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HARDY-TYPE INEQUALITIES WITH WEIGHTS

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Abstract. Hardy-type inequality with weights is derived in abstract form. Particular cases are presented to demonstrate the applicability of the method and to show generalizations of existing results. Sharpness of inequalities is proved and the results are illustrated with several examples.

Introduction. The aim of this paper is to prove new Hardy–type inequalities with weights. Let Ω be a bounded domain, $\Omega \subset R^n$, $n \geq 1$ with a boundary $\partial\Omega \in C^1$. Suppose that f is a vector function defined in Ω, $|f| \neq 0$ and with components $f_i \in C^1(\Omega)$, $i = 1, \dots, n$. Let $p > 1$ and assume there exist in Ω functions $v > 0$, $v^{1-p} \in L^1(\Omega)$ and $w \geq 0$ such that

$$-\operatorname{div} f - (p-1)v|f|^{p'} \geq w, \text{ in } \Omega, \tag{1}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Let $\partial\Omega$ be divided into two parts $\partial\Omega = \Gamma_- \cup \Gamma_+$, where

$$\Gamma_- = \{x \in \partial\Omega : \langle f, \nu \rangle < 0\}, \quad \Gamma_+ = \{x \in \partial\Omega : \langle f, \nu \rangle \geq 0\}.$$

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Here ν is the unit outward to Ω normal vector on $\partial\Omega$ and $\langle .,. \rangle$ is the scalar product in R^n. For functions $u \in C^\infty_{\Gamma_-}(\Omega)$, where $C^\infty_{\Gamma_-} = \{u \in C^\infty, u = 0$ in a neigbourhood of $\Gamma_-\}$ we obtain as a consequence of Theorem 1 below, the following Hardy–type inequalities

$$\int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u\rangle}{|f|} \right|^p dx \geq \int_\Omega w|u|^p dx, \tag{2}$$

and

$$\begin{aligned} \left(\int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u\rangle}{|f|} \right|^p dx\right)^{1/p} &\geq \left(\frac{1}{p'}\right)\left(\int_\Omega v|f|^{p'}|u|^p dx\right)^{1/p} \\ + \ &\left(\frac{1}{p}\right)\int_{\Gamma_+} \langle f, \nu\rangle |u|^p dS \left(\int_\Omega v|f|^{p'}|u|^p dx\right)^{-1/p'}, \end{aligned} \tag{3}$$

where dS is $n-1$ dimensional surface measure. The form of Hardy inequality (3) is not the usual one, it depends on derivative of u in the direction of the unit vector $\dfrac{f}{|f|}$, on two functions v, w satisfying (1) and on additional term including boundary integral.

Since $\langle f, \nu\rangle \geq 0$ on Γ_+ and $|\nabla u|^p \geq \left|\dfrac{\langle f, \nabla u\rangle}{|f|}\right|^p$, in (2) and (3) we can replace their left hand sides correspondingly with

$$\int_\Omega v^{1-p}|\nabla u|^p dx \quad \text{and} \quad \left(\int_\Omega v^{1-p}|\nabla u|^p dx\right)^{1/p}.$$

The classical Hardy inequality in R^1_+, see Hardy [1], states

$$\int_0^\infty |u'(x)|^p x^\alpha dx \geq \left(\frac{p-1-\alpha}{p}\right)^p \int_0^\infty x^{-p+\alpha}|u(x)|^p dx, \tag{4}$$

where $1 < p < \infty$, $\alpha < p-1$ and $u(x)$ is absolutely continuous on $[0, \infty)$ with $u(0) = 0$. Note that with $f = \left(\dfrac{p-1-\alpha}{p-1}\right)^{p-1} x^{\alpha-p+1}$, $v = x^{\alpha(1-p')}$ and $w = 0$ condition (1) is satisfied and on $\Omega = [0, T]$ inequality (3) has the form

$$\begin{aligned} \left(\int_0^T |u'(x)|^p x^\alpha dx\right)^{1/p} &\geq \left(\frac{p-1-\alpha}{p}\right)\left(\int_0^T x^{\alpha-p}|u|^p dx\right)^{1/p} \\ + \ &\left(\frac{1}{p}\right) T^{\alpha-p+1}|u(T)|^p \left(\int_0^T x^{\alpha-p}|u|^p dx\right)^{-1/p'}. \end{aligned} \tag{5}$$

Moreover with $u(x) = x^k$, $k > \dfrac{p-1-\alpha}{p}$ inequality (5) becomes an equality.

There are generalizations of (4) for the n-dimensional case, $n \geq 2$, for bounded domains and for different weights (kernels of the integrals), see for more details the reviews in Davies [2], Opic and Kufner [3].

One direction of the investigations with respect to the domain concerns the optimal properties of the domain $\Omega \subset R^n$, where the Hardy inequality

$$(6) \qquad \int_\Omega |\nabla u(x)|^p d(x)^\alpha dx \geq C_\Omega \int_\Omega d(x)^{-p+\alpha} |u(x)|^p dx, \quad u \in C_0^\infty(\Omega),$$

holds with $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\alpha < p - 1$.

It is an open question what are the optimal, say necessary and sufficient conditions on $\partial\Omega$ in order to have (6). The inequality (6) was proved by Neĉas [4] for bounded domains Ω with Lipschitz boundary $\partial\Omega$ and $u \in C_0^\infty(\Omega)$. Next generalizations of (6) are by Kufner [5] for Hölder $\partial\Omega$ and by Wannebo [6] for Ω with generalized Hölder conditions. Detailed description of these results can be found in Opic and Kufner [3], Maz'ja [7], Hajlasz [8]. Further generalizations are made in Ancona [9], Lewis [10], Hajlasz [8], Koskela and Lehrback [11]. They are based on the investigation of the pointwise Hardy inequalities with capacity methods, see the review in Koskela and Lehrback [11]. Note that in [8] inequality (6) was proved in the domain $\Omega_t = \{x \in \Omega, d(x) < t\}$ for $u \in C_0^\infty(\Omega)$ and without zero conditions for u on the set $\{x \in \Omega, d(x) = t\}$. More general result was proved in Koskela and Lehrback [11].

