EMPIRICAL BAYES TWO-SIDED TEST
FOR THE PARAMETER OF LINEAR EXPONENTIAL DISTRIBUTION FOR RANDOM INDEX *

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Abstract. In the case of random index, the empirical Bayes two-side test rule for the parameter of linear exponential distribution is constructed. The asymptotically optimal property for the proposed EB test is obtained under suitable conditions. It is shown that the convergence rates of the proposed EB test rules can arbitrarily close to $O(n^{-\frac{1}{2}})$.

1. Introduction. Estimation and test of empirical Bayes (EB) have been investigated in many papers, in the particular for the exponential and scale exponential family, the readers are referred to literature (see [2, 4, 5, 6, 8]). Recently, Empirical Bayes test for the parameter of linear exponential distribution have been discussed [9, 10]. Usually, EB approaches are concerned with non-random index of size of the historical samples. In fact, we may meet the problem

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that size of the historical samples is a random index which takes positive integer valued random variable. Study on limit theory with random index has been acquired some results [11]. Up to now, Empirical Bayes test problem for the parameter of distribution family for random index hasn’t been studied. Differing from the previous many study, in the case of random index, we will renew to construct empirical Bayes test rule for the parameter of linear exponential distribution is constructed.

Let \( X \) have a conditional density function for given \( \theta \)

\[
(1.1) \quad f(x|\theta) = (\mu x + \theta) \exp \left\{ -\theta x - \frac{1}{2} \mu x^2 \right\},
\]

where \( \mu \) is a known constant and \( \theta \) is an unknown parameter. Denote sample space \( \Omega = \{ x \mid x > 0 \} \) and parameter space \( \Theta = \left\{ \theta > 0 \mid \int_{\Omega} f(x|\theta)dx = 1 \right\} \).

In this paper, we discuss the following two-sided test problem

\[
(1.2) \quad H_0 : \theta_1 \leq \theta \leq \theta_2 \Leftrightarrow H_1 : \theta < \theta_1 \text{ or } \theta > \theta_2,
\]

where \( \theta_1 \) and \( \theta_2 \) are given positive constant, taking \( \theta_0 = \frac{\theta_1 + \theta_2}{2} \) and \( \gamma_0 = \frac{\theta_2 - \theta_1}{2} \), then two-sided test problem of (1.2) is equivalent with

\[
(1.3) \quad H^*_0 : |\theta - \theta_0| \leq \gamma_0 \Leftrightarrow H^*_1 : |\theta - \theta_0| > \gamma_0.
\]

For hypothesis test problem (1.3), we have loss function

\[
L_i(\theta, d_i) = (1-i)a[(\theta-\theta_0)^2 - \gamma_0^2]I[|\theta-\theta_0| > \gamma_0] + ia[\gamma_0^2 - (\theta-\theta_0)^2]I[|\theta-\theta_0| \leq \gamma_0], \quad i = 0, 1
\]

where \( a > 0, \ d = \{d_0, d_1\} \) is action space, \( d_0 \) and \( d_1 \) imply acceptance and rejection of \( H^*_0 \).

Assume that the prior distribution \( G(\theta) \) of \( \theta \) is unknown, we obtain randomized decision function

\[
(1.4) \quad \delta(x) = P(\text{accept } H^*_0 \mid X = x).
\]

Then, the Bayes risk function of \( \delta(x) \) is shown by

\[
(1.5) \quad R(\delta(x), G(\theta)) = \int G(\theta) \left[ \int \left[ L_0(\theta, d_0)f(x|\theta)\delta(x) + L_1(\theta, d_1)f(x|\theta)(1 - \delta(x)) \right]dx \right]dG(\theta)
\]

\[
= a \int \beta(x)\delta(x)dx + C_G,
\]
Empirical Bayes two-sided test . . .

(1.6) \[ C_G = \int_\Theta L_1(\theta, d_1) dG(\theta), \beta(x) = \int_\Theta \left[ (\theta - \theta_0)^2 - \gamma_0^2 \right] f(x|\theta) dG(\theta). \]

The marginal density function of \( X \) is given by

\[ f_G(x) = \int_\Theta f(x|\theta) dG(\theta) = \int_\Theta (\mu x + \theta) \exp\left(-\theta x - \frac{1}{2} \mu x^2 \right) dG(\theta). \]

Let

\[ p_G(x) = \int_\Theta \exp\left(-\theta x - \frac{1}{2} \mu x^2 \right) dG(\theta). \]

