le m} \zeta(t_j)_+/x_j\Big)^\alpha\Big\}, \tag{26}$$

for all $x_j > 0,\ t_j \in [0,1],\ 1 \le j \le m$. For more examples and details on the representations of max–stable processes see Section 5 below.

The next result provides simple sufficient conditions for the convergence in (25) to hold also in a Hölder space.

Proposition 4.1. *Suppose that ζ is an $\mathcal{H}_\gamma$–valued random element, $0 < \gamma \le 1$. Let $M_n(t) := a_n^{-1}\max_{1\le i\le n} X_i\zeta_i(t)$, with a_n as in (3), where the X_i's and ζ_i's are independent copies of X and ζ as in (24). If*

$$\mathbb{E}\|\zeta_+\|_{\mathcal{H}_\gamma}^{\alpha+\delta} \equiv \mathbb{E}\Big(\sup_{t\in[0,1]} \zeta(t)_+ \vee \sup_{t,s\in[0,1]} \frac{|\zeta(t)_+ - \zeta(s)_+|}{|t-s|^\gamma}\Big)^{\alpha+\delta} < \infty, \tag{27}$$

with some $\delta > 0$, then

$$M_n \overset{\mathcal{H}_\beta}{\Longrightarrow} \eta, \quad \textit{as } n \to \infty, \tag{28}$$

for all $\beta \in (0,\gamma)$, where '$\overset{\mathcal{H}_\beta}{\Rightarrow}$' stands for weak convergence of distributions in $(\mathcal{H}_\beta, \|\cdot\|_{\mathcal{H}_\beta})$, and η satisfies (26). In particular, a max–stable process η with *finite–dimensional distributions as in* (26) *has a version with paths in* $\mathcal{H}_\beta, 0 < \beta < \gamma$.

Proof. Let $x_j > 0,\ t_j \in [0,1],\ 1 \le j \le m$ be arbitrary and observe that

$$\mathbb{P}\{M_n(t_j) \le x_j,\ 1 \le j \le m\} = \mathbb{P}\Big\{ \bigvee_{1\le i\le n} X_i Y_i \le a_n \Big\} = \mathbb{P}\{X_1 Y_1 \le a_n\}^n, \tag{29}$$

where

$$Y_i := \bigvee_{1\le j\le m} \zeta_i(t_j)/x_j,\ 1 \le i \le n.$$

On the other hand, since $a_n > 0,\ x_j > 0,\ 1 \le j \le m$, and $X_1 > 0$, we have that $\mathbb{P}\{X_1 Y_1 \le a_n\} = \mathbb{P}\{X_1 Y_1^+ \le a_n\}$, where

$$Y_1^+ := \bigvee_{1\le j\le m} \zeta(t_j)_+/x_j,\ 1 \le i \le n.$$

By (27) we have $\mathbb{E}|\zeta(t)|^{\alpha+\delta} < \infty$, for all $t \in [0,1]$, and hence $\mathbb{E}(Y_1^+)^{\alpha+\delta} < \infty$. Thus, in view of the independence of X_1 and Y_1^+, Breiman's Lemma (see e.g. Lemma 1.1 in [16]) implies that

$$\mathbb{P}\{X_1 Y_1^+ > a_n\} \sim \mathbb{E}(Y_1^+)^\alpha L(a_n) a_n^{-\alpha}, \quad \text{as } n \to \infty.$$

Therefore, by using the fact that $\log(c_n^n) \sim -n(1 - c_n)$, as $c_n \to 0,\ c_n > 0$, we obtain

$$\log(\mathbb{P}\{X_1 Y_1^+ \le a_n\}^n) \sim -n\mathbb{E}(Y_1^+)^\alpha L(a_n) a_n^{-\alpha} \longrightarrow \mathbb{E}(Y_1^+)^\alpha, \tag{30}$$

as $n \to \infty$, since the choice of the normalizing sequence a_n in (3) ensures that $nL(a_n)/a_n^\alpha \to 1,\ n \to \infty$.