Another way to describe the properties of Ω is to connect the validity of inequality (6) with the existence of the solutions of a certain boundary value problem for second order elliptic equation with a singular weight. In Ancona [9] it was proved that a necessary and sufficient condition for (6) when $p = 2$, $\alpha = 0$ is the existence of a positive super–harmonic function v in Ω and a positive number δ such that $\Delta v + \dfrac{\delta}{d(x)^2} v \leq 0$. Moreover, it was proved in [9] that $\max \delta = C_\Omega$.

One more direction of generalization of (4) is an inequality with a kernel, singular in an internal point of Ω

$$(7) \qquad \int_\Omega |\nabla u(x)|^p dx \geq C_\Omega \int_\Omega \frac{|u(x)|^p}{|x|^p} dx,$$

where $u \in C_0^\infty(\Omega)$, $\Omega \subseteq R^n$, $0 \in \Omega$, $n \geq 3$ and $p > 1$. The best constant $C_\Omega = \left(\dfrac{n-2}{2}\right)^2$ is obtained in Leray [12] for $\Omega = R^n$ and $p = 2$, see also Peral and Vazquez [13]. Let us mention that in all these papers the constant in Hardy inequality is optimal, but the inequality is not sharp. This is due to the fact that there is no function of the admissible class of functions for which the Hardy inequality becomes equality. That is why in the work of Brezis and Vazquez [14] the question about the existence of a remainder term in the right hand side is posed. In [14] for $p = 2$ the generalization of (7)

$$\int_\Omega |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u(x)|^2}{|x|^2} dx + \lambda(\Omega) \int_\Omega |u(x)|^2 dx, \tag{8}$$

is obtained where Ω is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda(\Omega) = z_0^2 \omega_n^{2/n} |\Omega|^{-2/n}$, z_0 is the first zero of the Bessel's function $J_0(z)$. The question of Brezis and Vazquez [14] leads to several improvements of (7), see Admurthi et al. [15], Barbatis et al. [16, 17], Alvino et al. [18]. These improvements concern also the regularity conditions of $\partial\Omega$.

Let us note that the possibility to use a vector function f and two functions v and w in inequalities (2), (3) serve for many new Hardy–type inequalities.

In Section 2 we prove our main results and in particular inequalities (2), (3). In Section 3 we study the sharpness of Hardy inequalities. In Section 4 as an application some particular cases of a vector function f and functions v, w are given. Here we comment the possibilities to obtain the results in Barbatis et al. [16, 17] and their generalization.

2. Main results.

2.1. General case. We start with a general Hardy-type inequality. For this purpose we introduce the notations

$$\begin{aligned} L(u) &= \int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx, \\ K_0(u) &= \int_{\Gamma_+} \langle f, \nu \rangle |u|^p dS, \\ K(u) &= \int_\Omega v |f|^{p'} |u|^p dx, \\ N(u) &= \int_\Omega w |u|^p dx, \end{aligned} \tag{9}$$

where f, v, w, Γ_+ are given in Section 1.

Theorem 1. *Under condition* (1), *for every* $u \in C^\infty_{\Gamma_-}(\Omega), u \neq 0$, *the following inequality holds*

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{(K_0(u) + (p-1)K(u) + N(u))^p}{K^{p-1}(u)}. \tag{10}$$

Proof. Since

$$\int_\Omega \langle f, \nabla |u|^p \rangle dx = p \int_\Omega |u|^{p-2} u \langle f, \nabla u \rangle dx, \tag{11}$$

applying the Hölder inequality on the right hand side with

$$v^{-1/p'} \frac{\langle f, \nabla u \rangle}{|f|} \quad \text{and} \quad v^{1/p'} |f| |u|^{p-2} u,$$

as factor of the integrand we get

$$\int_\Omega \langle f, \nabla |u|^p \rangle dx \leq p \left(\int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \right)^{1/p} \left(\int_\Omega v |f|^{p'} |u|^p dx \right)^{1/p'}. \tag{12}$$

Rising both sides of (12) to power p it follows that

$$\int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \geq \frac{\left| \frac{1}{p} \int_\Omega \langle f, \nabla |u|^p \rangle dx \right|^p}{\left(\int_\Omega v |f|^{p'} |u|^p dx \right)^{p-1}}. \tag{13}$$

Integrating by parts the numerator of the right hand side of (13), from (1) and $u|_{\Gamma_-} = 0$ we get

$$\begin{aligned}
&\frac{1}{p} \int_\Omega \langle f, \nabla |u|^p \rangle dx = \frac{1}{p} \int_{\partial\Omega} \langle f, \nu \rangle |u|^p dS - \frac{1}{p} \int_\Omega \operatorname{div} f |u|^p dx \\
&= \frac{1}{p} \int_{\partial\Omega} \langle f, \nu \rangle |u|^p dS - \frac{1}{p} \int_\Omega (\operatorname{div} f + (p-1) v |f|^{p'}) |u|^p dx \\
&+ \left(\frac{p-1}{p} \right) \int_\Omega v |f|^{p'} |u|^p dx \geq \frac{1}{p} (K_0(u) + N(u) + (p-1)K(u)).
\end{aligned}$$

So from (13) we obtain (10). □

Remark 1. The idea of the proof of Theorem 1 comes from Boggio [19] (for $p = 2$), Flekinger et al. [20] (Theorem II.1) and Barbatis et al. [17] (Theorem 4.1). In difference with the mentioned above works in our case we consider functions not necessary zero on the whole boundary $\partial\Omega$ and due to this there is an additional boundary term K_0 in (10). Also in L and K there is a weight v, which is 1 in the mention above works.

Remark 2. Applying the Young inequality

$$\frac{Q^p}{H^{p-1}} \geq pk^{p-1}Q - (p-1)k^pH, \tag{14}$$

with $H > 0$, $Q \geq 0$ and constant $k \geq 0$ to the right hand side of (10) we get

$$L(u) \geq k^{p-1}(K_0(u) + N(u)) + (p-1)k^{p-1}(1-k)K(u). \tag{15}$$

In particular, with $k = 1$ in (15) and ignoring $K_0(u)$ since $K_0(u) \geq 0$ we obtain (2).