Hence, \( p_G^{(1)}(x) = -\int_\Theta (\mu x + \theta) \exp\left(-\theta x - \frac{1}{2} \mu x^2 \right) dG(\theta) = -f_G(x) \), where \( p_G^{(1)}(x) \) is derivative of \( p_G(x) \), then

\[ \int_x^\infty f_G(x) dx = p_G(x). \]

By (1.6) and simple calculation, we have

\[ \beta(x) = f_G^{(2)}(x) + Q(x) f_G^{(1)}(x) - \mu Q(x) p_G(x) + S(x) f_G(x), \]

where \( Q(x) = 2\mu x + 2\theta_0, S(x) = \mu^2 x^2 + 2\mu \theta_0 x + 3\mu + \theta_0^2 - \gamma_0^2 \), and \( f_G^{(1)}(x) \) and \( f_G^{(2)}(x) \) are first and second order derivative of \( f_G(x) \).

Using (1.5), Bayes test function is obtained as follows

\[ \delta_G(x) = \begin{cases} 1, & \beta(x) \leq 0, \\ 0, & \beta(x) > 0. \end{cases} \]

Further, we can get minimum Bayes risk

\[ R(G) = \inf_\delta R(\delta, G) = R(\delta_G, G) = a \int_{\Omega} \beta(x) \delta_G(x) dx + C_G. \]

When the prior distribution of \( G(\theta) \) is known and \( \delta(x) = \delta_G(x) \), \( R(G) \) is achieved. However, where \( G(\theta) \) is unknown, so \( \delta_G(x) \) can not be made use of, we need to introduce EB method.

The rest of this paper is structured as follows. Section 2 presents an EB test. In section 3, we obtain asymptotic optimality and the optimal rate of convergence of the EB test.
2. Construction of EB test function for random index. Under the following condition, we need to construct EB test function. We make the following assumptions: let $(X_1, \theta_1), \ldots, (X_{\tau_n}, \theta_{\tau_n})$ and $(X, \theta)$ be independent random vectors, the $\theta_i$ $(i = 1, \ldots, \tau_n)$ and $\theta$ are independently identically distributed (i.i.d.) and have the common prior distribution $G(\theta)$. Let $X_1, X_2, \ldots, X_{\tau_n}, X$ be mutually independent random variable sequence with the common marginal density function $f_G(x)$, where $X_1, X_2, \ldots, X_{\tau_n}$ are historical samples and $X$ is present sample and $\tau_n$ is a discrete random index which takes positive integer values with known distribution. Assume $f_G(x) \in C_{s,\alpha}, x \in \mathbb{R}$, where $C_{s,\alpha} = \{g(x) \mid g(x)$ is probability density function and has continuous $s$-th order derivative $g^{(s)}(x)$ with $|g^{(s)}(x)| \leq \alpha, s \geq 3, \alpha$ and $s$ are natural numbers}. First construct estimator of $\beta(x)$.

Let $K_r(x)$ $(r = 0, 1, \ldots, s - 1)$ be a Borel measurable real function vanishing off $(0, 1)$ such that

\begin{equation}
(A1) \quad 1 \int_0^1 v^t K_r(v)dv = \begin{cases} (-1)^t, & t = r, \\ 0, & t \neq r, t = 0, 1, \ldots, s - 1. \end{cases}
\end{equation}

Denote $f_G^{(0)}(x) = f_G(x)$, $f_G^{(r)}(x)$ is the $r$ order derivative of $f_G(x)$ $(r = 0, 1, \ldots, s - 1)$. Similar to Prakasa [7], kernel estimation of $f_G^{(r)}(x)$ is defined by

\begin{equation}
(2.1) \quad f_G^{(r)}(x) = \frac{1}{\tau_n h_n^{(1+r)}} \sum_{j=1}^{\tau_n} K_r \left( \frac{x - X_j}{h_n} \right),
\end{equation}

where $h_n$ is smoothing bandwidth and $\lim_{n \to \infty} h_n = 0$.

