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REGULARITY OF SET-VALUED MAPS AND THEIR SELECTIONS THROUGH SET DIFFERENCES. PART 1: LIPSCHITZ CONTINUITY*

Robert Baier, Elza Farkhi

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Dedicated to the 65th birthday of Asen L. Dontchev
and to the 60th birthday of Vladimir M. Veliov.

Abstract. We introduce Lipschitz continuity of set-valued maps with respect to a given set difference. The existence of Lipschitz selections that pass through any point of the graph of the map and inherit its Lipschitz constant is studied. We show that the Lipschitz property of the set-valued map with respect to the Demyanov difference with a given constant is characterized by the same property of its generalized Steiner selections. For a univariate multifunction with only compact values in $\mathbb{R}^n$, we characterize its Lipschitz continuity in the Hausdorff metric (with respect to the metric difference) by the same property of its metric selections with the same constant.

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Key words: Lipschitz continuous set-valued maps, selections, generalized Steiner selection, metric selection, set differences, Demyanov metric, Demyanov difference, metric difference.

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1. Introduction. The question of existence of selections of set-valued maps that inherit regularity properties of these maps has been attracting the attention of researchers for a long time. The positive answers of this question may have essential impact on analysis and numerics in various fields using set-valued analysis (see e.g. [3, Chap. 9], [2, 1]). For instance, the question of existence of continuous selections passing through every point of the graph of a continuous set-valued function is well-known (see e.g. [3, Theorem 9.5.2]). In the case of a Lipschitz multifunction with compact convex values, there exists a Lipschitz selection through any point of the graph [3, Theorem 9.5.3], [15, 23, 1] with a Lipschitz constant depending on the dimension and the Lipschitz constant of the multifunction.

For a multifunction with only compact images (not necessarily convex), this question has in general a negative answer [2, Sec. 1.6], [15, Sec. 3]. But, for continuous mappings of one variable (univariate) having in addition bounded variation, the answer is positive [19, 15]. In particular, a Lipschitz mapping (with respect to the Hausdorff metric) defined on a compact interval has a Lipschitz selection with the same Lipschitz constant [19, Theorem 2], which may also pass through every point of its graph [15, 22], [17, Chap. 8].

Introducing a new general framework, we define various Lipschitz-type properties of set-valued functions using various subtraction operations on sets. Our approach is based on the representation of some distances in the space of compact (or convex compact) subsets of $\mathbb{R}^n$ by set differences. To be more specific, for any "good" notion of a difference of two sets, $A \ominus_\Delta B$, we can define a distance (or even a metric),

$$\mathrm{d}_\Delta(A,B) := \|A \ominus_\Delta B\| \tag{1}$$

with the common set norm $\|A\| = \sup_{a\in A} \|a\|$. The corresponding Lipschitz continuity of the map F is defined as

$$\|F(x) \ominus_\Delta F(y)\| \le L\|x-y\|.$$

In particular, the Hausdorff metric can be represented in (1) with the metric difference of sets [16], [17, Sec. 2.1]. Another example is the Demyanov metric in the set of convex compacts in $\mathbb{R}^n$ [14], which may be also expressed in the above way with the Demyanov difference [6].

In this paper we focus our attention on Lipschitz properties induced by various set differences. We review known notions of Lipschitz continuity and present them with known set differences. We also obtain new Lipschitz notions based on set differences. A main advantage of this approach is that the inclusion hierarchy between set differences or the inequality between their norms immediately implies the hierarchy of the corresponding Lipschitz conditions.

Special attention is given to Lipschitz conditions with respect to the metric difference (identical to Lipschitz condition in the Hausdorff metric) or with respect to the Demyanov difference of convex compacts in $\mathbb{R}^n$. It is shown that Lipschitz conditions on the set-valued functions with respect to metric or Demyanov difference of sets are equivalent to the same conditions satisfied uniformly by certain families of special selections. For the metric difference, this is the family of the so-called metric selections constructed initially by Hermes [19], [17, Sec. 8.1]. The selections corresponding to the Demyanov difference are the generalized Steiner selections (see [11, 6]).

The paper is organized as follows. In the next section we define some notions of set differences and discuss some axioms (basic properties) of such differences. In Section 3 various Lipschitz conditions with respect to given set differences are introduced and compared, and their properties are studied. Special cases and properties, arithmetic operations, as well as the hierarchy of these notions are studied. The characterization of D-Lipschitz mappings by their Lipschitzian generalized Steiner selections in Section 4 is followed by the corresponding characterization of Lipschitz univariate maps by uniform Lipschitzian metric selections in Section 5. In the last section a collection of examples is presented illustrating the hierarchy of different Lipschitz notions.

2. Set differences and their properties. We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of nonempty compact subsets of $\mathbb{R}^n$, and by $\mathcal{C}(\mathbb{R}^n)$ the set of nonempty convex compact subsets of $\mathbb{R}^n$. By $\|\cdot\|$ we denote some vector norm in $\mathbb{R}^n$ and by $\|\cdot\|_2$ the Euclidean norm, the spectral norm of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $\|M\|_2$, and for a set $A \in \mathcal{K}(\mathbb{R}^n)$ we denote $\|A\| := \sup\{\|a\| \,:\, a \in A\}$. By definition, $\|\emptyset\| = -\infty$. The notation $\|\cdot\|_\infty$ is used for the maximum norm in $\mathbb{R}^n$. The convex hull of the set A is denoted by $\mathrm{co}(A)$, $\overline{\mathrm{co}}(A)$ is the closed convex hull of A.

The *support function* for a set $A \in \mathcal{K}(\mathbb{R}^n)$ is defined as

$$\delta^*(l, A) := \max_{a\in A} \langle l, a\rangle \quad (l \in \mathbb{R}^n),$$

the *supporting face*

$$Y(l, A) := \{a \in A \,:\, \langle l, a\rangle = \delta^*(l, A)\}$$

is the set of maximizers (and the subdifferential of the support function). A supporting point (an element of the supporting face) is denoted by $y(l, A)$.

Recall that the *Hausdorff distance* between two sets in $\mathcal{K}(\mathbb{R}^n)$ is

$$\mathrm{d_H}(A,B) := \max\left\{\max_{a\in A}\operatorname{dist}(a,B),\ \max_{b\in B}\operatorname{dist}(b,A)\right\},$$

where the distance from a point $a \in \mathbb{R}^n$ to a set $B \in \mathcal{K}(\mathbb{R}^n)$ is defined as

$$\operatorname{dist}(a,B) := \min_{b\in B}\|a-b\|_2.$$

It is well-known that the spaces $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}^n)$ are complete metric spaces with respect to the Hausdorff metric [26, Theorem 1.8.2 and 1.8.5]. We will also use the *Demyanov distance* between the sets A, B, defined by

$$\mathrm{d_D}(A,B) := \sup\{\|y(l,A)-y(l,B)\|_2 \,:\, l \in T_A \cap T_B\}, \tag{2}$$

where $T_A \subset S_{n-1}$ is the set of full measure (in the unit sphere $S_{n-1} \subset \mathbb{R}^n$) such that the supporting face $Y(l,A)$ consists of a single point $y(l,A)$ for all $l \in T_A$ (see [14]).

Recall the notation of the *multiplication of a set by a scalar* and the *Minkowski sum* of sets:

$$\lambda A := \bigcup_{a\in A}\{\lambda a\} \quad (\lambda\in\mathbb{R}), \quad -A := (-1)\cdot A, \quad A+B := \bigcup_{\substack{a\in A\\ b\in B}}\{a+b\}$$

The *translation of a set A by a vector $b \in \mathbb{R}^n$* is denoted by $A+b := A+\{b\}$. We now recall the definitions of some known differences of compact, nonempty subsets of $\mathbb{R}^n$, all of them do not lead to a vector space.