Relations (29) and (30) imply that

$$\mathbb{P}\{M_n(t_j) \le x_j,\ 1 \le j \le m\} \longrightarrow \exp\{-\mathbb{E}(Y_1^+)^\alpha\} = \exp\Big\{ -\mathbb{E}\Big(\max_{1\le j\le m} \zeta(t_j)_+/x_j \Big)^\alpha \Big\},$$

as $n \to \infty$. In view of (26), the right–hand side of the last expression equals $\mathbb{P}\{\eta(t_j) \le x_j,\ 1 \le j \le m\}$, where the $\eta = \{\eta(t)\}_{t\in T}$ is the α–Fréchet process in (26). We have thus shown the convergence of the finite–dimensional distributions. Condition (27) and Theorem 2.1 imply (28). □

Remarks 2.1 and 3.6 suggest that the functional limit results for maxima in Hölder spaces can be extended to the case of Besov spaces. At this point, we can provide only a partial result in Proposition 4.2 below for the case when the tail exponent α is confined to the interval $(0, 1)$.

The Besov space $B^{\gamma}_{p,q}([0,1]),\ 1 \le p, q \le \infty$ consists of $L^p([0,1])$ functions f with finite

$$|f|_{B^{\gamma}_{p,q}} := \Big(\int_0^1 (t^{-\gamma}\omega_p(f;t))^q \frac{dt}{t} \Big)^{1/q}, \tag{31}$$

where

$$\omega_p(f;t) := \sup_{0<h\le t} \Big(\int_0^{1-h} |f(x+h) - f(x)|^p dx \Big)^{1/p},\ t \in (0,1). \tag{32}$$

If $q = \infty$, the right–hand side in (31) is interpreted as $\sup_{t\in(0,1)} t^{-\gamma}\omega_p(f;t)$, while $\omega_\infty(f;t) := \sup_{0<h<t} \|f(\cdot + h) - f(\cdot)\|_\infty$. The function space $B^{\gamma}_{p,q} \equiv B^{\gamma}_{p,q}([0,1])$, $1 \le p, q \le \infty,\ 0 < \gamma \le 1$, equipped with the norm

$$\|f\|_{B^{\gamma}_{p,q}} := \|f\|_{L^p} + |f|_{B^{\gamma}_{p,q}},\quad f \in B^{\gamma}_{p,q}, \tag{33}$$

becomes a Banach space. It follows that $B^{\gamma}_{\infty,\infty} = \mathcal{H}_\gamma$ (see e.g. Theorem 1.122 in [28]) and therefore, the Besov spaces may be viewed as natural extensions of Hölder spaces, where the notion of regularity is understood in an average (L^p, L^q)–sense.

In our context, if $1 \le p, q < \infty$, then the inequality $\|f \vee g\|_{B^{\gamma}_{p,q}} \le \|f\|_{B^{\gamma}_{p,q}} \vee \|g\|_{B^{\gamma}_{p,q}}$ is not valid for all $f,\ g \in B^{\gamma}_{p,q}$. Thus, the method of proof in Theorem 2.1 does not extend in full generality to the case of Besov spaces. Nevertheless, some partial results are possible. The next lemma shows that $B^{\gamma}_{p,q}$ is closed with respect to point–wise maxima.

Lemma 4.1. *For all* $f, g \in B^{\gamma}_{p,q},\ 1 \le p, q \le \infty,\ 0 < \gamma \le 1$, *we have*

$$\|f \vee g\|_{B^{\gamma}_{p,q}} \le \|f\|_{B^{\gamma}_{p,q}} + \|g\|_{B^{\gamma}_{p,q}}. \tag{34}$$

Proof. Relation (4) implies that $|(f \vee g)(x+h) - (f \vee g)(x)| \le |f(x+h) - f(x)| + |g(x+h) - g(x)|$. Thus, the Minkowski inequality applied to (32) yields

$$\omega_p(f \vee g; t) \le \omega_p(f;t) + \omega_p(g;t),$$

for all $t \in [0, 1-h]$, and hence $|f \vee g|_{B^\gamma_{p,q}} \le |f|_{B^\gamma_{p,q}} + |g|_{B^\gamma_{p,q}}$. On the other hand, since $|f \vee g| \le |f| + |g|$, we obtain $\|f \vee g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$, which in view of (33) implies the desired inequality (34). $\Box$

Now, for $0 < \alpha < 1$, we obtain the following extension of Proposition 4.1.