As $p_G(x) = \int_x^{\infty} f_G(x)dx = E\{I(X_i > x)\}$, hence, $p_G(x)$ is defined as follows

\begin{equation}
(2.2) \quad p_G(x) = \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} I(X_i > x).
\end{equation}

Thus, estimator of $\beta(x)$ is obtained by

\begin{equation}
(2.3) \quad \beta_{\tau_n}(x) = f_G^{(2)}(x) + Q(x)f_G^{(1)}(x) - \mu Q(x)p_{\tau_n}(x) + S(x)f_{\tau_n}(x).
\end{equation}

Hence, EB test function is defined by

\begin{equation}
(2.4) \quad \delta_{\tau_n}(x) = \begin{cases} 1, & \beta_{\tau_n}(x) \leq 0, \\ 0, & \beta_{\tau_n}(x) > 0. \end{cases}
\end{equation}
Let $E$ denote the mathematical expectation with respect to the joint distribution of $X_1, X_2, \ldots, X_{\tau_n}$, we get overall Bayes risk of $\delta_{\tau_n}(x)$

\begin{equation}
R(\delta_{\tau_n}(x), G) = a \int_{\Omega} \beta(x)E[\delta_{\tau_n}(x)]dx + C_G.
\end{equation}

If \( \lim_{n \to \infty} R(\delta_{\tau_n}, G) = R(\delta_{G}, G) \), \( \{\delta_{\tau_n}(x)\} \) is asymptotic optimality of EB test function; if \( R(\delta_{\tau_n}, G) - R(\delta_{G}, G) = O(n^{-q}) \), \( q > 0 \), \( O(n^{-q}) \) is asymptotic optimality convergence rates of EB test function of \( \{\delta_{\tau_n}(x)\} \). Before proving the theorems, we give a series of lemmas.

Let $c, c_1, c_2, c_3, \ldots, c_8$ be different constants in different cases even in the same expression.

**Lemma 2.1** (Lu, [3]). Let \( \{X_i, i \geq 1\} \) be independent identical distribution random samples, with \( EX_i = 0 \) and \( E|X_i|^r < \infty \), \( r \geq 2 \), then

\[ E|\sum_{i=1}^{n} X_i|^r \leq cn^r E|X_i|^r. \]

**Lemma 2.2.** $f_{\tau_n}^{(r)}(x)$ is defined by (2.1). Let $X_1, X_2, \ldots, X_{\tau_n}$ be independent identically distributed random samples. Assume (A1) holds, $\forall x \in \Omega$,

1. If $f_{\tau_n}^{(r)}(x)$ is continuous function, \( \lim_{n \to \infty} h_n = 0 \) and \( \lim_{n \to \infty} \frac{1}{h_n^{2+s}} E\left( \frac{1}{\tau_n} \right) = 0 \), we have
   \[ \lim_{n \to \infty} E|f_{\tau_n}^{(r)}(x) - f_{G}^{(r)}(x)|^2 = 0. \]

2. If $f_G(x) \in C_{s, \alpha}$, $h_n = n^{-\frac{1}{s+\alpha}}$, $E\left( \frac{1}{\tau_n} \right) = o(n^{-\gamma})$, where $\gamma = \frac{s - 1}{s + 1}$, \( s \geq 2 \), for $0 < \lambda \leq 1$, we get
   \[ E|f_{\tau_n}^{(r)}(x) - f_{G}^{(r)}(x)|^{2\lambda} \leq c \cdot n^{-\frac{\lambda(s-\lambda)}{1+s}}. \]

**Proof.** Proof of (1):

\begin{equation}
E|f_{\tau_n}^{(r)}(x) - f_{G}^{(r)}(x)|^2 \leq 2|Ef_{\tau_n}^{(r)}(x) - f_{G}^{(r)}(x)|^2 + 2Var(f_{\tau_n}^{(r)}(x)) := 2(I_1^2 + I_2),
\end{equation}
where

\[
Ef_{\tau_n}^{(r)}(x) = E \left\{ E \left[ \frac{1}{mh_n^{1+r}} \sum_{i=1}^{m} K_r \left( \frac{x-X_i}{h_n} \right) | \tau_n = m \right] \right\} \\
= E \left\{ \frac{1}{h_n^{1+r}} E \left[ K_r \left( \frac{x-X_i}{h_n} \right) | \tau_n = m \right] \right\} \\
= h_n^{-(1+r)} E \left[ K_r \left( \frac{x-X_i}{h_n} \right) \right] \\
= h_n^{-(1+r)} \int_{0}^{\infty} K_r \left( \frac{x-y}{h_n} \right) f_G(y) dy \\
= h_n^{-r} \int_{0}^{1} K_r(u) f_G(x-h_nu) du.
\]

We obtain the following Taylor expansion of \( f_G(x-h_nu) \) in \( x \)

\[
f_G(x-h_nu) - f_G(x) = \frac{f_G'(x)}{1!} (-h_nu) + \frac{f_G''(x)}{2!} (-h_nu)^2 + \cdots + \frac{f_G^{(s)}(x)}{s!} (-h_nu)^s,
\]

where \( 0 < \xi < 1 \).