Let us illustrate in the following example the possibility to choose a vector function f in order to obtain sharp Hardy inequality.

Example 1. Let ϕ be the first eigenfunction of the p-Laplacian in $\Omega \subset R^n$, $p > 1$, $n \geq 2$ with first eigenvalue λ

$$\left| \begin{array}{l} -\Delta_p \phi = \lambda |\phi|^{p-2}\phi, \quad \text{in } \Omega, \\ \phi|_{\partial\Omega} = 0. \end{array} \right. \tag{16}$$

Let us define the vector function $f = \dfrac{|\nabla\phi|^{p-2}\nabla\phi}{|\phi|^{p-2}\phi}$. Then (1) becomes

$$-\operatorname{div} f = -\frac{\Delta_p\phi}{|\phi|^{p-2}\phi} + (p-1)\frac{|\nabla\phi|^p}{|\phi|^p} = \lambda + (p-1)|f|^{p'},$$

with $\lambda = w > 0$ and $v \equiv 1$. From Theorem 1 we obtain

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{[(p-1)K(u) + N(u)]^p}{K^{p-1}(u)}, \quad u \in C_0^\infty(\Omega), \tag{17}$$

where

$$L(u) = \int_\Omega \left| \frac{\langle \nabla\phi, \ \nabla u\rangle}{|\nabla\phi|} \right|^p dx, \quad K(u) = \int_\Omega \left| \frac{\nabla\phi}{\phi} \right|^p |u|^p dx, \quad N(u) = \lambda \int_\Omega |u|^p dx.$$

Simple computation gives that inequality (17) is sharp, i.e. becomes an equality, for $u(x) = \phi(x)$.

2.2. Formulation with level function. Since condition (1) is not easy checkable we will replace it with sufficient conditions (18), (19) below and in Theorem 2 we will reformulate the result of Theorem 1. In fact all applications shown in the Section 4 are consequences of Theorem 2.

Suppose for a fixed bounded domain Ω that there exist a $C^{0,1}(\Omega)$ function F and a vector-function h with components $h_i \in C^{0,1}(\Omega)$, $i = 1, \dots, n$, such that for all intervals $(\varepsilon, \tau) \subset (0, T)$ the strip $G_{\varepsilon,\tau} = \{x \in \Omega : |F(x)| \in (\varepsilon, \tau)\} \subset \Omega$, $\bar{G}_{0,T} = \bar{\Omega}$ and a.e. in Ω

$$-F \operatorname{div} h \geq 0, \tag{18}$$

$$\langle h, \nabla F\rangle > 0. \tag{19}$$

Let us illustrate with an example the existence of a function F and vector function h satisfying (18), (19) when Ω is an annular domain.

Example 2. Let $\Omega = B_R \backslash B_r \subset R^n$, where $R > r$ and B_R, B_r are balls centered at 0 with radius R and r respectively. With a number $m = \dfrac{p-n}{p-1}$, $p > 1$, $p \neq n$, let us define a function $\psi(x) = \dfrac{R^m - |x|^m}{R^m - r^m}$. Then function $F = \psi$ and vector function $h = |\nabla\psi|^{p-2}\nabla\psi$ satisfy conditions (18), (19) in Ω.

Denote by $\Gamma_s = \{x \in \bar{\Omega} : |F(x)| = s\}, s \in [0, T]$ the level surfaces of F. Then for a.e. $s \in [0, T]$, Γ_s is smooth $(n-1)$-dim manifold. For such $\varepsilon < \tau$ the outward with respect to the strip $G_{\varepsilon,\tau}$ unit normals on Γ_ε and Γ_τ are

$$\nu|_{\Gamma_\varepsilon} = -\operatorname{sgn} F \left.\frac{\nabla F}{|\nabla F|}\right|_{\Gamma_\varepsilon}, \quad \nu|_{\Gamma_\tau} = \operatorname{sgn} F \left.\frac{\nabla F}{|\nabla F|}\right|_{\Gamma_\tau}. \tag{20}$$

Note that $\partial\Omega = \Gamma_0 \cup \Gamma_T$, $0 < T \leq \infty$ where

$$\begin{aligned} \Gamma_0 &= \{x \in \partial\Omega : \exists x_\varepsilon \in \Gamma_\varepsilon, x_\varepsilon \to x \text{ a.e. for } \varepsilon \to 0\}, \\ \Gamma_T &= \{x \in \partial\Omega : \exists x_\tau \in \Gamma_\tau, x_\tau \to x \text{ a.e. for } \tau \to T\}. \end{aligned}$$

We will choose f in (1) as $f = |F|^{-p} F.h$.

By (20) and the choice of f we get

$$\langle f, \nu\rangle|_{\Gamma_\varepsilon} = -|F|^{1-p} \frac{\langle h, \nabla F\rangle}{|\nabla F|}|_{\Gamma_\varepsilon} \leq 0 \quad \text{and } \langle f, \nu\rangle|_{\Gamma_\tau} \geq 0,$$

and hence $\Gamma_0 = \Gamma_-$, $\Gamma_T = \Gamma_+$.

Define

$$M_{\Gamma_0}^\infty = \{u \in C^\infty : \varepsilon^{1-p} \int_{\Gamma_\varepsilon} \frac{\langle h, \nabla F\rangle}{|\nabla F|} |u|^p dS \to 0 \text{ for } \varepsilon \to 0\}. \tag{21}$$

Obviously $C^\infty_{\Gamma_-} \subset M^\infty_{\Gamma_0}$. From the equality

$$-\operatorname{div} f = -|F|^{-p} F \operatorname{div} h + (p-1)|F|^{-p}\langle h, \nabla F\rangle,$$

and (18), (19) we choose the weights v, w as

$$v = \frac{\langle h, \nabla F\rangle}{|h|^{p'}}, \quad w = -|F|^{-p} F \operatorname{div} h. \tag{22}$$

Note that $v > 0$, $v^{1-p} \in L^1(\Omega)$ and $w \geq 0$.