Definition 2.1. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$. We define the*

(i) algebraic difference *as*

$$A \ominus_A B := A + (-1)\cdot B,$$

(ii) geometric/star-shaped/Hadwiger-Pontryagin difference [18, 25] *as*

$$A \ominus_G B := \{x \in \mathbb{R}^n \,:\, x + B \subset A\},$$

(iii) Demyanov difference [10, *Subsec.* III.1.5] *as*

$$A \ominus_D B := \overline{\operatorname{co}}\{y(l,A) - y(l,B) \,:\, l \in T_A \cap T_B\},$$

where $T_A, T_B \subset S_{n-1}$ are as in (2)*,* *(iv)* metric difference *of sets* [16] *and* [17, *Sec.* 2.1]

$$A \ominus_M B := \{a - b \,:\, \|a-b\|_2 = \operatorname{dist}(a,B) \ \mathit{or}\ \|b-a\|_2 = \operatorname{dist}(b,A)\}. \tag{3}$$

Let us note that all these differences are compact sets. The geometric difference can be empty, contrary to the other differences.

In the special case when $B = \{b\}$ is a singleton, all these differences coincide and are equal to the translated set $A - b := A + \{-b\}$.

The Demyanov difference is always convex. Since for $A, B \in \mathcal{K}(\mathbb{R}^n)$, $A \ominus_D B = \operatorname{co}(A) \ominus_D \operatorname{co}(B)$, we use this difference in practice for convex sets $A, B \in \mathcal{C}(\mathbb{R}^n)$.

Rewriting the algebraic difference and the geometric difference with the help of the translations of A as

$$A \ominus_A B = \bigcup_{b\in B} (A-b)\,, \qquad A \ominus_G B = \bigcap_{b\in B} (A-b)\,,$$

and with [5, proof of Lemma 3.17], we easily get the following inclusions between the above differences:

$$A \ominus_M B \subseteq A \ominus_A B, \tag{4}$$

$$A \ominus_G B \subseteq A \ominus_D B \subseteq A \ominus_A B, \tag{5}$$

$$\delta^*(l, A \ominus_G B) \le \delta^*(l, A) - \delta^*(l, B) \le \delta^*(l, A \ominus_D B) \quad (l \in S_{n-1}) \tag{6}$$

The following lemma does not provide any inclusion between the geometric difference and the metric one, but together with (4) it yields the norm inequalities

$$\|A \ominus_G B\|_2 \le \|A \ominus_M B\|_2 \le \|A \ominus_A B\|_2. \tag{7}$$

Lemma 2.2. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, then*

$$\|A \ominus_G B\|_2 \le \|A \ominus_M B\|_2 = \mathrm{d}_H(A,B). \tag{8}$$

P r o o f. If the geometric difference $A \ominus_G B$ is empty, then the norm equals $-\infty$ by convention and the inequality holds trivially.

Otherwise, let $\delta := \|A \ominus_G B\|_2$. Due to compactness, there is a vector $x \in A \ominus_G B \subset \mathbb{R}^n$ with $\|x\|_2 = \delta$ such that $x + B \subset A$. We now prove that there exist vectors $\widehat{a} \in A$, $\widehat{b} \in B$ with $\widehat{b} = \widehat{a} - x$,

$$\|\widehat{a} - \widehat{b}\|_2 = \delta = \operatorname{dist}(\widehat{a}, B).$$

Consider an element $\widehat{b} \in Y(x, B)$, the supporting face of B, and define the corresponding vector $\widehat{a} := \widehat{b} + x$. Clearly, $\widehat{a} \in A$, since $x + B \subset A$.

We show that $\delta = \|\widehat{a} - \widehat{b}\|_2 = \operatorname{dist}(\widehat{a}, B)$. Since $\widehat{b} \in Y(x, B)$, we have

$$\langle x, b \rangle \leq \delta^*(x, B) = \langle x, \widehat{b} \rangle \quad (b \in B)$$

and therefore $\langle b - \widehat{b}, x \rangle = \langle b - \widehat{b}, \widehat{a} - \widehat{b} \rangle \leq 0$. Thus, we arrive at the estimate

$$\begin{aligned} \|\widehat{a} - b\|_2^2 &= \|(\widehat{a} - \widehat{b}) + (\widehat{b} - b)\|_2^2 = \|\widehat{a} - \widehat{b}\|_2^2 + 2\langle \widehat{a} - \widehat{b}, \widehat{b} - b \rangle + \|\widehat{b} - b\|_2^2 \\ &\geq \|\widehat{a} - \widehat{b}\|_2^2 + \|\widehat{b} - b\|_2^2 \geq \|\widehat{a} - \widehat{b}\|_2^2 = \delta^2. \end{aligned}$$

Hence, $\widehat{b}$ is a projection of $\widehat{a}$ on B and $\widehat{a} - \widehat{b} \in A \ominus_M B$ so that

$$\|A \ominus_G B\|_2 = \|\widehat{a} - \widehat{b}\|_2 \leq \|A \ominus_M B\|_2. \qquad \square$$

Similarly, we can also establish a result analogous to (6) for the metric difference.

Lemma 2.3. *Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, then*

$$\delta^*(l, A \ominus_G B) \leq \delta^*(l, A) - \delta^*(l, B) \leq \delta^*(l, A \ominus_M B) \quad (l \in S_{n-1}), \tag{9}$$

$$\operatorname{co}(A \ominus_G B) \subset \operatorname{co}(A \ominus_M B). \tag{10}$$

P r o o f. The inclusion (10) follows from the inequality between the support functions in (9). To prove (9), consider an arbitrary direction $l \in S_{n-1}$.

The case that the geometric difference is empty yields the first inequality, since the support function equals $-\infty$ by convention. In this case it remains to prove only the right inequality which is done in step (ii).

(i) Let $A \ominus_G B$ be nonempty and choose $\widetilde{x} \in A \ominus_G B$ such that

$$\langle l, \widetilde{x} \rangle = \delta^*(l, A \ominus_G B).$$

Let us choose $\widetilde{b} \in Y(l, B)$ so that $\langle l, \widetilde{b} \rangle = \delta^*(l, B)$. Clearly, $\widetilde{x} + B \subset A$ so that $\widetilde{a} := \widetilde{x} + \widetilde{b} \in A$ and $\langle l, \widetilde{a} \rangle \leq \delta^*(l, A)$. Hence,

$$\delta^*(l, A \ominus_G B) = \langle l, \widetilde{x} \rangle = \langle l, \widetilde{a} \rangle - \langle l, \widetilde{b} \rangle \leq \delta^*(l, A) - \delta^*(l, B).$$

(ii) Now, take $\widehat{a} \in Y(l, A)$ such that $\langle l, \widehat{a} \rangle = \delta^*(l, A)$. Define $\widehat{b} \in B$ so that

$$\|\widehat{a} - \widehat{b}\|_2 = \operatorname{dist}(\widehat{a}, B).$$

Then, $\widehat{a} - \widehat{b} \in A \ominus_M B$ so that $\langle l, \widehat{a} - \widehat{b}\rangle \leq \delta^*(l, A \ominus_M B)$. Thus,

$$\delta^*(l, A) - \delta^*(l, B) \leq \langle l, \widehat{a}\rangle - \langle l, \widehat{b}\rangle = \langle l, \widehat{a} - \widehat{b}\rangle \leq \delta^*(l, A \ominus_M B). \qquad \square$$

There is no general result on an inclusion between the Demyanov and the metric difference, which is illustrated by the following example.