Proposition 4.2. *Let* X, ζ, *and* M_n *be as in Proposition* 4.1 *where now* ζ *takes values in the Besov space* $B^\gamma_{p,q}[0,1]$, *for some* $1 \le p, q \le \infty$ *and* $0 < \gamma \le 1$. *Suppose that* $0 < \alpha < 1$ *and* $\mathbb{E}\|\zeta\|^{\alpha+\delta}_{B^\gamma_{p,q}} < \infty$, *for some* $\delta > 0$. *Then, for all* $0 < \beta < \gamma$, *we have*

$$M_n \overset{B^\beta_{p,q}}{\Longrightarrow} \eta, \quad \text{as } n \to \infty, \tag{35}$$

where '$\overset{B^\beta_{p,q}}{\Rightarrow}$' *denotes weak convergence in* $B^\beta_{p,q}[0,1]$ *and the limit* η *satisfies* (26). *In particular, a max–stable process* η *with finite–dimensional distributions as in* (26) *has a version with paths in* $B^\beta_{p,q}[0,1], 0 < \beta < \gamma$.

Proof. Relation (34) implies that

$$\|M_n\|_{B^\gamma_{p,q}} \le \frac{1}{a_n}\sum_{i=1}^n \|X_i\zeta_i\|_{B^\gamma_{p,q}} = \frac{1}{n^{1/\alpha}\ell(n)}\sum_{i=1}^n X_i\|\zeta_i\|_{B^\gamma_{p,q}}.$$

Now, Relation (24), the fact that $\mathbb{E}\|\zeta_i\|^{\alpha+\delta}_{B^\gamma_{p,q}} < \infty$ and Breiman's Lemma imply that

$$\mathbb{P}\{X_i\|\zeta_i\|_{B^\gamma_{p,q}} > x\} \sim cL(x)x^{-\alpha}, \quad \text{as } x \to \infty, \tag{36}$$

with $c = \mathbb{E}\|\zeta_i\|^\alpha_{B^\gamma_{p,q}}$ (see e.g. Lemma 1.1 in [16]).

Since $0 < \alpha < 1$, the positive random variables $Z_i := X_i\|\zeta_i\|_{B^\gamma_{p,q}}$, $i = 1, \ldots, n$ belong the domain of attraction of a totally skewed, positive α–stable random variable. Moreover, Relation (36) and the fact that $nL(a_n)/a_n^\alpha \to 1$, as $n \to \infty$ (recall (3)) imply that the right–hand side of (36) converges in distribution to a proper α–stable random variable (see e.g. Theorem XIII.6.2 in [7]).

This shows that the laws of $\|M_n\|_{B^\gamma_{p,q}}$, $n \ge 1$ are tight in $\mathbb{R}$. Now, the fact that the unit ball in $B^\gamma_{p,q}[0,1]$ is *compact* in $B^\beta_{p,q}[0,1]$, for all $0 < \beta < \gamma$ implies that the laws of $M_n(\cdot)$, $n \ge 1$ are tight in $B^\beta_{p,q}$, $0 < \beta < \gamma$. See also the proof of Theorem 2.1, and Remarks 2.1 and 2.2, above. This completes the proof of (35) since the convergence of the finite–dimensional distributions of M_n was already established in Proposition 4.1. $\Box$

5. Examples and applications. Condition (27) in Proposition 4.1 may be hard to check in practice. In this section, we examine several interesting classes of stochastic processes for which (27) can be verified with the help of the results of Section 3. We provide also an application of Proposition 4.2, where the paths of the resulting max–stable process are discontinuous. We start with some preliminaries on the spectral representations of max–stable processes.