With the use of function F and vector function h the notations (9) become

$$\begin{aligned} L_T(u) &= \int_{G_{0,T}} (\langle h, \nabla F\rangle)^{1-p} \, |\langle h, \nabla u\rangle|^p \, dx, \\ K_{0T}(u) &= T^{1-p} \int_{\Gamma_T} \frac{\langle h, \nabla F\rangle}{|\nabla F|} |u|^p dS, \\ K_T(u) &= \int_{G_{0,T}} |F|^{-p}\langle h, \nabla F\rangle |u|^p dx, \\ N_T(u) &= -\int_{G_{0,T}} |F|^{-p} F \operatorname{div} h |u|^p dx. \end{aligned} \tag{23}$$

Following the derivation of Theorem 1 we get:

Theorem 2. *Under the conditions* (18), (19) *and for the weights v and w satisfying* (22) *the inequality* (10) *holds for all $u \in M^\infty_{\Gamma_0}$, i.e.*

$$L_T(u) \geq \left(\frac{1}{p}\right)^p \frac{(K_{0T}(u) + (p-1)K_T(u) + N_T(u))^p}{K_T^{p-1}(u)}. \tag{24}$$

Particular cases of (24) *are*

$$L_T(u) \geq K_{0T}(u) + N_T(u), \tag{25}$$

$$L_T(u) \geq \left(\frac{1}{p'}\right)^p K_T(u), \tag{26}$$

$$L_T(u) \geq \left(\frac{1}{p}\right)^p \frac{(K_{0T}(u) + (p-1)K_T(u))^p}{K_T^{p-1}(u)}, \tag{27}$$

P r o o f. Following the proof of Theorem 1 we obtain (24).

Inequalities (25) and (26) are consequences of the Young inequality (15) with $k = 1$ and $k = \left(\frac{1}{p'}\right)$ respectively.

Since $N_T(u) \geq 0$ we get (27). $\square$

2.3. Cases $p = 1$ and $p = \infty$. It is easy to see that we can extrapolate the inequality (26) in Theorem 2 to the limit cases: $p = 1$ and $p = \infty$ and to obtain the following corollary.

Corollary 1. *Under the conditions of Theorem* 2 *for the limit cases of* p *we get the inequalities:*

(i) For $p = 1$*:*

$$\int_{G_{0,T}} \ln \frac{T}{|F|} |\langle h, \nabla u\rangle| dx \geq \int_{G_{0,T}} \langle h, \nabla F\rangle \left|\frac{u}{F}\right| dx, \tag{28}$$

(ii) For $p = \infty$*:*

$$\frac{1}{T} \int_0^T \operatorname*{ess\,sup}_{G_{0,t}} \left|\frac{\langle h, \nabla u\rangle}{\langle h, \nabla F\rangle}\right| dt \geq \operatorname*{ess\,sup}_{G_{0,T}} \left|\frac{u}{F}\right|. \tag{29}$$

P r o o f. When we get $v = \dfrac{\langle h, \nabla F\rangle}{|h|^{p'}}$ then

$$\begin{aligned} L_T^{1/p}(u) &= \left(\int_{G_{0,T}} \langle h, \nabla F\rangle \left|\frac{\langle h, \nabla u\rangle}{\langle h, \nabla F\rangle}\right|^p dx\right)^{1/p}, \\ K_T^{1/p}(u) &= \left(\int_{G_{0,T}} |F|^{-p} \langle h, \nabla F\rangle |u|^p dx\right)^{1/p}. \end{aligned}$$

So it holds

$$\begin{aligned} &L_T^{1/p}(u) \to_{p\to 1} \int_{G_{0,T}} |\langle h, \nabla u\rangle| dx; \quad L_t^{1/p}(u) \to_{p\to\infty} \operatorname*{ess\,sup}_{G_{0,t}} \left|\frac{\langle h, \nabla u\rangle}{\langle h, \nabla F\rangle}\right|, \\ &K_T^{1/p}(u) \to_{p\to 1} \int_{G_{0,T}} \langle h, \nabla F\rangle \left|\frac{u}{F}\right| dx; \quad K_T^{1/p}(u) \to_{p\to\infty} \operatorname*{ess\,sup}_{G_{0,T}} \left|\frac{u}{F}\right|, \end{aligned}$$

where ess sup is under the measure $\eta(\omega) = \displaystyle\int_\omega \langle h, \nabla F\rangle dx$. Then

$$\begin{aligned} \lim_{p\to 1} \int_0^T t^{-1/p} L_t^{1/p}(u) dt &= \int_0^T t^{-1} \lim_{p\to 1} L_t^{1/p}(u) dt \\ &= \int_0^T t^{-1} \int_{G_{0,t}} |\langle h, \nabla u\rangle| dx dt = \int_{G_{0,T}} \ln \frac{T}{|F|} |\langle h, \nabla u\rangle| dx, \end{aligned}$$

and hence we get (i) and (ii). $\square$

3. Sharpness of Hardy inequalities. In the following Theorem we prove that inequality (26) is ε-sharp.

Theorem 3. *Let* $v = \dfrac{\langle h, \nabla F\rangle}{|h|^{p'}}$ *in condition* (22) *and* $u_\varepsilon = |F|^{\frac{1+\varepsilon}{p'}}, \varepsilon > 0$ *then the inequality* (26) *in Theorem* 2 *for* $T < \infty$ *is* ε*-sharp, i.e.*

$$L_T(u_\varepsilon) = \left(\frac{1+\varepsilon}{p'}\right)^p K_T(u_\varepsilon).$$

P r o o f. Let $\varepsilon > 0$ and $u_\varepsilon = |F|^{\frac{1+\varepsilon}{p'}}$, then we have

$$\begin{aligned} K_T(u_\varepsilon) &= \int_{G_{0,T}} |F|^{\varepsilon(p-1)-1} < h, \nabla F > dx \\ &= \int_{G_{0,T}} a(|F|) < h, \nabla F > dx = H_a(0,T), \end{aligned}$$

where $a(s) = s^{\varepsilon(p-1)-1}$.