Example 2.4. Let $A = [3, 6] \subset \mathbb{R}$ and $B = [0, 1] \subset \mathbb{R}$. Then

$$A \ominus_D B = [3, 5] \subset A \ominus_M B = [2, 5],$$

while for $A = [3, 4] \cup [5, 6] \subset \mathbb{R}$ and $B = [0, 1] \subset \mathbb{R}$,

$$A \ominus_D B = [3, 5], \quad A \ominus_M B = [2, 3] \cup [4, 5].$$

The following example in $\mathbb{R}^2$ shows that even for convex sets the Demyanov difference does not have to be a subset of the metric one:

$$A = \{(x, y) : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}, \quad B = \mathrm{co}(\{(-1, 0), (1, 0)\})$$

Then,

$$A \ominus_M B = \mathrm{co}\left\{\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right)\right\} \cup \mathrm{co}\left\{\left(0, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)\right\} \subset A = A \ominus_D B.$$

Some of the properties of the set differences listed below, called here axioms, are used further in some proofs and allow to formulate the main conditions independent from a specific set difference. For compact sets $A, B, C, A_i, B_i \in \mathcal{K}(\mathbb{R}^n)$, $i = 1, 2$, the following axioms should hold:

(A1) $\quad A \ominus_\Delta B = \{0\} \iff A = B,$

(A2) $\quad \|B \ominus_\Delta A\| = \|A \ominus_\Delta B\|,$

(A3) $\quad \|A \ominus_\Delta B\| \leq \|A \ominus_\Delta C\| + \|C \ominus_\Delta B\|,$

(A4) $\quad \|(\alpha A) \ominus_\Delta (\alpha B)\| = |\alpha| \cdot \|A \ominus_\Delta B\| \quad (\alpha \in \mathbb{R}),$

(A5) $\quad \|(\alpha A) \ominus_\Delta (\beta A)\| \leq |\alpha - \beta| \cdot \|A\| \quad (\alpha \geq \beta \geq 0),$

(A6) $\quad \|(A_1 + A_2) \ominus_\Delta (B_1 + B_2)\| \leq \|A_1 \ominus_\Delta B_1\| + \|A_2 \ominus_\Delta B_2\|$

For all the above differences "$\ominus_\Delta$", except for the geometric one, converting the order of the sets A, B in (A2) leads to multiplying the difference by -1.

If the set difference "$\ominus_\Delta$" satisfies the axioms (A1)–(A3), then the definition

$$\mathrm{d}_\Delta(A,B) := \|A \ominus_\Delta B\| \quad (A,B \in \mathcal{X}) \tag{11}$$

gives a metric and this leads to the following lemma.

Lemma 2.5.
(i) The space $\mathcal{X} = \mathcal{K}(\mathbb{R}^n)$ is a metric space with the metric [16]

$$\mathrm{d}_H(A,B) = \|A \ominus_M B\|_2 \qquad (A,B \in \mathcal{K}(\mathbb{R}^n)). \tag{12}$$

The space $\mathcal{X} = \mathcal{C}(\mathbb{R}^n)$ is a metric space with the metric [14, *Sec.* 4], [6]

$$\mathrm{d}_D(C,D) = \|C \ominus_D D\|_2 \qquad (C,D \in \mathcal{C}(\mathbb{R}^n)), \tag{13}$$

since in both cases (A1)–(A3) *are satisfied.*

(ii) The metric difference also satisfies (A4)–(A5)*, the algebraic one satisfies* (A2)–(A4), (A6)*.*

(iii) The Demyanov difference satisfies the axioms (A2)–(A6) *in $\mathcal{K}(\mathbb{R}^n)$ with*

$$A \ominus_\Delta B = \{0\} \iff \mathrm{co}(A) = \mathrm{co}(B) \tag{14}$$

replacing (A1)*.*
(iv) The geometric difference satisfies the axioms (A4)–(A5) *for $\mathcal{X} = \mathcal{C}(\mathbb{R}^n)$.*

We further present properties which are stronger forms of some of the axioms listed above and indicate for which set difference they hold.

$$B \ominus_\Delta A = -(A \ominus_\Delta B), \tag{A2'}$$
$$A \ominus_\Delta B \subset (A \ominus_\Delta C) \ + (C \ominus_\Delta B), \tag{A3'}$$
$$(\alpha A) \ominus_\Delta (\alpha B) = \alpha(A \ominus_\Delta B) \quad (\alpha \geq 0), \tag{A4'}$$
$$(\alpha A) \ominus_\Delta (\beta A) = (\alpha - \beta)A \quad (\alpha \geq \beta \geq 0), \tag{A5'}$$
$$(A_1 + A_2) \ominus_\Delta (B_1 + B_2) \subset (A_1 \ominus_\Delta B_1) + (A_2 \ominus_\Delta B_2) \tag{A6'}$$

The Demyanov difference satisfies (A2')–(A6') on $\mathcal{K}(\mathbb{R}^n)$, for proving (A6') we use [14, Lemma 3.1]. (A2') and (A4') are satisfied by the metric and algebraic difference. The algebraic difference also satisfies (A3') and (A6'), the geometric one fulfills (A4') and (17), (A5') holds only in $\mathcal{C}(\mathbb{R}^n)$.

The following property holds for $\Delta \in \{\mathrm{M}, \mathrm{D}, \mathrm{A}\}$ and follows from (A2') and (A4'):

$$(\alpha A) \ominus_\Delta (\alpha B) = |\alpha|(B \ominus_\Delta A) \qquad (\alpha < 0) \tag{15}$$

We can also weaken axiom (A6) resp. (A6') by only considering translation of sets:

$$\|(A_1 + a_2) \ominus_\Delta (B_1 + b_2)\| \le \|A_1 \ominus_\Delta B_1\| + \|a_2 - b_2\|, \tag{16}$$

$$(A_1 + a_2) \ominus_\Delta (B_1 + b_2) = (A_1 \ominus_\Delta B_1) + (a_2 - b_2), \tag{17}$$

where $a_2, b_2 \in \mathbb{R}^n$. (16) holds for the metric difference, whereas the stronger form (17) is fulfilled by $\Delta \in \{G, D, A\}$,

Instead of (A4') the geometric difference satisfies

$$(\alpha A) \ominus_G (\alpha B) = \alpha(A \ominus_G B) \quad (\alpha < 0). \tag{18}$$

Furthermore, it satisfies (A6') with the opposite inclusion "$\supset$" (while the algebraic difference satisfies (A6') even as an equality).

Example 2.6. The metric difference satisfies the weaker axioms (A5) and (A6), but not the stronger forms (A5') and (A6').
To see this, consider in the special forms (16) resp. (17) of (A6) resp. (A6')

$$\begin{aligned} & A_1 = [1,2], \quad B_1 = [0,1], \quad a_2 = 0, \quad b_2 = 1, \\ & (A_1 + a_2) \ominus_M (B_1 + b_2) = [1,2] \ominus_M [1,2] = \{0\}, \\ & (A_1 \ominus_M B_1) + (a_2 - b_2) = [0,1] - 1 = [-1,0], \end{aligned}$$

so that equality in (17) does not hold. Nevertheless, the estimate (16) still holds, since

$$\begin{aligned} & \|(A_1 + a_2) \ominus_\Delta (B_1 + b_2)\|_2 = 0 \\ & \le 1 = \|[-1,0]\|_2 = \|(A_1 \ominus_M B_1) + (a_2 - b_2)\|_2. \end{aligned}$$

For the following choices

$$\begin{aligned} & A_1 = [1,2], \quad B_1 = [1,2], \quad a_2 = 0, \quad b_2 = -1, \\ & (A_1 + a_2) \ominus_M (B_1 + b_2) = [1,2] \ominus_M [0,1] = [0,1], \\ & (A_1 \ominus_M B_1) + (a_2 - b_2) = \{0\} + 1 = \{1\}, \end{aligned}$$

even the inclusion "$\subset$" in (17) is prevented.