Since \( f_G(x) \) is continuous in \( x \) and \((A1)\), it is easy to see that

\[
0 \leq \lim_{n \to \infty} |Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)| = \lim_{n \to \infty} \left| \frac{1}{h_n^{1+r}} \int_{0}^{1} K_r(u) f_G(x-h_nu) du - f_G^{(r)}(x) \right|
\leq \frac{1}{r!} \int_{0}^{1} u^r |K_r(u)| \lim_{n \to \infty} \left| f_G^{(r)}(x-\xi h_nu) - f_G^{(r)}(x) \right| du = 0,
\]

we have

\[
(2.7) \quad \lim_{n \to \infty} I_1^2 = \lim_{n \to \infty} \left| Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x) \right|^2 = 0.
\]
by Lemma 2.1, we get

\[ I_2 = \frac{1}{h_n^{2r+2}} D \left[ \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} K_r \left( \frac{x - X_i}{h_n} \right) \right] \]

\[ = \frac{1}{h_n^{2r+2}} E \left[ \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} \left( K_r \left( \frac{x - X_i}{h_n} \right) - EK_r \left( \frac{x - X_i}{h_n} \right) \right)^2 | \tau_n = m \right] \]

\[ \leq c_1 \frac{1}{h_n^{2r+2}} E \left( \frac{1}{\tau_n} \right). \]

Then, when \( \frac{1}{h_n^{2r+2}} E \left( \frac{1}{\tau_n} \right) \to 0 \), we have

\[ \lim_{n \to \infty} J_2 = 0. \]

Substituting (2.7) and (2.9) into (2.6), the proof of (1) is completed.

Proof of (2): Similar to (2.6), we can show that

\[ E \left| f^{(r)}_{\tau_n}(x) - f^{(r)}_G(x) \right|^{2\lambda} \leq 2[E f^{(r)}_{\tau_n}(x) - f^{(r)}_G(x)]^{2\lambda} + 2[Var f^{(r)}_{\tau_n}(x)]^{\lambda} := 2(J_{1,2}^{2\lambda} + J_{2,2}^{\lambda}). \]

We obtain the following Taylor expansion of \( f_G(x - h_nu) \) in \( x \)

\[ f_G(x - h_nu) = f_G(x) + \frac{f'_G(x)}{1!}(-h_nu) + \frac{f''_G(x)}{2!}(-h_nu)^2 + \cdots + \frac{f^{(r)}_G(x - \xi h_nu)}{r!}(-h_nu)^r, \]

where \( 0 < \xi < 1 \).

Since (A1) and \( f_G(x) \in C_{s,\alpha} \), we have

\[ |Ef^{(r)}_{\tau_n}(x) - f^{(r)}_G(x)| \leq \int_0^1 |K_r(v)h_n^{s-r}v^s| \left| \frac{f^{(s)}_G(x - \xi h_nu)}{s!} \right| dv \leq c \cdot h_n^{s-r}. \]

when \( h_n = n^{-2r+2} \), we get

\[ J_{1,2}^{2\lambda} = |Ef^{(r)}_{\tau_n}(x) - f^{(r)}_G(x)|^{2\lambda} \leq c \cdot n^{-\frac{s(\lambda+1)}{2\lambda(r+1)}}. \]
By (2.7), when \( h_n = n^{-\frac{1}{s+2}} \), \( E \left( \frac{1}{\tau_n} \right) = o(n^{-\gamma}) \), where \( \gamma = \frac{s-1}{s+1} \), we have

\[
J_2^\lambda \leq c_1 \left( (h_n^{2r+2})^{-1} \right)^\lambda \left[ E \left( \frac{1}{\tau_n} \right) \right]^\lambda \leq c \cdot n^{-\frac{\lambda(s-r)}{s+1}}.
\]

Substituting (2.11) and (2.12) into (2.9), obviously, proof of (2) is completed. □

**Lemma 2.3** (Van, [2]). \( R(\delta_G, G) \) and \( R(\delta_{\tau_n}, G) \) are defined by (1.9) and (2.4), then

\[
0 \leq R(\delta_{\tau_n}, G) - R(\delta_G, G) \leq a \int_{\Omega} |\beta(x)| \mathbb{P}(|\beta_{\tau_n}(x) - \beta(x)| \geq |\beta(x)|) dx.
\]

