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THE MAXIMUM PRINCIPLE IN TIME-INCONSISTENT LQ EQUILIBRIUM CONTROL PROBLEM FOR JUMP DIFFUSIONS

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Abstract. In this paper, we discuss a class of stochastic linear quadratic dynamic decision problems of a general time-inconsistent type, in the sense that, it does not satisfy the Bellman optimality principle. More precisely, the dependence of the running and the terminal costs in the objective functional on some general discounting coefficients, as well as on some quadratic terms of the conditional expectation of the state process, makes the problem time-inconsistent. Open-loop Nash equilibrium controls are then constructed instead of optimal controls, this has been accomplished through the stochastic maximum principle approach that includes a flow of forward-backward stochastic differential equations under a maximum condition. Then by decoupling the flow of the adjoin process, we derive an explicit representation of the equilibrium strategies in feedback form. As an application, we study some concrete examples. We emphasize that; this method can provide the necessary and sufficient conditions to characterize the equilibrium strategies.

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Key words: stochastic maximum principle, time inconsistency, linear quadratic control problem, equilibrium control, variational inequality.

While most existing results which are based on the dynamic programming principle and the extended HJB equation can create only the sufficient condition to characterize the equilibrium strategies.

Introduction. Stochastic optimal control problems with linear dynamics and quadratic stage costs are one of the most important classes of optimal control ones. They have wide applications in engineering and financial mathematics, etc. A major approach for studying such stochastic control problems is the dynamic programming principle which expresses the optimal policy in terms of an optimization problem involving the value function (or a sequence of value functions in the time-varying case). The proof of the dynamic programming principle is technical and has been studied by different methods. The value function can be create, by an iteration connecting to the Bellman operator, which maps functions on the state space into functions on the state space and involves an expectation and a minimization step.

A number of studies have been devoted to this topic by different methods. Wu and Wang [23] discussed a kind of stochastic LQ problem for system driven by a Brownian motion and an independent Poisson jump process then a linear feedback regulator for the optimal control problem is given by the solution of a generalized Riccati equation system. In view of completing of squares technique, Hu and Øksendal [13] studied the stochastic LQ problem for a general stochastic differential equation with random coefficients, under partial information. Meng [12] investigate the stochastic maximum principle in LQ control problem for multidimensional stochastic differential equation driven by a Brownian motion and a Poisson random martingale measure and obtain the existence and uniqueness result for a class of backward stochastic Riccati equations. For more information on LQ control models for stochastic dynamic systems, we refer to [20], [28], [23], [25] and [28].

Time-inconsistent stochastic control problems have received remarkable attention in the recent years. The risk aversion attitude of a mean-variance investor [2], [3] and [9], and the portfolio optimization model with non-exponential discount function [6] and [7], provide two well-known examples of time-inconsistency in mathematical finance. The main difficulty when facing a time-inconsistent optimal control problem is that, we cannot use the dynamic programming and the standard HJB techniques, meaning that an optimal strategy might not remains optimal as time goes. However, the main approach to handle the time-inconsistent optimal control problems, is by viewing them within a game theoretic framework. Nash equilibriums are therefore considered instead of optimal solutions, see e.g. [2], [4], [5], [6], [7], [9], [10], [16], [17], [18], [24], [25] and [26].

The fundamental idea is that the control action that the controller makes at every instant of time, is considered as a game against all the control actions that describes the future incarnations of the controller. Strotz [24], was the first who used this game perspective to handle the dynamic time-inconsistent decision problem on the deterministic Ramsay problem [18]. Then by capturing the idea of non-commitment, by letting the commitment period being infinitesimally small, he characterized a Nash equilibrium strategy. Further work which extend [18] are [10], [18], [17] and [8]. Ekland and Lazrak [6] and Ekland and Pirvu [7] apply this game perspective to investigate the optimal investment-consumption problem under general discount functions, in both, deterministic and stochastic framework. Then, by means of the so-called "local spike variation" they provide a formal definition of feedback Nash equilibrium controls in continuous time. The work [2] extends the idea to the stochastic framework where the controlled process is Markovian. In addition, an extended HJB equation is derived, along with a verification argument that characterizes a Markov subgame perfect Nash equilibium.