We note that the remarkable property (A1), i.e.

$$A \ominus_M B = \{0\} \iff A = B \tag{19}$$

holds for the metric difference in $\mathcal{K}(\mathbb{R}^n)$, whereas for the Demyanov difference it holds in $\mathcal{C}(\mathbb{R}^n)$. For general compact sets we can only claim (A1'), i.e.

$$A \ominus_D B = \{0\} \iff \mathrm{co}(A) = \mathrm{co}(B).$$

For the geometric difference one can observe that $A \ominus_G B = \{0\}$ whenever $A \supseteq B$ and there is no other nonempty set C such that $B + C \subseteq A$.
For the algebraic difference, it is straightforward to see that only

$$A \ominus_A B = \{0\} \iff A = B = \{a\} \tag{20}$$

holds, i.e. both sets must be singletons.

Further, one can express the Hausdorff and the Demyanov metric in terms of metric and Demyanov difference respectively.

Remark 2.7. Although one cannot establish an inclusion relation between the Demyanov and the metric difference of two given sets, one can get an inequality between their norms. Namely, it is proved in [14, Lemma 4.1] and [24, Proposition 2.4.5] that for $A, B \in \mathcal{C}(\mathbb{R}^n)$, it holds

$$\mathrm{d_H}(A, B) \le \mathrm{d_D}(A, B) = \sup_{l \in S_{n-1}} \mathrm{d_H}(Y(l, A), Y(l, B)) \tag{21}$$

which implies that $\|A \ominus_M B\|_2 \le \|A \ominus_D B\|_2$. The topology induced by the Demyanov metric is stronger than the Hausdorff one (see [14, Example 3.1 and Sec. 4]).

Taking advantage of (12) or (13), we can express regularity notions of multifunctions with respect to the Hausdorff metric in $\mathcal{K}(\mathbb{R}^n)$ resp. the Demyanov metric in $\mathcal{C}(\mathbb{R}^n)$ in terms of the corresponding set differences. We discuss this in a general setting in the next section.

3. Regularity notions for multimaps through set differences. As remarked earlier in Lemma 2.5, the definition $\mathrm{d}_\Delta(A, B) := \|A \ominus_\Delta B\|$ for $\Delta \in \{M, D\}$ defines a metric space $\mathcal{X} = \mathcal{K}(\mathbb{R}^n)$ (resp. $\mathcal{X} = \mathcal{C}(\mathbb{R}^n)$).

Throughout the paper we consider a closed set $X \subset \mathbb{R}^m$ and set-valued maps $F : X \rightrightarrows \mathbb{R}^n$.

3.1. Lipschitz continuity.

Definition 3.1. *A set-valued function $F : X \to \mathcal{K}(\mathbb{R}^n)$ is called* Lipschitz *on X* with respect to the set difference "$\ominus_\Delta$" (*or shortly* Δ-Lipschitz) *with a constant $L \geq 0$ if*

$$\|F(x) \ominus_\Delta F(y)\| \leq L\|x - y\| \quad (\textit{for all } x, y \in X).$$

Therefore, in view of (12), a multifunction is *Lipschitz* (*in the Hausdorff metric*) resp. *satisfies the Lipschitz condition* (*LC*) iff it is Lipschitz with respect to the metric difference. Similarly, a multifunction $F : X \to \mathcal{C}(\mathbb{R}^n)$ is *Lipschitz in the Demyanov metric* (*D-Lipschitz*) iff it is Lipschitz with respect to the Demyanov difference resp. *Lipschitz with respect to geometric difference* (*G-Lipschitz*) iff the geometric difference is chosen for the set difference and we set $-\infty$ as the norm of the empty set.

In a similar way one can introduce continuity, modulus of continuity and the variation of multifunctions with respect to any given set difference by applying (11), e.g. with respect to the Demyanov differences of sets [6]. These notions with respect to the metric difference coincide with the classical notions of variation and moduli of continuity in the Hausdorff metric.

To demonstrate the power of regularity with respect to set differences, we prove the single-valuedness of a set-valued map at the points of continuity with respect to the algebraic difference (A-continuity), similarly to results on monotone maps in [27, 20].

Proposition 3.2. *Let $F : X \rightrightarrows \mathbb{R}^n$ be continuous (with respect to $\ominus_A$) at the point $x_0 \in X$ with nonempty images. Then, $F(\cdot)$ is single-valued at x_0.*

Proof. Due to the A-continuity in x_0, for $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}^n$ with $\|x - x_0\| \leq \delta$ it follows that

$$\|F(x) \ominus_A F(x_0)\| \leq \varepsilon.$$

Clearly, $\|F(x_0) \ominus_A F(x_0)\| = 0$ and the algebraic difference $F(x_0) \ominus_A F(x_0)$ equals the origin. Thus, by (20), $F(x_0)$ is a singleton. $\square$

Corollary 3.3. *If $F : X \rightrightarrows \mathbb{R}^n$ is Lipschitz with respect to the algebraic difference* (*A-Lipschitz*), *then $F(x) = \{f(x)\}$* (*i.e. $F(\cdot)$ is single-valued*) *and $f(\cdot)$ is Lipschitz.*
If $F(x) = \{f(x)\}$ and $f : X \to \mathbb{R}^n$ is Lipschitz, then $F(\cdot)$ is A-Lipschitz.

Remark 3.4. If one closely looks at the proof of the theorem that every monotone map is a.e. single-valued in [20], one can see that in fact it is proved there that $F(x) \ominus_A F(x) = \{0\}$ for a.e. x, which by (20) implies that $F(x)$ is a singleton for a.e. x.

Next we give examples of classes of set-valued maps which are D-Lipschitz (a stronger property than Lipschitz continuity with respect to the Hausdorff distance as we will see in Proposition 3.10).

Lemma 3.5. *Consider a convex, compact, nonempty set $U \subset \mathbb{R}^n$ and a Lipschitz function $r : X \to [0,\infty)$ with constant L.*
Then, $F(x) := r(x)U$ for $x \in X$ is D-Lipschitz with constant $L\|U\|_2$.

Proof. We rewrite the Demyanov metric by (21) as

$$\begin{aligned} &\mathrm{d_D}(F(x), F(y)) = \sup\nolimits_{l \in S_{n-1}} \mathrm{d_H}(Y(l, r(x)U), Y(l, r(y)U)) \\ &\le |r(x) - r(y)| \cdot \sup\nolimits_{l \in S_{n-1}} \|Y(l,U)\|_2 \le L \cdot \|U\|_2 \cdot \|x-y\|_2. \end{aligned}$$ □

Setting $r(x) = 1$ we get that constant set-valued maps are D-Lipschitz with constant 0. Another example class is given by moving ellipsoids.

Lemma 3.6. *Let $R : X \to \mathbb{R}^{n\times n}$ be a Lipschitz function such that uniform invertibility for the transposed matrices holds, i.e. there exists $\varepsilon > 0$ with*

$$\|R(x)^\top l\|_2 \ge \varepsilon \quad (x \in X,\ l \in S_{n-1}), \tag{22}$$

and let us set $U := PB_1(0)$ with an invertible matrix $P \in \mathbb{R}^{n\times n}$ and the closed unit ball $B_1(0) \subset \mathbb{R}^n$.
Then, $F(x) := R(x)U$ for $x \in X$ is D-Lipschitz with constant $L_\varepsilon := L\left(\frac{1}{\varepsilon^2}\right)$.

Proof. Using [6, Remark 2.1] we use the formulas

$$\begin{aligned} Y(l, R(x)U) &= \left\{ \frac{1}{\|P^\top R(x)^\top l\|_2} \cdot R(x)PP^\top R(x)^\top l \right\}, \\ M(x) &:= R(x)PP^\top R(x)^\top. \end{aligned}$$

Hence, the assumptions guarantee Lipschitz continuity of the function $x \mapsto y(l, R(x)U)$ uniformly in $l \in S_{n-1}$. Equation (21) yields

$$\begin{aligned} &\mathrm{d_D}(F(x), F(y)) = \sup_{l \in S_{n-1}} \mathrm{d_H}(Y(l, R(x)U), Y(l, R(y)U)) \\ &= \sup_{l \in S_{n-1}} \left\| \frac{1}{\|P^\top R(x)^\top l\|_2} M(x)l - \frac{1}{\|P^\top R(y)^\top l\|_2} M(y)l \right\|_2 \le L_\varepsilon \|x-y\|_2. \end{aligned}$$ □

Note that (22) holds e.g. for orthogonal matrices $R(x)$ with $\varepsilon = 1$.