**Lemma 2.4.** \( p_G(x) \) and \( p_{\tau_n}(x) \) are defined by (1.7) and (2.2). Let \( X_1, X_2, \ldots, X_{\tau_n} \) be independent identical random samples, then, for \( 0 < \lambda \leq 1 \),

\[
E \left[ \left( \frac{1}{\tau_n} \right)^\lambda \right] = O(n^{-\lambda}), \text{ we have}
\]

\[
E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \leq cn^{-\lambda}.
\]

**Proof.** Since

\[
E p_{\tau_n}(x) = E \left\{ I(x_1 > x) \right\} = \int_x^\infty \frac{f_G(y)}{m(y)} dy = \int_x^\infty \frac{p(y)}{m(y)} dy = p_G(x),
\]

we get \( \phi_{\tau_n} \) is an unbiased estimator of \( p_G(x) \).

Applying moment monotone inequality, we have

\[
\left( E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \right)^{\frac{1}{2\lambda}} \leq \left( E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \right)^{\frac{1}{2}}.
\]

That is to say

\[
E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \leq \left( E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \right)^{\frac{1}{\lambda}} := J.
\]

By Lemma 2.1, we can easily get

\[
J = E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} \leq E \left[ E |p_{\tau_n}(x) - p_G(x)|^{2\lambda} | \tau_n = m \right]
\]

\[
\leq cE \left[ \left( \frac{1}{\tau_n} \right)^\lambda \right] \leq cn^{-\lambda}.
\]

The proof of Lemma 2.4 is completed. □
3. Asymptotic optimality and convergence rates.

**Theorem 3.1.** \( \delta_{\tau_n}(x) \) is defined by (2.4). Let \( X_1, X_2, \ldots, X_{\tau_n} \) be independent identical random sample. Assume (A1) and the following regularity conditions hold.

1. \( \lim_{n \to \infty} h_n = 0 \) and \( \lim_{n \to \infty} \frac{1}{n} E \left( \frac{1}{\tau_n} \right) = 0 \),

2. \( \int \theta^2 dG(\theta) < \infty \),

3. \( f^{(2)}_G(x) \) is continuous function of \( x \),

we get

\[
\lim_{n \to \infty} R(\delta_{\tau_n}, G) = R(\delta_G, G).
\]

**Proof.** By Lemma 2.2, we have

\[
0 \leq R(\delta_{\tau_n}, G) - R(\delta_G, G) \leq a \int |\beta(x)|p(|\beta_{\tau_n}(x) - \beta(x)| \geq |\beta(x)|)dx.
\]

Witting \( \Psi_{\tau_n}(x) = |\beta(x)|p(|\beta_{\tau_n}(x) - \beta(x)| \geq |\beta(x)|) \). Hence, \( \Psi_{\tau_n}(x) \leq |\beta(x)| \).

Again by (1.6) and Fubini theorem, we can get

\[
\int_{\Omega} |\beta(x)|dx \leq \theta_0^2 - \gamma_0^2 + \int_{\Theta} \theta^2 dG(\theta) + 2|\theta_0| \int_{\Theta} \theta dG(\theta) < \infty.
\]

Applying domain convergence theorem, then

\[
0 \leq \lim_{n \to \infty} R(\delta_{\tau_n}, G) - R(\delta_G, G) \leq \int_{\Omega} \lim_{n \to \infty} \Psi_{\tau_n}(x)dx,
\]

If Theorem 3.1 holds, we only need to prove \( \lim_{n \to \infty} \Psi_{\tau_n}(x) = 0 \) a.s.x,

By Markov’s and Jensen’s inequations, then

\[
\Psi_{\tau_n}(x) \leq \left[ E \left| f^{(2)}_{\tau_n}(x) - f^{(2)}_G(x) \right|^2 \right]^{\frac{1}{2}} + |Q(x)| \left[ E \left| f^{(1)}_{\tau_n}(x) - f^{(1)}_G(x) \right|^2 \right]^{\frac{1}{2}} + \mu |Q(x)\left[ E \left| p_{\tau_n}(x) - p_G(x) \right|^2 \right]^{\frac{1}{2}} + |S(x)| \left[ E \left| f_{\tau_n}(x) - f_G(x) \right|^2 \right]^{\frac{1}{2}}.
\]