To the best of our knowledge, there is little work in the literature concerning equilibrium strategy for time-inconsistent LQ control problems. In [24] Yong studied a general discounting time-inconsistent deterministic LQ model, and he derive a closed-loop equilibrium strategies, via a forward ordinary differential equation coupled with a backward Riccati-Volterra integral equation. Hu et al [9] investigate open loop equilibrium strategies for time inconsistent LQ control problem with random coefficients by adopting a Pontryagin type stochastic maximum principle approach, we refer to [22] for partially observed recursive optimization problem. Yong [26] investigate a time-inconsistent stochastic LQ problem for mean-field type stochastic differential equation and closed-loop solutions are presented by means of multi-person differential games, the limit of which leads to the equilibrium Riccati equation. As far as we know, there is no literature on the time-inconsistent stochastic linear-quadratic optimal control problems incorporating stochastic jumps.

Novelty and contribution. Motivated by these points, this paper studies optimality conditions for time-inconsistent linear quadratic stochastic control problem, in the sense that, it does not satisfy the Bellman optimality principle, since a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair. The state is described by a n-dimensional non homogeneous controlled SDE with jump processes, defined on a complete filtered probability space. The objective functional includes the cases of hyperbolic discounting, as well as, the continuous-time

Markowitz's mean-variance portfolio selection problem, with state-dependent risk aversion.

Our objective, is to investigate a characterization of Nash equilibrium controls instead of optimal controls. The novelty of this work lies in the fact that, our calculations are not limited to the exponential discounting framework, the time-inconsistency of the LQ optimal control in this situation, is due to the presence of some general discounting coefficients, involving the so-called hyperbolic discounting situations. In addition, the presence of some quadratic terms of the expected controlled state process, in both the running and the terminal costs, (this can be motivated by the reward term in the mean-variance portfolio choice model), and the presence of the risk aversion term, whish stems from the state-dependent utility function in economics [9], make the problem time-inconsistent. Each of these terms introduces time-inconsistency of the underlying model, in somewhat different ways.

We accentuate that, our model covers some class of time-inconsistent stochastic LQ optimal control problem studied by [9], and some relevant cases appeared in [25]. Note that, in [9] the weighting matrices do not depend on current time t and in [25] the terminal cost do not depend on current state ξ. Moreover, we have defined the equilibrium controls in open-loop sense, in a manner similar to [9], which is different from the feedback form, see e.g. [2], [3], [4], [6], [7], [24], [26] and [23].

Structure of the paper. The rest of the paper is organized as follows. In Section 2, we describe the model and formulate the objective. In Section 3 we present the first main result of this work (Theorem 3.2), which characterizes the equilibrium control via a stochastic system, which involves a flow of forward-backward stochastic differential equation with jumps (FBSDEJ in short), along with some equilibrium conditions. In Section 4, by decoupling the flow of the FBSDEJ, we investigate a feedback representation of the equilibrium control, via some class of ordinary differential equations, which do not have a symmetry structure. Section 5 is devoted to some applications, we solve a continuous time mean–variance portfolio selection model and some one-dimensional general discounting LQ problems. The paper ends with Appendix containing some proofs.

2. Problem setting. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. We assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by a d-dimensional standard Browian motion $(W(t))_{t\in[0,T]}$ and an independent Poisson measure N on $[0,T] \times Z$ where $Z \subset \mathbb{R} - \{0\}$. We assume that the compensator of N has the form $\mu(dt,dz) = \theta(dz)\,dt$ for some positive and $\sigma-$finite Levy measure on Z, endowed with it's Borel $\sigma-$field $\mathcal{B}(Z)$. We suppose that $\int_Z 1 \wedge |z|^2 \theta(dz) < \infty$ and write $\tilde{N}(dt,dz) = N(dt,dz) - \theta(dz)\,dt$ for the compensated jump martingal rondom measure of N. Obviously, we have

$$\mathcal{F}_t = \sigma\left[\int\int_{A\times(0,s]} N(dr,de)\,; s \le t, A \in \mathcal{B}(Z)\right] \vee \sigma\left[B_s; s \le t\right] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\theta-$null sets, and $\sigma_1 \vee \sigma_2$ denotes the $\sigma-$field generated by $\sigma_1 \cup \sigma_2$.