3.2. Properties and hierarchy of Lipschitz maps. The properties in the next proposition are well-known for the case of Lipschitz maps in the Hausdorff metric.

Proposition 3.7. *Let $F_1, F_2 : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{K}(\mathbb{R}^n)$ be Lipschitz with respect to the set difference "$\ominus_\Delta$" and $\alpha \in \mathbb{R}$. We set $F(\cdot) = \alpha F_1(\cdot)$ and $G(\cdot) = F_1(\cdot) + F_2(\cdot)$.*
(i) If (A4) *holds for the set difference "$\ominus_\Delta$", then $F(\cdot)$ is Lipschitz with respect to the set difference "$\ominus_\Delta$" for $\alpha \geq 0$.*
(ii) If either (A2) *with* (A4) *or* (18) *holds for the set difference "$\ominus_\Delta$", then $F(\cdot)$ remains Lipschitz with respect to the set difference "$\ominus_\Delta$" even for $\alpha < 0$.*
(iii) If (A6) *holds for "$\ominus_\Delta$", then $G(\cdot) = F_1(\cdot) + F_2(\cdot)$ is Lipschitz with respect to the set difference "$\ominus_\Delta$".*

Proof. (i) For $\alpha \geq 0$

$$\|F(x) \ominus_\Delta F(y)\| \leq |\alpha| \cdot \|F_1(x) \ominus_\Delta F_1(y)\| \leq |\alpha| \cdot L \cdot \|x - y\|.$$

(ii) For $\alpha < 0$ and $\Delta \in \{G, D, A\}$ we have the same estimate due to (A2) and (A4) resp. (18), since

$$\|F(x) \ominus_\Delta F(y)\| \leq |\alpha| \cdot \max\{\|F_1(x) \ominus_\Delta F_1(y)\|, \|F_1(y) \ominus_\Delta F_1(x)\|\}.$$

(iii) The result for the sum follows from (A6):

$$\|G(x) \ominus_\Delta G(y)\| \leq \|F_1(x) \ominus_\Delta F_1(y)\| + \|F_2(x) \ominus_\Delta F_2(y)\| \qquad \square$$

Conditions (A2), (A4) resp. (18) hold for the geometric, the metric and the Demyanov difference, hence for G-Lipschitz, Lipschitz and D-Lipschitz maps, whereas condition (A6') or (A6) holds for the Demyanov and algebraic difference resp. metric one and not for the geometric difference (the opposite inclusion holds for the latter).

Let us note that since all differences in Definition 2.1 coincide for singletons, the Lipschitz property with respect to any of these differences coincides with the Lipschitz condition for single-valued $F(\cdot)$. Hence, there is no difference in the Lipschitz notions with respect to various differences for single-valued maps.

Proposition 3.8 (single-valued case)**.** *Let $f : X \to \mathbb{R}^n$ and set $F(x) := \{f(x)\}$. Then, the properties D-Lipschitz, Lipschitz and G-Lipschitz coincide with the usual Lipschitz condition for $f(\cdot)$.*

Proof. The claim follows from $F(x) \ominus_D F(y) = F(x) \ominus_A F(y)$ and

$$F(x) \ominus_G F(y) = F(x) \ominus_M F(y) = F(x) \ominus_D F(y) = \{f(x) - f(y)\},$$

$$\|F(x) \ominus_G F(y)\|_2 = \mathrm{d_H}(F(x), F(y)) = \mathrm{d_D}(F(x), F(y)) = \|f(x) - f(y)\|_2$$

which shows that D-Lipschitz and Lipschitz condition coincide in this case and the set-valued Lipschitz condition is equivalent to the pointwise case. □

In the 1d case several notions coincide.

Proposition 3.9 (1d case). *Let $F : I \to \mathcal{C}(\mathbb{R})$ be given with $I \subset \mathbb{R}$ closed. Then,*

(i) $F(\cdot)$ is G-Lipschitz with $F(s) \ominus_G F(t) \neq \emptyset$ for $s, t \in I$ if and only if $F(\cdot)$ is Lipschitz.

(ii) The properties D-Lipschitz and Lipschitz coincide.

(iii) If $F(\cdot)$ is given with $F(t) = [a(t), b(t)]$, then $F(\cdot)$ being Lipschitz is equivalent to $a(\cdot), b(\cdot)$ being both Lipschitz.

Proof. Consider $C = [c_1, c_2]$, $D = [d_1, d_2]$ with $C, D \in \mathcal{C}(\mathbb{R})$. Since the geometric and the Demyanov difference in $\mathcal{C}(\mathbb{R})$ are equal by

$$\begin{aligned} C \ominus_G D &= \{x \in I \,:\, \forall l = \pm 1 :\; l \cdot x \leq \delta^*(l, C) - \delta^*(l, D)\} \\ &= \{x \in I \,:\, -x \leq -c_1 + d_1,\; x \leq c_2 - d_2\} = [c_1 - d_1, c_2 - d_2] \\ &= \mathrm{co}\{y(-1, C) - y(-1, D), y(1, C) - y(1, D)\} = C \ominus_D D \end{aligned}$$

under the condition of nonemptiness in (i), we can apply this for $C = F(x)$, $D = F(y)$ and (6) yields

$$\mathrm{d_H}(F(x), F(y)) = \|F(x) \ominus_G F(y)\|_2 = \|F(x) \ominus_D F(y)\|_2 = \mathrm{d_D}(F(x), F(y)).$$

Hence, we have equality in (6) so that (i)–(ii) follow.
(iii) follows from (ii), since $y(1, F(t)) = b(t)$, $y(-1, F(t)) = a(t)$. □

The assumption in (i) that the geometric difference is never empty is quite restrictive and requires that the diameter of $F(\cdot)$ is constant. Example 6.1 shows such an example.

The next proposition generalizes [6, Sec. 5] to multivariate maps and shows the hierarchy of the Lipschitz notions.

Proposition 3.10 (hierarchy for Lipschitz maps). *Let $F : X \Rightarrow \mathbb{R}^n$ be a set-valued map with images in $\mathcal{K}(\mathbb{R}^n)$. Then, the following implications hold:*

D-Lipschitz $\Rightarrow$ Lipschitz $\Rightarrow$ G-Lipschitz

Proof. The left implication follows from (21), the right implication holds due to Lemma 2.2. □

4. Lipschitz generalized Steiner selections. We would like to adapt some results in [4, 6] about the representation and on selections of set-valued univariate maps to the multivariate case. Generalized Steiner selections are introduced and studied in [11, 12, 13]. They are defined for set-valued maps with convex images via generalized Steiner points which introduce a smooth measure in the original definition of the Steiner point. Thus, a *Castaing representation* of the set-valued map $F : X \rightrightarrows \mathbb{R}^n$ can be obtained in [12, Theorem 3.4] for $x \in X$, i.e.

$$F(x) = \overline{\bigcup_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(F(x))\}}, \tag{23}$$

where we define the *generalized Steiner (GS) selection* via the *generalized Steiner point* of the corresponding image of the set-valued map, i.e.