Since $\displaystyle\int_0^T a(s)ds < \infty$ it follows that $H_a(0,T) < \infty$. Then under the conditions above

$$\left|\frac{\langle h, \nabla u_\varepsilon\rangle}{|h|}\right| = \frac{1+\varepsilon}{p'}|F|^{\frac{\varepsilon(p-1)-1}{p}}\frac{\langle h, \nabla F\rangle}{|h|},$$

and

$$L_T(u_\varepsilon) = \left(\frac{1+\varepsilon}{p'}\right)^p \int_{G_{0,T}} |F|^{\varepsilon(p-1)-1}\langle h, \nabla F\rangle dx = \left(\frac{1+\varepsilon}{p'}\right)^p K_T(u_\varepsilon) < \infty. \quad \square$$

Let us illustrate the strictness results on two examples.

Example 3 (Singularity at 0)**.** Let $T > 0$, $m = \dfrac{p-n}{p-1}$, $p > n \geq 2$ and B be the ball with center 0 and radius $(mT)^{1/m}$. Define $F(x) = \dfrac{1}{m}|x|^m$ and $h = \nabla\phi$ with $\phi = \dfrac{1}{2-n}|x|^{2-n}$ in B.

We have (18) because $F \operatorname{div} h = 0$ and since $m - n = \dfrac{p(1-n)}{p-1} < 0$ then $\langle h, \nabla F\rangle = |x|^{m-n} > 0$ in B which is (19). Also $|F| \leq s$ is equivalent to $|x| \leq (ms)^{1/m}$, so $G_{0,T} = B$ and $\Gamma_s = \{x : |x| = (ms)^{1/m}\}$, $\Gamma_T = \partial B$, $0 \leq s \leq T$.

Choosing v, w satisfying (22) as $v = \dfrac{\langle h, \nabla F\rangle}{|h|^{p'}}$, $w = 0$ for $L_T(u)$, $K_{0T}(u)$, $K_T(u)$ we obtain

$$
\begin{aligned}
L_T(u) &= \int_B \left|\frac{\langle x, \nabla u\rangle}{|x|}\right|^p dx,\\
K_{0T}(u) &= m^{\frac{1-n}{m}} T^{\frac{1-p}{m}} \int_{\partial B} |u|^p dS, \qquad (30)\\
K_T(u) &= m^p \int_B |x|^{-p} |u|^p dx.
\end{aligned}
$$

From Theorem 2, we get

$$
(L_T)^{1/p}(u) \geq \left(\frac{1}{p}\right)[K_{0T}(u) K_T^{-1/p'} + (p-1) K_T^{1/p}]. \qquad (31)
$$

Now, inequality (31) becomes

$$
\begin{aligned}
\left(\int_B \left|\frac{< x, \nabla u >}{|x|}\right|^p\right)^{1/p} &\geq \left(\frac{1}{p}\right)\left((mT)^{-\frac{p-1}{m}} \int_{\partial B} |u|^p d\sigma\right)\left(\int_B \frac{|u|^p}{|x|^p} dx\right)^{-1/p'} \qquad (32)\\
&+ \left(\frac{p-n}{p}\right)\left(\int_B \frac{|u|^p}{|x|^p} dx\right)^{1/p}.
\end{aligned}
$$

Simple computation gives that inequality (32) is equality for the functions

$$
u = |F|^k = \frac{1}{m^k}|x|^{mk}, \quad k > \frac{1}{p'}.
$$

Note that $u \in W^{1,p}(B)$ because

$$
\int_B |\nabla u|^p dx = \left(\frac{k}{m^{k-1}}\right)^p \int_B |x|^{mkp-p} dx < \infty,
$$

since $mkp - p > \dfrac{p-n}{p-1}\dfrac{p-1}{p}p - p = -n$.

Let us check that $u \in M_{\Gamma_0}^{\infty}$, where $M_{\Gamma_0}^{\infty}$ is defined in (21). Indeed,

$$
\begin{aligned}
\varepsilon^{1-p} \int_{\Gamma_\varepsilon} \frac{\langle h, \nabla F\rangle}{|\nabla F|} |u|^p dS &= \varepsilon^{1-p} \int_{\Gamma_\varepsilon} \frac{|x|^{mkp+1-n}}{m^{kp}} dS\\
&= \varepsilon^{1-p+kp} |S_1| \to 0,
\end{aligned}
$$

for $\varepsilon \to 0$, since $1-p+kp>0$ when $k>\dfrac{1}{p'}$.

Recall that the classical Hardy inequality for $p>n$ is, see Opic and Kufner [3]

$$\int_B |\nabla u|^p \geq \left(\frac{p-n}{p}\right)^p \int_B \frac{|u|^p}{|x|^p}dx, u\in W_0^{1,p}(B), \tag{33}$$

and the constant $\left(\dfrac{p-n}{p}\right)^p$ is optimal. It is well known that equality in (33) is not possible for a function $u\in C_0^\infty(B)$. In contrast with (33), inequality (32) with the additional term

$$\left(\frac{1}{p}\right)\left((mT)^{-\frac{p-1}{m}}\int_{\partial B}|u|^p dS\right)\left(\int_B \frac{|u|^p}{|x|^p}dx\right)^{-1/p'}$$

is sharp, see also Fabricant et al. [21].