By the seminal work of [6], every continuous in probability max–stable process has a *spectral representation* in terms of certain functionals of a Poisson point process. These functionals may be also viewed as extremal stochastic integrals with respect to a random sup–measure. In fact, as shown in [25], for every separable in probability max–stable process $\{\eta(t)\}_{t\in[0,1]}$ with $\alpha-$Fréchet marginals, $\alpha > 0$, we have:

$$\{\eta(t)\}_{t\in[0,1]} \stackrel{d}{=} \Big\{ {\int^{e}}_{E} f(t,u) M_\alpha(du) \Big\}_{t\in[0,1]}. \tag{37}$$

Here $(E, \mathcal{E}, \mu)$ is a measure space that can be chosen to be standard Lebesgue, the $f(t,\cdot)$'s are non–negative measurable functions with $\int_E f(t,u)^\alpha \mu(du) < \infty$ and $M_\alpha(du)$ is a random sup–measure with control measure $\mu(du)$. That is, (i) M_α is *independently scattered*, i.e. it assigns independent α–Fréchet variables $M_\alpha(A_i)$, $1 \le i \le n$ to disjoint sets $A_i \in \mathcal{E}$, $1 \le i \le n$; (ii) the measure μ controls the scale of the random variables $M_\alpha(A)$: $\mathbb{P}\{M_\alpha(A) \le x\} = \exp\{-\mu(A)x^{-\alpha}\}$, $x > 0$; (iii) M_α is *σ–sup–additive*, i.e. $M_\alpha(\cup_{n\ge1} A_n) = \sup_{n\ge1} M_\alpha(A_n)$, almost surely, for all $A_n \in \mathcal{E}$, $n \in \mathbb{N}$.

For a simple function, the above extremal integrals are defined as follows:

$${\int^{e}}_{E} \Big(\sum_{i=1}^{n} a_i 1_{A_i}(u) \Big) M_\alpha(du) := \bigvee_{i=1}^{n} a_i M_\alpha(A_i),$$

where $a_i \ge 0$ and the A_i's are disjoint measurable sets. It then follows that

$${\int^{e}}_{E} af(u) \vee bg(u) M_\alpha(du) = a {\int^{e}}_{E} f(u) M_\alpha(du) \vee b {\int^{e}}_{E} g(u) M_\alpha(du), \quad \text{(max–linearity)}$$

for all $a, b \ge 0$ and simple non–negative functions f and g. We have moreover that

$$\mathbb{P}\Big\{ {\int^{e}}_{E} f(u) M_\alpha(du) \le x \Big\} = \exp\{-\|f\|^\alpha_{L^\alpha(\mu)} x^{-\alpha}\}, \quad x > 0, \quad \text{(isometry)}$$

and therefore, the definition of the extremal integral extends by continuity in probability to all deterministic integrands in $L^\alpha_+(E, \mathcal{E}, \mu)$. By using the max–

linearity and isometry properties above, one can readily obtain the finite–dimensional distributions of the process:

$$\mathbb{P}\{\eta(t_j) \le x_j,\ 1 \le j \le n\} = \exp\Big\{ - \int_E \Big(\max_{1\le j\le n} \frac{f(t_j, u)}{x_j}\Big)^{\alpha} \mu(du)\Big\},$$

for all $t_j \in [0,1], x_j > 0,\ 1 \le j \le n$ (recall also (26)). For more details, see Proposition 5.11' in [22], [6], [25], and [9].

If the control measure μ is a probability measure, then the integrands $f(t,\cdot)$ may be interpreted as random variables defined on the probability space $(E, \mathcal{E}, \mu)$. Conversely, given a non–negative stochastic process $\zeta(t,\omega'),\ t \in [0,1]$, defined on a probability space $(\Omega', \mathcal{F}', P')$, with $\mathbb{E}_{P'}\zeta(t)^{\alpha} < \infty$, one can define the max–stable process

$$\eta = \{\eta(t)\}_{t\in[0,1]} \stackrel{d}{=} \Big\{ {\int_{\Omega'}}^{\!\!\!\!\!\!\mathrm{e}} \zeta(t,\omega')_+ M_{\alpha}(d\omega')\Big\}_{t\in[0,1]}. \tag{38}$$

We refer to the latter representations as *doubly stochastic*. The stochasticity in η is due to the randomness of the sup–measure M_{α}, which is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) \neq (\Omega', \mathcal{F}', P')$. On the other hand, the stochasticity in ζ governs the dependence structure of η.