$$\mathrm{St}_\alpha(F(\cdot))(x) := \mathrm{St}_\alpha(F(x)).$$

Here as in [12], $\mathcal{SM}$ is the set of probability measures α with $\mathcal{C}^1(B_1(0))$-density functions. In [6] this representation result is extended to a set $\mathcal{AM}$ of atomic measures $\alpha_{[l]}$ which is concentrated in a single point $l \in S_{n-1}$ via

$$\mathrm{St}_{\alpha_{[l]}}(F(x)) := \mathrm{St}(Y(l, F(x))).$$

For abbreviation we denote $\mathcal{M}_{\mathrm{sp}}$ to be either $\mathcal{AM}$ or $\mathcal{SM}$. The representation (23) also holds for $\mathcal{AM}$ as

$$F(x) = \overline{\mathrm{co}} \Big\{ \bigcup_{l \in S_{n-1}} \{\mathrm{St}_{\alpha_{[l]}}(F(x))\} \Big\}. \tag{24}$$

We first discuss Lipschitz continuous selections of special type which inherit the Lipschitz continuity of the set-valued function in the Hausdorff metric. In [11, Theorem 3.6] it is proved that each GS-selection for smooth measures is Lipschitz with a varying Lipschitz constant depending on the measure provided that the set-valued map is Lipschitz. If we require that the mapping $F : X \to \mathcal{C}(\mathbb{R}^n)$ is even D-Lipschitz, the Lipschitz constant of all GS-selections will be uniformly bounded which is proved for univariate maps in [6, Proposition 5.1].

Proposition 4.1. *The set-valued map $F : X \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ is D-Lipschitz with a constant $L \geq 0$, if and only if the GS-selections $(\operatorname{St}_\alpha(F(\cdot)))_{\alpha \in \mathcal{M}_{sp}}$ are uniformly Lipschitz with the same constant, i.e.*

$$\sup_{\alpha \in \mathcal{M}_{sp}} \| \operatorname{St}_\alpha(F(x)) - \operatorname{St}_\alpha(F(y))\|_2 \leq L\|x - y\|_2 \quad (x, y \in X).$$

Proof. The assertion follows immediately from [6, Corollary 4.8], since

$$\mathrm{d_D}(F(x), F(y)) = \sup_{\alpha \in \mathcal{M}_{\mathrm{sp}}} \| \operatorname{St}_\alpha(F(x)) - \operatorname{St}_\alpha(F(y))\|_2. \qquad \square$$

5. Lipschitz metric selections. We recall the known result (see e.g. [9, Lemma 9.2]) that a Lipschitz univariate map $F : \mathbb{R}^1 \to \mathcal{K}(\mathbb{R}^n)$ has a family of selections, passing through every point of its graph, which are Lipschitz with the same Lipschitz constant as F. We give here a proof which is a modification of the proof of Hermes [19] for the existence of a Lipschitz selection of such a map (see [16] and [17, Sec. 8.1]). The constructed selections using the Arzelà-Ascoli theorem are called *metric selections.*

The GS selections from the previous section are uniformly Lipschitz only if the stronger condition of D-Lipschitz continuity of the set-valued map F is satisfied while the metric selections are uniformly Lipschitz whenever F is Lipschitz in the Hausdorff metric.

Let us recall the construction of *metric selections.*

Definition 5.1. *Let $F : [a, b] \to \mathcal{K}(\mathbb{R}^n)$. We take a uniform partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_N = b$ with $x_i = a + i(b - a)/N$, $i = 0, \ldots, N$. For a given (x, y), $x \in [x_k, x_{k+1}]$, $y \in F(x)$, we define y_k as a projection of y on $F(x_k)$, and then, starting from y_k, we find subsequently for any given y_i, $i \geq k$, a point y_{i+1} satisfying $\|y_{i+1} - y_i\|_2 = \operatorname{dist}(y_i, F(x_{i+1}))$. Similarly, starting backwards from y_k, we project for any $i \leq k$ the vector y_i onto $F(x_{i-1})$. Thus we construct a sequence of points $y_i \in F(x_i)$, $i = 0, \ldots, N$, such that for any $i = 0, \ldots, N - 1$,*

$$(25) \qquad \|y_{i+1} - y_i\|_2 = \operatorname{dist}(y_{i+1}, F(x_i)) \text{ or } \|y_{i+1} - y_i\|_2 = \operatorname{dist}(y_i, F(x_{i+1})).$$

A sequence $\{(x_i, y_i)\}_{i=0,\ldots,N}$ satisfying (25) *is called* metric chain. *Any piecewise-linear interpolant $y^N(x)$ of such points (x_i, y_i), $i = 0, N$, of a metric chain is called* metric piecewise-linear interpolant.

Remark 5.2. If $F : [a,b] \to \mathcal{K}(\mathbb{R}^n)$ is Lipschitz continuous (with respect to the Hausdorff metric), this piecewise linear function is also Lipschitz continuous with the same Lipschitz constant. In this case, by Arzelà-Ascoli theorem, the constructed sequence of functions $y^N(\cdot)$ has a uniformly convergent subsequence. Then any (pointwise) limit function of a convergent subsequence of metric piecewise-linear interpolants is a selection of F, and is called *metric selection*. Since F is Lipschitz, it is easily verified that the metric selections are also Lipschitz with the same Lipschitz constant as F (see e.g. [17, Sec. 8.1]).

One can formulate the following characterization of Lipschitz mappings $F : [a,b] \to \mathcal{K}(\mathbb{R}^n)$.

Proposition 5.3. *$F : \mathbb{R} \to \mathcal{K}(\mathbb{R}^n)$ is Lipschitz (in the Hausdorff metric) with constant μ if and only if all metric selections of F are uniformly Lipschitz with constant μ.*

Proof. We have sketched in Remark 5.2 the proof of one (the non-trivial) direction of the claim. The second direction is easier. Indeed, let all metric selections be Lipschitz with the constant L. We have to show that F is Lipschitz in the Hausdorff metric with the same constant. Take arbitrary $x', x'' \in [a,b]$, and $y' \in F(x')$, $y'' \in F(x'')$ such that $\|y' - y''\|_2 = \mathrm{d_H}(F(x'), F(x''))$, for instance let $\|y' - y''\|_2 = \mathrm{dist}(y', F(y''))$. There is a metric selection $y(x)$ passing through (x', y'), such that $y(x') = y'$ and $\|y(x') - y(x'')\|_2 \le L|x' - x''|$. Thus

$$\mathrm{d_H}(F(x'), F(x'')) = \|y(x') - y''\|_2 \le \|y(x') - y(x'')\|_2 \le L|x' - x''|,$$

which completes the proof. $\square$

Remark 5.4. In a similar way one can prove necessary and sufficient conditions for a set-valued map to be of bounded variation (in the Hausdorff metric) via the uniform bounded variation (with the same bound on the variation) of its metric selections. In this case, in the proof of the necessity (the non-trivial direction), one cannot use the Arzelà-Ascoli theorem, but the Helly's selection principle ([21, Chap. 10, Subsec. 36.5]). Results in this spirit can be found e.g. in Chistyakov [9, 7, 8].

6. Examples. In this section we present examples illustrating different notions of Lipschitz continuity as well as the obtained theorems on Lipschitz selections.

6.1. Examples for different Lipschitz notions.

Example 6.1. Set $F : \mathbb{R} \to \mathbb{R}$ as

$$F(t) = [\sin(t), \sin(t) + 1] \quad (t \in \mathbb{R}).$$

Then, $F(\cdot)$ is G-Lipschitz and Lipschitz, but not A-Lipschitz.

The Lipschitz property follows from Propositions 3.7, 3.8 and 3.9 for $F(t) = [0, 1] + \sin(t)$, since the sine function is Lipschitz. Although Proposition 3.9 (i) holds with

$$\begin{aligned} \operatorname{diam}(F(t)) &= (\sin(t) + 1) - \sin(t) = 1 \quad (t \in \mathbb{R}), \\ F(s) \ominus_G F(t) &= \{\sin(s) - \sin(t)\}, \end{aligned}$$

let us directly check the G-Lipschitz property:

$$\|F(s) \ominus_G F(t)\| = |\sin(s) - \sin(t)| \leq |s - t|$$

Hence, $F(\cdot)$ is G-Lipschitz with constant $L = 1$. Since the map is not everywhere single-valued, it cannot be A-Lipschitz (see Proposition 3.2).