Example 4 (Singularity on the boundary.)**.** Let Ω, F, h are defined as in Example 2 then under Theorem 2, Hardy inequality (27) has the form

$$\begin{aligned}
&\left(\int_{B_R\backslash B_r}\left|\frac{\langle x,\nabla u\rangle}{|x|}\right|^p\right)^{1/p}\\
&\geq \frac{1}{p'}\left(\int_{B_R\backslash B_r}\frac{|u|^p}{|x|^{(n-1)p'}\left|\dfrac{R^m-|x|^m}{m}\right|^p}dx\right)^{1/p}\\
&+\frac{1}{p}r^{1-n}\left|\frac{R^m-r^m}{m}\right|^{1-p}\int_{\partial B_r}|u|^p dS\left(\int_{B_R\backslash B_r}\frac{|u|^p}{|x|^{(n-1)p'}\left|\dfrac{R^m-|x|^m}{m}\right|^p}dx\right)^{-1/p'}
\end{aligned} \tag{34}$$

for all functions $u\in C^\infty(B_R\backslash B_r)$ such that $\left(\dfrac{R^m-\delta^m}{R^m-r^m}\right)^{1-p}\delta^{1-n}\displaystyle\int_{S_\delta}|u|^p d\sigma \to 0$, when $\delta\to R$.

It is easy to see that inequality (34) is an equality on functions $u_k(x) = \left(\dfrac{R^m-|x|^m}{m}\right)^k$, $k>\dfrac{1}{p'}$.

Note that inequality (34) has meaning also for $p\to n$ i.e., for $m\to 0$ in the form

$$\left(\int_{B_R\setminus B_r}\left|\frac{\langle x,\nabla u\rangle}{|x|}\right|^n\right)^{1/n}$$

$$\geq \frac{1}{n'}\left(\int_{B_R\setminus B_r}\frac{|u|^p}{|x|^n\left|\ln\dfrac{R}{|x|}\right|^n}dx\right)^{1/n} \tag{35}$$

$$+ \frac{1}{n}\left|\ln\frac{R}{r}\right| r^{1-n}\int_{\partial B_r}|u|^n dS\left(\int_{B_R\setminus B_r}\frac{|u|^n}{|x|^n\left|\ln\dfrac{R}{|x|}\right|^n}dx\right)^{-1/n'}$$

and this inequality is sharp for functions $u(x)=\left|\ln\dfrac{R}{|x|}\right|^k, \quad k>\dfrac{1}{p'}$.

4. Applications. Below we will study two particular cases.

The first one in Section 4.1 is inequality (15) with $k=1$. This situation arrives when, for a suitable choice of f or F and h we can efficiently estimate w. Such scheme gives remainder terms, as it was done in Barbatis et al. [16, 17]. Based on these results we generalize them when $v\neq 1$ and in the case when Ω is a strip near the boundary.

The second application in Section 4.2 is when the only hypothesis is $w\geq 0$. Then it is possible to ignore the quantity N in (10) or (24). Nevertheless we get an optimal Hardy inequality in the second case under some additional conditions.

4.1. Inequality with singularities on the boundary. It is well known, see for example [16], that there exists a smooth positive function $z(s), s\in[-\infty,\ln T]$ which is a solution of the inequality

$$Pz\equiv -z'+(p-1)z-(p-1)z^{p'}\geq H(s) \tag{36}$$

with some positive $H(s)$. Note that in [16]

$$\begin{aligned} z(s) &= \left(\frac{1}{p'}\right)^{p-1}\left(1+O\left(\frac{1}{|s|}\right)\right) \text{ and}\\ H(s) &= \left(\frac{1}{p'}\right)^{p}\left(1+O\left(\frac{1}{|s|^2}\right)\right), \quad s\to\infty. \end{aligned}$$

For instance, when $p = 2$

$$z(s) = \frac{1}{2}\left(\frac{\ln T - s}{1 + \ln T - s}\right) \text{ and } H(s) = \frac{1}{4}\left(1 + \frac{1}{(1 + \ln T - s)^2}\right).$$

In Barbatis et al. [16, 17] it was proved, in notations used above, the inequality

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega H(\ln t)\frac{|\nabla t|^p}{t^p}|u|^p dx \tag{37}$$

under the conditions: $t > 0, -\Delta_p t \geq 0$ when $t = d^\lambda$, d is the distance to some manifold.

Our first aim is to generalize the results in [16, 17]. We will use the inequality (25), proved in Theorem 2, i.e.

$$L_T \geq R_{0T} + N_T. \tag{38}$$

For this purpose, suppose that there exists a function $t = t(x)$ defined in Ω, sufficiently smooth, satisfying the following assumptions:

$$\begin{aligned} &\text{(i) } \Gamma_0 = \{x \in \partial\Omega : t(x) = 0\} \text{ and } G_{0,\tau} = \{x : 0 < |t(x)| < \tau\} \subset \Omega, \\ &S_\tau = \{x : |t(x)| = \tau\}; \\ &\text{(ii) } -t\Delta_p t \geq 0; \\ &\text{(iii) } v = v(t) > 0 \text{ is defined in } (0,\tau) \text{ and } T = \int_0^\tau v(s)ds < \infty. \end{aligned} \tag{39}$$

Proposition 1. *Under the conditions* (39) *and* (36) *for all* $u \in C^\infty$ *such that* $\varepsilon^{1-p}\int_{|t(x)|=\varepsilon} |\nabla t|^{p-1}|u|^p dx \to 0$ *for* $\varepsilon \to 0$, *we have the inequality*

$$\begin{aligned} &\int_{0<|t(x)|<\tau} v^{1-p}(t(x))\left|\frac{\langle\nabla t, \nabla u\rangle}{|\nabla t|}\right|^p \geq T^{1-p}z(\ln T)\int_{S_\tau} |\nabla t|^{p-1}|u|^p dS \\ &+ \int_{0<|t(x)|<\tau} \left(\frac{1}{t}\int_0^t v(s)ds\right)^{-p} v(t)H\left(\ln\left|\int_0^t v(s)ds\right|\right)\left|\frac{\nabla t}{t}\right|^p |u|^p dx. \end{aligned} \tag{40}$$

Proof. We need only to determine F and h satisfying (18) and (19). Let

$$F = \left[z\left(\ln\left|\int_0^t v(s)ds\right|\right)\right]^{1-p'}\int_0^t v(s)ds,$$

where z is a positive solution of (36) and $h = |\nabla t|^{p-2}\nabla t$. Then $F(x) = 0$ *iff* $x \in S_0 = \{x : |t(x)| = 0\}$ and $|F| = T$ *iff* $x \in S_\tau$. Now from (39) (ii) we have

$$-F \operatorname{div} h = -\left(\frac{1}{t}\int_0^t v(s)ds\right)\left[z\left(\ln\left|\int_0^t v(s)ds\right|\right)\right]^{1-p'} t\Delta_p t \geq 0.$$