Every separable in probability max–stable process has a doubly stochastic representation, which is typically not unique. The stochastic properties of the integrands in (38), however, are often helpful in establishing the properties of the resulting max–stable process (see, e.g. [26], [9], [10], and [29]). The examples in the rest of this section are based on doubly stochastic representations.

- **(Schlather processes)** Doubly stochastic max–stable processes (and random fields) driven by stationary Gaussian processes ζ were proposed and studied in [24]. Suppose that $\{\zeta(t),\ t \in [0,1]\}$ is a zero–mean Gaussian process with continuous paths defined on the probability space $(\Omega', \mathcal{F}', P')$. Let

$$d^2 := \sup_{t,s\in[0,1]} \frac{\mathbb{E}(\zeta(t) - \zeta(s))^2}{|t-s|^{2H}} < \infty, \tag{39}$$

for some $0 < H \le 1$. Then, Relation (10) readily follows with $\delta := 2$, the Hölder norm $\|\zeta\|_{\gamma}$ is finite and has finite moments of all order, for all $0 < \gamma < H$. Indeed, (39) implies that $\sigma^2(t,s) := \mathrm{Var}(\zeta(t) - \zeta(s)) \le$

$d|t-s|^{2H}$, $t,s \in [0,1]$, and hence

$$(40)\quad \sup_{t,s\in[0,1]} \mathbb{P}\{|\zeta(t)-\zeta(s)|/|t-s|^H > u\} \le \mathbb{P}\{|Z| > u/d\} \le \frac{\sqrt{2}d}{\sqrt{\pi}u} e^{-u^2/2d^2},$$

for $u > 0$, where Z denotes a standard Normal random variable.

By Proposition 4.1 the max–stable process $\eta(t) := {}^{\mathrm{e}}\!\!\int_{\Omega'} \zeta(t,\omega')_+ M_\alpha(d\omega')$, $t \in [0,1]$, has a version with paths in $\mathcal{H}_\beta$, for all $0 < \beta < H$.

- **(Brown–Resnick processes)** Let again $\zeta(t) \equiv \zeta(t,\omega')$, $t \in \mathbb{R}$ be a zero–mean Gaussian process with variance $\sigma^2(t) = \mathrm{Var}(\zeta(t))$ and continuous paths, defined on the probability space $(\Omega', \mathcal{F}', P')$. [10], extending results of [4], showed that, surprisingly, the max–stable process

$$(41)\qquad \eta(t) := {}^{\mathrm{e}}\!\!\int_{\Omega'} e^{\zeta(t,\omega')-\alpha\sigma^2(t)/2} M_\alpha(d\omega'), \quad t \in \mathbb{R},$$

is stationary provided that the increments of ζ are stationary. The process η in (41) is referred to as a *Brown–Resnick stationary* max–stable process. For more details on Brown–Resnick stationary random fields, see [10].

The following result establishes the Hölder regularity of the process η.

Proposition 5.1. *If* (39) *holds, then the Brown–Resnick process* $\eta = \{\eta(t)\}_{t\in[0,1]}$ *in* (41) *has a version with paths in* $\mathcal{H}_\beta$*, for all* $0 < \beta < H$*.*

P r o o f. Observe that $\eta(t) = e^{-\alpha\sigma^2(t)/2}\widetilde{\eta}(t)$, where $\widetilde{\eta}(t) := {}^{\mathrm{e}}\!\!\int_{\Omega'} e^{\zeta(t,\omega')} M_\alpha(d\omega')$. By the Cauchy–Schwartz inequality and (39), we have that

$$|\sigma^2(t)-\sigma^2(s)| \le (\mathbb{E}_{P'}(\zeta(t)-\zeta(s))^2)^{1/2}(\mathbb{E}_{P'}(\zeta(t)+\zeta(s))^2)^{1/2} \le \mathrm{const}\,|t-s|^H,$$

$t,s \in [0,1]$. Thus, the deterministic function $t \mapsto e^{-\alpha\sigma^2(t)/2}$ belongs to $\mathcal{H}_H$, and to complete the proof, it is enough to show that the process $\widetilde{\eta}(t)$, $t \in [0,1]$ has a version with paths in $\mathcal{H}_\beta$, for all $0 < \beta < H$. This is because $fg \in \mathcal{H}_\beta$ for all $f, g \in \mathcal{H}_\beta$.