Again by Lemma 2.1 (1) and Lemma 2.4, for fixed \( x \in \Omega \), when \( r = 0, 1, 2 \).
and $\lambda = 1$, we get

$$0 \leq \lim_{n \to \infty} \Psi_{\tau_n}(x)$$

$$\leq \left[ \lim_{n \to \infty} E|f_{\tau_n}^{(2)}(x) - f_G^{(2)}(x)|^2 \right]^{\frac{1}{2}}$$

$$+ |Q(x)| \left[ \lim_{n \to \infty} E|f_{\tau_n}^{(1)}(x) - f_G^{(1)}(x)|^2 \right]^{\frac{1}{2}}$$

$$+ \mu|Q(x)| \left[ \lim_{n \to \infty} E|p_n(x) - p_G(x)|^2 \right]^{\frac{1}{2}}$$

$$+ |S(x)| \left[ \lim_{n \to \infty} E|f_{\tau_n}(x) - f_G(x)|^2 \right]^{\frac{1}{2}} = 0.$$ (3.2)

Substituting (3.2) into (3.1), proof of Theorem 3.1 is completed. □

**Theorem 3.2.** $\delta_{\tau_n}(x)$ is defined by (2.4). Let $X_1, X_2, \ldots, X_n$ be independent identical random samples. Assume (A1) and the following regularity conditions hold.

1. $f_G(x) \in C_{s,\alpha}$, where $s \geq 3$,
2. $h_n = n^{-\frac{s}{s+2}}, E(\frac{1}{\tau_n}) = o(n^{-\gamma})$, where $\gamma = \frac{s-1}{s+1}$,
3. for $0 < \lambda \leq 1$ and $m = 0, 1, 2$, $\int_{\Omega} x^m |\beta(x)|^{1-\lambda} dx < \infty$.

we have

$$R(\delta_{\tau_n}, G) - R(\delta_G, G) = O \left( n^{-\frac{\lambda(s-1)}{2(s+2)}} \right).$$
Proof. By Lemma 2.2 and Markov’s inequations, then

\[
0 \leq R(\delta_{\tau_n}, G) - R(\delta_G, G) \leq c_1 \int_{\Omega} |\beta(x)|^{1-\lambda} E|f^{(2)}_{\tau_n}(x) - f^{(2)}_{G}(x)|^\lambda dx
+ c_2 \int_{\Omega} |\beta(x)|^{1-\lambda} |Q(x)|^\lambda E|f^{(2)}_{\tau_n}(x) - f^{(2)}_{G}(x)|^\lambda dx
+ c_3 \int_{\Omega} |\beta(x)|^{1-\lambda} |Q(x)|^\lambda E|p_{\tau_n}(x) - p_G(x)|^\lambda dx
+ c_4 \int_{\Omega} |\beta(x)|^{1-\lambda} |S(x)|^\lambda E|f_{\tau_n}(x) - f_G(x)|^\lambda dx
= A_n + B_n + C_n + D_n.
\]

By Lemma 2.2 (2) and condition (6), we get

\[
A_n \leq c_5 n^{-\frac{(s-2)}{2(s+1)}} \int_{\Omega} |\beta(x)|^{1-\lambda} dx \leq c_5 n^{-\frac{(s-2)}{2(s+1)}}.
\]

\[
B_n \leq c_6 n^{-\frac{(s-2)}{2(s+2)}} \int_{\Omega} |\beta(x)|^{1-\lambda} |Q(x)|^\lambda dx \leq c_6 n^{-\frac{(s-2)}{2(s+1)}}.
\]

\[
D_n \leq c_7 n^{-\frac{(s-2)}{2(s+1)}} \int_{\Omega} |\beta(x)|^{1-\lambda} |S(x)|^\lambda dx \leq c_7 n^{-\frac{(s-2)}{2(s+1)}}.
\]

Using Lemma 2.4 and condition (6), we can obtain

\[
C_n \leq c_8 n^{-\frac{\lambda}{2}} \mu^{-\frac{\lambda}{2}} \int_{\Omega} |\beta(x)|^{1-\lambda} |Q(x)|^\lambda dx \leq c_7 n^{-\frac{\lambda}{2}}.
\]

Substituting (3.4)–(3.7) into (3.3), we have

\[
R(\delta_{\tau_n}, G) - R(\delta_G, G) = O \left( n^{-\frac{(s-2)}{2(s+1)}} \right).
\]

Proof of Theorem 3.2 is completed. □

Remark. When \( \lambda \to 1 \), \( O \left( n^{-\frac{(s-2)}{2(s+1)}} \right) \) is arbitrarily close to \( O \left( n^{-\frac{1}{2}} \right) \).

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