Also

$$\begin{aligned}\nabla F &= v(t)\left[z\left(\ln\left|\int_0^t v(s)ds\right|\right)\right]^{1-p'} \\ &+ (1-p')\left[z\left(\ln\left|\int_0^t v(s)ds\right|\right)\right]^{-p'} z'\left(\ln\left|\int_0^t v(s)ds\right|\right)\nabla t,\end{aligned}$$

and using (36) we get

$$\langle \nabla F, h\rangle = \left(-1 - \frac{H}{p-1} z^{-p'}\right) v|\nabla t|^p > 0.$$

So, from the definition of w in (22) we obtain

$$\begin{aligned} w &= (p-1)(h\nabla F - v|h|^{p'})|F|^{-p} \\ &= (p-1)v\left|\int_0^t v(s)ds\right|^{-p} z^{p'}[z^{1-p'} + (1-p')z^{-p'}z' - 1]|\nabla t|^p \\ &= v\left|\frac{1}{t}\int_0^t v(s)ds\right|^{-p} Pz|\nabla t|^p \\ &= v\left|\frac{1}{t}\int_0^t v(s)ds\right|^{-p} H\left(\ln\left|\int_0^t v(s)ds\right|\right)|\nabla t|^p \geq 0.\end{aligned}$$

Finally applying (25) in Theorem 2 we get (40). □

The result in Proposition 1 is a generalization of that in Neĉas[4], Kufner[5], Wannebo [6], see inequality (5).

Example 5. If $v(t) = t^{\beta-1}$, $\beta > 0$, the weight in the left hand side of (40) becomes $v^{1-p}(t) = t^\alpha$ and $\alpha = (\beta-1)(1-p) < p-1$. Let z in (36) is a constant, $z = \left(\frac{1}{p'}\right)^{p-1}$, then $Pz = H = \left(\frac{1}{p'}\right)^p$ and since

$$|t|^{-p}\left(\frac{1}{t}\int_0^t v(s)ds\right)^{-p} v(t)|\nabla t|^p = \beta^p t^{\alpha-p}|\nabla t|^p,$$

inequality (40) becomes

$$\begin{aligned}\int_{0<|t(x)|<\tau} t^\alpha \left|\frac{\langle\nabla t, \nabla u\rangle}{|\nabla t|}\right|^p &\geq \tau^{1-p}\left(\frac{1}{p'}\right)^{p-1}\int_{S_\tau}|\nabla t|^{p-1}|u|^p dS \\ &+ \left(\frac{\beta}{p'}\right)^p \int_{0<|t(x)|<\tau}|\nabla t|^p t^{\alpha-p}|u|^p dx.\end{aligned} \tag{41}$$

Note that $\left(\dfrac{\beta}{p'}\right)^p = \left(\dfrac{p-1-\alpha}{p}\right)^p$. For the special case $t(x) = d(x) = \operatorname{dist}(x, \partial\Omega)$, so $|\nabla d| = 1$ in the strip $\Omega_\tau = \{x : 0 < d(x) < \tau\}$, the inequality (41) is

$$\begin{aligned} \int_{\Omega_\tau} d^\alpha(x)|\langle \nabla d(x), \nabla u\rangle|^p dx &\geq \left(\frac{1}{\tau p'}\right)^{p-1} \int_{S_\tau} |u|^p dS \\ + \quad &\left(\frac{p-1-\alpha}{p}\right)^p \int_{\Omega_\tau} d^{\alpha-p}(x)|u|^p dx. \end{aligned} \tag{42}$$

and is a generalization of inequality (5).

Here the following question arises: are the conditions on $\partial\Omega = \Gamma_0$ in the studies above enough to conclude that from assumptions (39) (i), (ii) it follows that $\left|\dfrac{\nabla t}{t}\right| \geq \dfrac{C}{d_{\Gamma_0}}$, where d_{Γ_0} is the distance to Γ_0?

4.2. Inequalities with double singularities. We will show how to choose functions h and F satisfying (18), (19), in order to apply inequality (26) in Theorem 2. Let us use the following notation: α is a subset of numbers $\{1, 2, \ldots, n\}$; $[\alpha]$ is the number of elements in α; X_α is an element of $R^{[\alpha]}$; $|X_\alpha|$ is the length of vector X_α.

Let $\varphi(s)$ be a nondecreasing $C^1(R_+)$ function, such that the function $s^{-\delta}\varphi(s)$ for $1 > \delta > 0$, is increasing in a neighborhood of $s = 1$ and

$$s^{-\delta}\varphi(s) = 0 \quad \textit{iff} \quad s = 1. \tag{43}$$

Fix α, β such that $\alpha \cap \beta \neq \emptyset$ and let function $\mu = \mu(x) \in C^1(\Omega)$, $\mu(x) > 0$ is such that $\langle X_\alpha, \nabla \mu(x)\rangle \leq 0$. Define

$$h = |X_\alpha|^{-[\alpha]} X_\alpha, \quad F = |X_\beta|^{-\delta} \varphi\left(\frac{|X_\beta|}{\mu(x)}\right), \quad \Gamma_0 = \{|X_\beta| = \mu(x)\}. \tag{44}$$