We shall verify that $X(t) := e^{\zeta(t)}$, $t \in [0,1]$ satisfies Condition (18) of Theorem 3.3 with $c = \infty$. Indeed, by the mean value theorem, applied to the function $x \mapsto e^x$, we have that

$$(42)\qquad |e^{\zeta(t)} - e^{\zeta(s)}| \le |\zeta(t)-\zeta(s)| e^{\|\zeta\|},$$

where $\|\zeta\| := \sup_{t\in[0,1]} \zeta(t)$. Since $\zeta(t),\ t \in [0,1]$ has continuous paths, $\|\zeta\| < \infty$, P'–almost surely. Therefore, by the Borel–TIS inequality (see e.g. Theorem 2.1.1 in [1]), we obtain that

$$P'\{\|\zeta\| > v\} \le C_2 e^{-v^{\kappa}}, \quad v > 0, \tag{43}$$

for some (any) $\kappa \in (0,2)$ and some $C_2 > 0$.

Now, in view of (42), we obtain

$$\begin{aligned} P'\Big\{\frac{|e^{\zeta(t)} - e^{\zeta(s)}|}{|t-s|^H} > u\Big\} &\le P'\Big\{\frac{|\zeta(t) - \zeta(s)|}{|t-s|^H} > u^{1/2}\Big\} + P'\{e^{\|\zeta\|} > u^{1/2}\} \\ &\le C_1 e^{-du} + C_2 e^{-\log(u^{1/2})^{\kappa}} =: \Lambda(u), \end{aligned} \tag{44}$$

for some positive constants C_1, C_2, d and κ that do not depend on $t, s \in [0,1]$. The last inequality follows from Relations (39) and (43). Indeed, the first term in the right–hand side of (44) follows from (39) by using bounds for standard Gaussian tails as in (40). On the other hand, the second term therein follows from (43) by replacing v with $\log(u^{1/2})$.

Since the exponent κ in the right–hand side of (44) can be taken to be greater than 1, we obtain that (18) and (19) hold with $c = \infty$. Theorem 3.3 then implies that the paths of $X = e^{\zeta}$ belong to $\mathcal{H}_\beta$, for all $\beta \in (0,H)$. Since $c = \infty$, by (20), we have moreover that $\mathbb{E}_{P'}\|e^{\zeta}\|^k_{\mathcal{H}_\beta} \equiv \mathbb{E}_{P'}\|X\|^k_\beta < \infty$, for all $k \in \mathbb{N}$ and $\beta \in (0,H)$. Proposition 4.1, thus implies that the max–stable process $\widetilde{\eta}(t) = \int^{e}_{\Omega'} e^{\zeta(t,\omega')} M_\alpha(d\omega')$ has a version with paths in $\mathcal{H}_\beta,\ 0 < \beta < H$. □

Remark 5.1. An anonymous referee has pointed out an elegant alternative approach to establishing path regularity for Brown–Resnick processes based on their de Haan spectral representation. Following [10], let $\xi_i(t) := \zeta_i(t) - \sigma^2(t)/2$ and without loss of generality assume that $\alpha = 1$. Consider a Poisson point process $\{U_i\}_{i\in\mathbb{N}}$ on $\mathbb{R}$ with intensity $e^{-u}\mathrm{d}u$, independent of the $\xi_i(t)$'s. The process

$$\eta(t) := \bigvee_{i\in\mathbb{N}} \exp\Big\{U_i + \xi_i(t)\Big\}, \quad t \in \mathbb{R} \tag{45}$$

equals in law to the Brown–Resnick process in Proposition 5.1 (with $\alpha = 1$).