We next state an example of a G-Lipschitz map which is not Lipschitz which shows that G-Lipschitz is a weaker assumption than Lipschitz continuity.

Example 6.2. Set $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ as

$$F(x) = \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ \sqrt{|x_1|} \end{pmatrix} \right\} \quad (x \in \mathbb{R}^2).$$

Then, $F(\cdot)$ is G-Lipschitz, but not Lipschitz.

(i) If we assume that $F(\cdot)$ is Lipschitz, there exists $L \geq 0$ forming the Lipschitz constant. The special choice

$$h_m = \frac{1}{m}, \ x^m = \begin{pmatrix} 2h_m \\ 0 \end{pmatrix}, \ y^m = \begin{pmatrix} h_m \\ 0 \end{pmatrix} \quad (m \in \mathbb{N})$$

yields

$$\operatorname{dist}\left(\begin{pmatrix} 2h_m \\ \sqrt{2}\sqrt{h_m} \end{pmatrix}, \operatorname{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} h_m \\ \sqrt{h_m} \end{pmatrix} \right\} \right) = \left\| \begin{pmatrix} 2h_m \\ \sqrt{2}\sqrt{h_m} \end{pmatrix} - \begin{pmatrix} h_m \\ \sqrt{h_m} \end{pmatrix} \right\|_2$$

$$= \left\| \begin{pmatrix} h_m \\ (\sqrt{2}-1)\sqrt{h_m} \end{pmatrix} \right\|_2 \le \mathrm{d_H}(F(x^m), F(y^m)) \le L\|x^m - y^m\|_2.$$

This leads to the contradiction

$$\sqrt{2} - 1 \le \sqrt{h_m + (\sqrt{2}-1)^2} \le L\sqrt{h_m}$$

for large $m \in \mathbb{N}$.

(ii) $F(\cdot)$ is G-Lipschitz with constant $L = 0$, since for $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$

$$F(x) \ominus_G F(y) = \begin{cases} \emptyset & \text{for } x, y \text{ with } x_1 \ne y_1, \\ \{0\} & \text{for } x, y \text{ with } x_1 = y_1, \end{cases}$$

$$\|F(x) \ominus_G F(y)\| = \left. \begin{cases} -\infty & \text{for } x, y \text{ with } x_1 \ne y_1 \\ 0 & \text{for } x, y \text{ with } x_1 = y_1 \end{cases} \right\} \le 0.$$

There exist Lipschitzian maps which are not D-Lipschitz, see [14, Example 3.1] and [6, Example 5.2].

6.2. Examples for Lipschitz selections. In [11, Theorem 3.6] it is shown that a Lipschitz set-valued map generates Lipschitz continuous generalized Steiner selection for smooth measures, but the Lipschitz constants of these selections are not uniformly bounded. The stronger requirement of D-Lipschitz continuity implies that the Lipschitz constants of the GS selections are the same as for the set-valued map. There is a Lipschitzian set-valued map $F : [a, b] \Rightarrow \mathbb{R}^2$ in [6, Example 5.2] which has even discontinuous generalized Steiner selections for an atomic measure. Obviously (in the view of Proposition 4.1), this set-valued map cannot be D-Lipschitz. The next example shows that the GS-selections corresponding to atomic measures need not be Lipschitz for a Lipschitz set-valued map (in the Hausdorff metric).

Example 6.3 ([14, Example 3.1])**.** Set $F : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F(x) = \mathrm{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(x_1) \\ \sin(x_1) \end{pmatrix} \right\} \quad (x \in \mathbb{R}^2).$$

We claim that $\mathrm{St}_{\alpha_{[l]}}(F(\cdot))$ is not Lipschitz for $l = \binom{1}{0}$ in $x = \frac{\pi}{2}\binom{1}{-1}$, it is even discontinuous while $F(\cdot)$ is Lipschitz.

From the example above we can calculate

$$\mathrm{St}_{\alpha_{[l]}}(F(x)) = \mathrm{St}(Y(l, F(x)))$$

$$= \begin{cases} \binom{\cos(x_1)}{\sin(x_1)}, & \text{if } l_1 \cos(x_1) + l_2 \sin(x_1) > 0, \\ \mathrm{St}(F(x)) = \frac{1}{2}\binom{\cos(x_1)}{\sin(x_1)}, & \text{if } l_1 \cos(x_1) + l_2 \sin(x_1) = 0, \\ \binom{0}{0}, & \text{else.} \end{cases}$$

(i) $\mathrm{St}_{\alpha_{[l]}}(F(\cdot))$ is not Lipschitz:
Let us consider the sequence $(x^m)_m$ with $x^m = (\frac{\pi}{2} - \frac{1}{m})\binom{1}{-1}$ for $m \in \mathbb{N}$ which converges to $x = \frac{\pi}{2}\binom{1}{-1}$.
For $l = \binom{1}{0}$ the above formula shows that

$$\mathrm{St}_{\alpha_{[l]}}(F(x^m)) = \begin{pmatrix} \cos(\frac{\pi}{2} - \frac{1}{m}) \\ \sin(\frac{\pi}{2} - \frac{1}{m}) \end{pmatrix} \underset{m\to\infty}{\longrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

But the value of the generalized Steiner selection for x does not coincide with this limit:

$$\mathrm{St}_{\alpha_{[l]}}(F(x)) = \frac{1}{2}\begin{pmatrix} \cos(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

As claimed the generalized Steiner selection is discontinuous, hence $F(\cdot)$ cannot be D-Lipschitz by Proposition 4.1.

Motivated by Lemma 3.5 we next give a D-Lipschitzian map which has uniform Lipschitz continuous GS-selections.

Example 6.4. Consider the set-valued map $F(t) = r(t)U$ for $t \in \mathbb{R}$ with $U = [-1, 1] \times \{1\}$ and $r : \mathbb{R} \to [0, \infty)$, e.g. $r(t) = |t|$, and $l = \binom{\cos(\phi)}{\sin(\phi)} \in S_1$. Then, the GS-selections are uniformly Lipschitz.

From Lemma 3.5 we know that $F(\cdot)$ is D-Lipschitz. By

$$\mathrm{St}_{\alpha_{[l]}}(U) = \begin{cases} \mathrm{St}(Y(l,U)) = \mathrm{St}(U) = \binom{0}{1} & (\phi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}), \\ y(l,U) = \binom{1}{1} & (\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})), \\ y(l,U) = \binom{-1}{1} & (\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})) \end{cases}$$

and the calculus rules in [6, Lemma 4.1] for generalized Steiner points we see that

$$\mathrm{St}_{\alpha_{[l]}}(F(t)) = r(t)\,\mathrm{St}_{\alpha_{[l]}}(U) = r(t)\,\mathrm{St}(Y(l,U)).$$

Hence, the generalized Steiner selections

$$\mathrm{St}_{\alpha_{[l]}}(F(t)) = \begin{cases} |t| \cdot \binom{0}{1} & (\phi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}), \\ |t| \cdot \binom{1}{1} & (\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})), \\ |t| \cdot \binom{-1}{1} & (\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})) \end{cases}$$

are different for various directions l, but uniformly Lipschitz with constant $\sqrt{2}$ which also follows from

$$\begin{aligned}&\| \operatorname{St}_{\alpha_{[l]}}(F(s)) - \operatorname{St}_{\alpha_{[l]}}(F(t))\|_2 = \|r(s)\operatorname{St}(Y(l,U)) - r(t)\operatorname{St}(Y(l,U))\|_2 \\ \leq\ & |\,|s| - |t|\,| \cdot \| \underbrace{\operatorname{St}(Y(l,U))}_{\in U} \|_2 \leq \|U\|_2 \cdot |s-t|.\end{aligned}$$

The following example is a slight variant of [17, Example 8.1.3]. One single metric selection which is not Lipschitz is enough to prevent a set-valued map from being Lipschitz as the following example demonstrates.