2.1. Notations. Throughout this paper, we use the following notations:

- S^n : the set of $n \times n$ symmetric real matrices.
- $C^\top$: the transpose of the vector (or matrix) C.
- $\langle .,. \rangle$: the inner product in some Euclidean space.

For any Euclidean space $H = \mathbb{R}^n$, $\mathbb{R}^{n\times m}$ or S^n with Frobenius norm $|.|$ we let for any $t \in [0,T]$

- $\mathbb{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}; H) := \{\xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t-\text{measurable, with } \mathbb{E}\left[|\xi|^p\right] < \infty\}$, for any $p \ge 1$.
- $$\mathbb{L}^2(Z, \mathcal{B}(Z), \theta; H) := \Big\{ r(.) : Z \to H \mid r(.) \text{ is } \mathcal{B}(Z)-\text{measurable} \quad , \text{ with } \int_Z |r(z)|^2 \theta(dz) < \infty \Big\}.$$
- $$\mathcal{S}^2_{\mathcal{F}}(t,T;H) := \Big\{ X(.) : [t,T] \times \Omega \to H \mid X(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]}-\text{adapted}, \quad s \mapsto X(s) \text{ is càdlàg, with } \mathbb{E} \sup_{s\in[t,T]} |X(s)|^2 ds < \infty \Big\}.$$
- $$\mathcal{L}^2_{\mathcal{F}}(t,T;H) := \Big\{ X(.) : [t,T] \times \Omega \to H | X(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]}-\text{adapted}, \quad \text{with } \mathbb{E}\left[\int_t^T |X(s)|^2 ds\right] < \infty \Big\}.$$

- $\mathcal{L}_{\mathcal{F}}^{\theta,2}([t,T]\times Z;H) := \Big\{ R(.,.) : [t,T]\times\Omega\times Z \to H \mid R(.) \text{ is } (\mathcal{F}_s)_{s\in[t,T]}$ $-$ adapted process on $[t,T]\times\Omega\times Z$, with $\mathbb{E}\left[\int_t^T \int_Z |R(s,z)|^2 \theta(dz)\, ds\right] < \infty \Big\}.$

- $\mathcal{C}([0,T];H) := \{ f : [0,T] \to H \mid f(.) \text{ is continuous} \}.$

- $\mathcal{D}[0,T] := \{ (t,s) \in [0,T]\times[0,T], \text{ such that } s \geq t \}.$

- $\mathcal{C}(\mathcal{D}[0,T];H) := \{ f(.,.) : \mathcal{D}[0,T] \to H(t,s) \mid f(.,.) \text{ is continuous} \}.$

- $\mathcal{C}^{0,1}(\mathcal{D}[0,T];H) := \Big\{ f(.,.) : \mathcal{D}[0,T] \to H \mid f(.,.) \text{ and } \frac{\partial f}{\partial s}(.,.)$ are continuous$\Big\}.$

2.2. Problem statement. We consider a n-dimensional non homogeneous linear controlled jump diffusion system

$$\begin{cases} dX(s) = \{A(s)X(s) + B(s)u(s) + b(s)\}\, ds \\ \qquad\qquad + \sum_{j=1}^{d} \{C_j(s)X(s) + D_j(s)u(s) + \sigma_j(s)\}\, dW^j(s) \\ \qquad\qquad + \int_Z \{E(s,z)X(s-) + F(s,z)u(s) + c(s,z)\}\, \tilde{N}(ds,dz), \\ \qquad\qquad\qquad\qquad s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{2.1}$$

where $(t,\xi,u(.)) \in [0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n)\times\mathcal{L}_{\mathcal{F}}^2(t,T;\mathbb{R}^m)$. Under some conditions, for any initial situation (t,ξ) and any admissible control $u(.)$ the state equation is uniquely solvable, we denote by $X(.) = X^{t,\xi}(.;u(.))$ its solution, for $s \in [t,T]$. Different controls $u(.)$ will lead to different solutions $X(.)$. Note that $\mathcal{L}_{\mathcal{F}}^2(t,T;\mathbb{R}^m)$ is the space of all admissible strategies.