Now, as argued in the proof of Proposition 13 of [10], one can show that with probability one, $\{\eta(t),\ t \in [0,1]\}$ is the maximum of a finite (but random) number of processes $\{\exp\{U_i + \xi_i(t)\},\ t \in [0,1]\}$. By (39), the latter may be taken to have paths in $\mathcal{H}_\beta$, for all $0 < \beta < H$. Thus, the claim of Proposition 5.1 follows, since Hölder regularity is preserved under point–wise maxima (Lemma 2.1).

Remark 5.2. The previous remark suggests also a method for the simulation of Brown–Resnick processes, which yields *exact paths* with high probability. A simple truncation of the Poisson point representation (45) is not practical. The recent work of [15] provides important alternative representations tailored for the efficient simulation of Brown–Resnick processes.

- **(Lévy–Fréchet)** Let now $0 < \beta < 1$ and consider the β–stable subordinator $\zeta = \{\zeta(t)\}_{t\geq 0}$ be defined on the probability space $(\Omega', \mathcal{F}', P')$. Namely, ζ is a positive Lévy β–stable process. Without loss of generality, we shall assume that ζ has right–continuous paths with left limits (*càdlág*).

  Now, consider an α–Fréchet random sup-measure M_α over the measurable space $(\Omega', \mathcal{F}')$ with control measure P'. The sup–measure is supported on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) \neq (\Omega', \mathcal{F}', P')$.

  If $0 < \alpha < \beta < 1$, then $\mathbb{E}_{P'}\zeta(t)^\alpha < \infty$, and then the following *doubly stochastic* max–stable process is well–defined:

$$\eta(t) := \int^{\mathrm{e}}_{\Omega'} \zeta(t,\omega') M_\alpha(d\omega'), \quad t \geq 0.$$

  We shall refer to $\eta = \{\eta(t)\}_{t\geq 0}$ as to a *Lévy–Fréchet* process.

  The process η does not have a version with continuous paths. Indeed, since ζ has *càdlág* paths, it is separable and measurable, where as a separant one can choose an arbitrary countable dense set D in $[0,\infty)$. The paths $t \mapsto \eta(t,\omega')$, however, are discontinuous with P'–probability one. Therefore, by the characterization of continuity in Theorem 3.2 of [21] it follows that η cannot have continuous paths. The following simple result establishes the Besov–space regularity of η.

Proposition 5.2. *For all* $0 < \alpha < \beta \leq 1$, *the process* $\eta = \{\eta(t)\}_{t\in[0,1]}$ *has a version with paths in* $B^\gamma_{p,\infty}([0,1])$, *for all* $1 \leq p < \infty$ *and* $0 < \gamma < 1/p$.

Proof. The monotonicity of $\eta(\cdot,\omega')$ and the inequality $a^p + b^p \leq (a+b)^p$, valid for all $a, b \geq 0$ and $p \geq 1$ imply that

$$v_p([0,1], \eta(\cdot,\omega')) = |\eta(1,\omega') - \eta(0,\omega')|^p = \eta(1,\omega')^p.$$

We also have that $\|\eta(\cdot,\omega')\|_{L^p} \le \eta(1,\omega')$ and therefore, by Lemma 5.1, we obtain

$$\|\eta(\cdot,\omega')\|_{B^\gamma_{p,\infty}} \le \eta(1,\omega') + cv_p^{1/p}([0,1],\eta(\cdot,\omega')) = (1+c)\eta(1,\omega'),$$

for P'–almost all ω', and some constant $c>0$ independent of η and ω'.

Thus, $\mathbb{E}_{P'}\|\eta\|^{\alpha+\delta}_{B^\gamma_{p,\infty}} \le \mathrm{const}\,\mathbb{E}_{P'}\eta^{\alpha+\delta} < \infty$, for some (any) $0<\delta<\beta-\alpha$ and for all $0<\gamma<1/p$. Proposition 4.2 yields the desired result. □

Remark 5.3. The logic in the proof of Proposition 5.2 applies to any other setting where one can control the norm of the p–variation. For the lack of a better example, we considered the case of *Lévy–Fréchet* processes here. A simple direct proof of Proposition 5.2 can be obtained without the use of Proposition 4.2 as follows.