Example 6.5. Consider $F : [0,\infty) \rightrightarrows \mathbb{R}$ with images in $\mathcal{K}(\mathbb{R}^n)$ defined as

$$F(t) = \begin{cases} [-1, 1-\sqrt{1-t^2}] & \text{for } t \in [0,1], \\ [-1,1] & \text{for } t > 1. \end{cases}$$

$F(\cdot)$ is not Lipschitz and has metric selections which are Lipschitz and at least one which is not Lipschitz.

(i) There exists a metric selection which is not Lipschitz.
The metric selection $\eta(\cdot)$ passing through the point $(1,1)$ of the graph is not Lipschitz (compare Figure 1), since

$$\eta(t) = \begin{cases} 1-\sqrt{1-t^2} & \text{for } t \in [0,1], \\ 1 & \text{for } t > 1. \end{cases}$$

For $h_m = \frac{1}{m}$ we have

$$|\eta(1) - \eta(1-h_m)| = \left|1 - \left(1 - \sqrt{1-(1-h_m)^2}\right)\right| = \sqrt{2h_m - h_m^2}.$$

Assuming the Lipschitz continuity in $t=1$, this expression must be bounded by $L|1-(1-h_m)| = Lh_m$ which leads to the contradiction $\sqrt{2-h_m} \leq L\sqrt{h_m}$ for large $m \in \mathbb{N}$. Hence, $|\eta(1) - \eta(1-h_m)| \leq Lh_m$ cannot hold.

(ii) All other metric selections would be Lipschitz, but if they approach the point $(1,1)$ in the graph with constant first coordinate $t=1$, their Lipschitz constants will explode.

(iii) $F(\cdot)$ is not Lipschitz
This follows directly from Proposition 3.9 (iii).

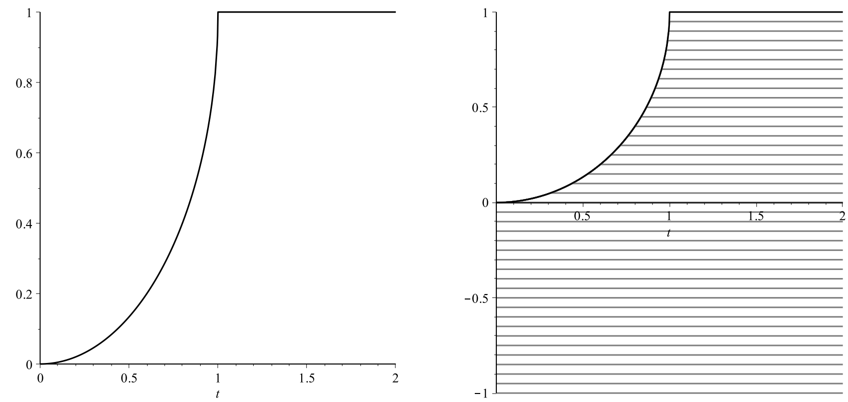

Fig. 1. Non-Lipschitz metric selection (left) and other metric selections (right) in Example 6.5

Only one metric selection of this map is non-Lipschitz (left picture in Figure 1), all the other metric selections are Lipschitz (right picture).

We consider a variant of Example 6.5 with a Lipschitz map that has uniform Lipschitz metric selections.

Example 6.6. Consider $F : [0,\infty) \Rightarrow \mathbb{R}$ with images in $\mathcal{K}(\mathbb{R}^n)$ defined as

$$F(t) = \begin{cases} [-1,t] & \text{for } t \in [0,1], \\ [-1,1] & \text{for } t > 1. \end{cases}$$

$F(\cdot)$ is Lipschitz and has metric selections which are uniformly Lipschitz.

(i) $F(\cdot)$ is Lipschitz
Obviously, $F(\cdot) = [a(t), b(t)]$ has Lipschitz functions $a(t) = -1$ and

$$b(t) = \begin{cases} t & \text{for } t \in [0,1], \\ 1 & \text{for } t > 1. \end{cases}$$

By Proposition 3.9 (iii) this assures the Lipschitz property (with constant 1).

(ii) all metric selections are uniformly Lipschitz
The metric selections $\eta_\alpha(\cdot)$ passing through the point $(0,\alpha)$ with $\alpha \in [-1,0]$ of the graph equal $\eta_\alpha(t) = \alpha$ (Lipschitz with constant 0). The ones passing through $(2,\beta)$ with $\beta \in (0,1]$ are

$$\eta^\beta(t) = \begin{cases} t & \text{for } t \in [0,\beta], \\ \beta & \text{for } t \geq \beta. \end{cases}$$

Hence, all selections are Lipschitz with uniform constant 1 (coinciding with the Lipschitz constant of $F(\cdot)$) which is guaranteed by Proposition 5.3.

All metric selections are Lipschitzian including the boundary selection (left picture in Figure 2). Other metric selections are depicted in the right picture.

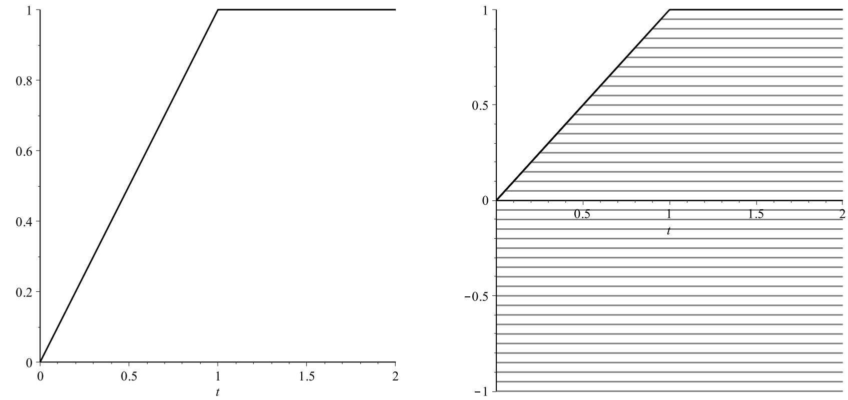

Fig. 2. Boundary metric selection (left) and other metric selections (right) in Example 6.6

Conclusions. In this paper we investigated the characterization of a set-valued Lipschitz map by uniformly Lipschitz selections in the two cases: for D-Lipschitz maps with convex images or for univariate Lipschitz (in the Hausdorff metric) maps with only compact images.

Part of our results may be easily extended to Hölder-continuous set-valued mappings (as Proposition 4.1). The case of metric selections is more complicated and requires more investigation.

As we already mentioned, in one dimension, the class of D-Lipschitz maps coincides with the class of Lipschitz convex-valued maps.

Generalized Steiner selections for the convex case give an interesting way to derive new selection results for set-valued maps. They are closely related to the Demyanov difference of sets. Uniformly Lipschitz GS selections provide a characterization of the class of D-Lipschitz set-valued maps. As we show in the second part of this paper, uniformly OSL generalized Steiner selections provide a characterization of the class of D-OSL mappings, for properly defined D-OSL condition for set-valued maps, with respect to the Demyanov difference.

Results for the rather weak notions G-Lipschitz and G-OSL set-valued maps remain a future task. The collection of examples presented here illustrates the established hierarchies and hopefully provide more insight in the various regularity classes for set-valued maps.

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Robert Baier
Chair of Applied Mathematics
University of Bayreuth
95440 Bayreuth, Germany
e-mail: `robert.baier@uni-bayreuth.de`

Elza Farkhi
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978, Israel
e-mail: `elza@post.tau.ac.il`

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