Conditions (18), (19) for h and F are satisfied, indeed:

$$\operatorname{div} h = -[\alpha]|X_\alpha|^{-[\alpha]-2} \frac{\langle X_\alpha, X_\alpha\rangle}{|X_\alpha|} + [\alpha]|X_\alpha|^{-[\alpha]} = 0,$$

$$\nabla_\beta F = |X_\beta|^{-\delta}\varphi'(s)\left[\frac{X_\beta}{|X_\beta|\mu(x)} - \frac{X_\beta}{\mu^2(x)}\nabla\mu(x)\right] - \delta\varphi(s)|X_\beta|^{-\delta-2}X_\beta,$$

$$\nabla_{\beta'} F = -|X_{\beta'}|^{1-\delta}\varphi'(s)\frac{\nabla_{\beta'}\mu}{\mu^2}, \quad s = \frac{|X_\beta|}{\mu(x)}$$

where ∇_β and $\nabla_{\beta'}$ are gradients with respect to β and β' variables, $\beta \cap \beta' = \emptyset$, $\beta \cup \beta' = (1, \dots, n)$. Then

$$\langle h, \nabla F\rangle \geq (s\varphi' - \delta\varphi)|X_\alpha|^{-[\alpha]}|X_\beta|^{-\delta-2}|X_{\alpha\cap\beta}|^2 > 0 \tag{45}$$

since $\varphi'(s) \geq 0$, $\langle X_\alpha, \nabla_\beta \mu\rangle + \langle X_\alpha, \nabla_{\beta'}\mu\rangle = \langle X_\alpha, \nabla\mu\rangle \leq 0$.

Denote by V and W the weights in L_T, K_T in Theorem 2

$$\begin{aligned} V &= (s\varphi' - \delta\varphi)^{1-p}|X_\alpha|^{p-[\alpha]}|X_\beta|^{(\delta+2)(p-1)}|X_{\alpha\cap\beta}|^{2(1-p)}, \\ W &= (s\varphi' - \delta\varphi)|\varphi|^{-p}|X_\alpha|^{-[\alpha]}|X_\beta|^{\delta(p-1)-2}|X_{\alpha\cap\beta}|^2. \end{aligned}$$

Then we have:

(i) If $\alpha \subset \beta$:

$$\begin{aligned} V &= (s\varphi' - \delta\varphi)^{1-p}|X_\alpha|^{2-p-[\alpha]}|X_\beta|^{(\delta+2)(p-1)}, \\ W &= (s\varphi' - \delta\varphi)|\varphi|^{-p}|X_\alpha|^{2-[\alpha]}|X_\beta|^{\delta(p-1)-2}. \end{aligned}$$

(ii) If $\alpha \supset \beta$:

$$\begin{aligned} V &= (s\varphi' - \delta\varphi)^{1-p}|X_\alpha|^{p-[\alpha]}|X_\beta|^{\delta(p-1)}, \\ W &= (s\varphi' - \delta\varphi)|\varphi|^{-p}|X_\alpha|^{-[\alpha]}|X_\beta|^{\delta(p-1)}. \end{aligned}$$

(iii) If $\alpha = \beta$:

$$\begin{aligned} V &= (s\varphi' - \delta\varphi)^{1-p}|X_\alpha|^{p-[\alpha]+\delta(p-1)}, \\ W &= (s\varphi' - \delta\varphi)|\varphi|^{-p}|X_\alpha|^{-[\alpha]+\delta(p-1)}. \end{aligned}$$

So for $u \in C^\infty_{\Gamma_0}$ applying inequality (26) in Theorem 2 we have:

Proposition 2. *The inequality holds:*

$$\int_{G_{0,T}} V\left|\frac{\langle h, \nabla u\rangle}{|h|}\right|^p dx \geq \left(\frac{1}{p'}\right)^p \int_{G_{0,T}} W|u|^p dx. \tag{46}$$

A particular case of (46), considered in Fabricant et al. [22] is with $\alpha = \beta = (1, \dots, n)$, i.e. $X_\alpha = X_\beta = x$, $[\alpha] = n$, $|X_\alpha| = |x|$ and

$$\langle x, \nabla\mu\rangle \leq 0, \quad g(s) = \begin{cases} \dfrac{1-s^k}{k}, & k \neq 0 \\ \ln\dfrac{1}{s}, & k = 0 \end{cases},$$

$$\Omega = \{x : s = \frac{|x|}{\mu(x)} \in (\sigma_1, \sigma_2), 0 \le \sigma_1 \le 1 \le \sigma_2 \le \infty\}.$$

Then for $b > 0$

$$(47)\qquad \begin{aligned} &\int_\Omega \left| g\left(\frac{|x|}{\mu(x)}\right)\right|^{(b-1)(1-p)} |x|^{p-n+kb(p-1)}|\nabla u|^p dx \\ \ge\ & \left(\frac{b}{p'}\right)^p \int_\Omega \left| g\left(\frac{|x|}{\mu(x)}\right)\right|^{b(1-p)-1} |x|^{-n+kb(p-1)}|u|^p dx, u \in C^\infty_{\Gamma_0}. \end{aligned}$$

If $\langle x, \nabla\mu(x)\rangle = 0$ then the ε-sharpness of (46) is a consequence of Theorem 3 when $\varphi = -\frac{1}{b}|g|^{b-1}g$ and $\delta = kb$. Indeed, we have to check (43) and (45). Since $sg' = kg - 1 < 0$ we get $s\varphi' - \delta\varphi = |g|^{b-1}(1 - kg) + k|g|^{b-1}g = |g|^{b-1} > 0$ and (45) is satisfied. From the equality $s^{-\delta}|\varphi(s)| = \frac{1}{b}s^{-kb}|g(s)|^b = 0$ and the definition of $g(s)$ it follows that $s^{-\delta}|\varphi(s)| = 0$ *iff* $s = 1$, i.e. (43) holds. Thus with the above choice of φ and δ, (46) and correspondingly (47) is ε-sharp.

Example 6. With $b = 1$, $k = \frac{n-p}{p-1} \ne 0$, $\mu(x) = 1$, $\Omega = \{|x| < 1\} = B_1$, the inequality (47) becomes

$$(48)\qquad \int_{B_1} |\nabla u|^p dx \ge \left(\frac{n-p}{p}\right)^p \int_{B_1} \frac{|u|^p}{|x|^p(1-|x|^k)^p} dx, u \in C_0^\infty(B_1).$$

Let us note that in (48) there are no relations between n and $p > 1$ and since $1 - |x|^k \le 1$, inequality (48) improves inequality (7), see [22].

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