Note that η is continuous in probability since ζ is continuous in the $L^\alpha(\Omega',P')$–sense (see e.g. [6]). Also, $\eta(t)<\eta(s)$, for all $0\le t\le s$, $\mathbb{P}$–almost surely, since the increments of ζ are positive P'–almost surely. Therefore, for any countable set D that is dense in $[0,1)$, defining

$$\widetilde{\eta}(t) := \inf_{s>t,\, s\in D\cup\{1\}} \eta(s),$$

one obtains a process $\widetilde{\eta} = \{\widetilde{\eta}(t)\}_{t\in[0,1]}$ with *càdlág* paths that is a version of η. Indeed, the continuity of probability of η implies that $\eta(t)=\widetilde{\eta}(t)$, $\mathbb{P}$–almost surely, for all $t\in[0,1]$.

Now, Lemma 5.1 (see below) applied to the paths $\widetilde{\eta}(\cdot,\omega)$ of the process $\widetilde{\eta} = \{\eta(t)\}_{t\in[0,1]}$ yields $\widetilde{\eta}\in B^\gamma_{p,\infty}$, with probability one, for all $0<\gamma<1/p$, $p\ge 1$.

The next result was used in the proof of Proposition 5.2. It is akin to Theorem 2 in [23], but here we provide a norm estimate in a more concrete setting with an elementary proof.

Lemma 5.1. *Let $1\le p\le\infty$ and $f:[0,1]\to\mathbb{R}$ be a* càdlág *function with finite $p-$variation $v_p([0,1],f)<\infty$, where*

$$v_p([a,b],f) := \sup\Big\{\sum_{k=0}^{m-1} |f(t_{k+1})-f(t_k)|^p,\ a=t_0<t_1<\cdots<t_m=b,\ m=1,2,\ldots\Big\}$$

Then, for all $0<\gamma<1/p$ and $1\le q\le\infty$, we have

$$|f|_{B^\gamma_{p,q}} \le c_{q,\gamma} v_p^{1/p}([0,1],f), \quad \textit{with } c_{q,\gamma} := ((1/p-\gamma)q)^{-1/q},$$

where $c_{\infty,\gamma} = 1$. *In particular,* $f \in B^{\gamma}_{p,\infty}$.

Proof. Consider the following function:

$$F(x) := \inf_{y>x} v_p([0,y],f), \quad (0 \le x < 1) \ \text{ with } \quad F(1) := v_p([0,1],f).$$

Thus, F is right–continuous, and monotone non–decreasing. Note that for all $x \in (0, 1-h)$, and sufficiently small $\delta > 0$, we have

$$v_p([0,x+\delta],f) + |f(x+h) - f(x+\delta)|^p \le v_p([0,x+h],f) \le F(x+h).$$

Therefore, by letting $\delta \to 0+$, the right–continuity of f and the fact that $v_p([0,x+\delta],f) \to F(x)$, $\delta \to 0+$, imply:

$$|f(x+h) - f(x)|^p \le F(x+h) - F(x) = \int_{(0,1)} 1_{(x,x+h]}(u) dF(u).$$

In view of (32) and the Tonelli–Fubini theorem, we obtain

$$\begin{aligned} \omega_p(f;t) &\le \Big(\sup_{0<h\le t} \int_{(0,1-h)} \Big(\int_{(0,1)} 1_{(x,x+h]}(u) dF(u) \Big) dx \Big)^{1/p} \\ &= \Big(\sup_{0<h\le t} \int_{(0,1)} h dF(u) \Big)^{1/p} \le t^{1/p} F^{1/p}(1) = t^{1/p} v_p^{1/p}([0,1],f). \end{aligned}$$

Integrating the last relation as in (31) yields $|f|_{B^{\gamma}_{p,q}} \le v_p^{1/p}([0,1],f)((1/p - \gamma)q)^{-1/q}$, which completes the proof of the lemma. □

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