Our aim is to minimize the following expected discounted cost functional

$$
\begin{aligned}
& J\left(t,\xi,u\left(.\right)\right) \\
& =\mathbb{E}^{t}\left[\int_{t}^{T}\frac{1}{2}\left(\left\langle Q\left(t,s\right)X\left(s\right),X\left(s\right)\right\rangle+\left\langle \bar{Q}\left(t,s\right)\mathbb{E}^{t}\left[X\left(s\right)\right],\mathbb{E}^{t}\left[X\left(s\right)\right]\right\rangle\right.\right. \\
& \qquad \left.+\left\langle R\left(t,s\right)u\left(s\right),u\left(s\right)\right\rangle\right)ds \qquad (2.2)\\
& \qquad +\left\langle \mu_{1}\left(t\right)\xi+\mu_{2}\left(t\right),X\left(T\right)\right\rangle \\
& \qquad \left.+\frac{1}{2}\left(\left\langle G\left(t\right)X\left(T\right),X\left(T\right)\right\rangle+\left\langle \bar{G}\left(t\right)\mathbb{E}^{t}\left[X\left(T\right)\right],\mathbb{E}^{t}\left[X\left(T\right)\right]\right\rangle\right)\right],
\end{aligned}
$$

over $u\left(.\right)\in\mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{m}\right)$, where $X\left(.\right)=X^{t,\xi}\left(.;u\left(.\right)\right)$ and $\mathbb{E}^{t}\left[.\right]=\mathbb{E}\left[.\left|\mathcal{F}_{t}\right.\right]$.

We need to impose the following assumptions about the coefficients

(H1) The functions $A\left(.\right),C_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n\times n}$, $B\left(.\right),D_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n\times m}$, $b\left(.\right),\sigma_{j}\left(.\right):\left[0,T\right]\rightarrow\mathbb{R}^{n}$, $E\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n\times n}$, $F\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n\times m}$, and $c\left(.,.\right):\left[0,T\right]\times Z\rightarrow\mathbb{R}^{n}$ are continuous and uniformly bounded. The coefficients on the cost functional satisfy

$$
\begin{cases}
Q\left(.,.\right),\bar{Q}\left(.,.\right)\in C\left(\mathcal{D}\left[0,T\right];S^{n}\right), \\
R\left(.,.\right)\in C\left(\mathcal{D}\left[0,T\right];S^{m}\right), \\
G\left(.\right),\bar{G}\left(.\right)\in C\left(\left[0,T\right];S^{n}\right), \\
\mu_{1}\left(.\right)\in C\left(\left[0,T\right];\mathbb{R}^{n\times n}\right), \\
\mu_{2}\left(.\right)\in C\left(\left[0,T\right];\mathbb{R}^{n}\right).
\end{cases}
$$

(H2) The functions $R\left(.,.\right),Q\left(.,.\right)$ and $G\left(.\right)$ satisfy

$$
R\left(t,t\right)\geq0,\ G\left(t\right)\geq0,\ \forall t\in\left[0,T\right],\text{and }Q\left(t,s\right)\geq0,\ \forall\left(t,s\right)\in\mathcal{D}\left[0,T\right].
$$

Under **(H1)** for any $\left(t,\xi,u\left(.\right)\right)\in\left[0,T\right]\times\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{n}\right)\times\mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{m}\right)$, the state equation (2.1) has a unique solution $X\left(.\right)\in\mathcal{S}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{n}\right)$, see for example [12]. Moreover, we have the following estimate

$$
\mathbb{E}\left[\sup_{t\leq s\leq T}\left|X\left(s\right)\right|^{2}\right]\leq K\left(1+\mathbb{E}\left[\left|\xi\right|^{2}\right]\right),
$$

for some positif constant K. The optimal control problem can be formulated as follows.

Problem (LQJ). *For any given initial pair* $\left(t,\xi\right)\in\left[0,T\right]\times\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t},\mathbb{P};\mathbb{R}^{n}\right)$, *find a control* $\hat{u}\left(.\right)\in\mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{m}\right)$ *such that*

$$
J\left(t,\xi,\hat{u}\left(.\right)\right)=\inf_{u\left(.\right)\in\mathcal{L}_{\mathcal{F}}^{2}\left(t,T;\mathbb{R}^{m}\right)}J\left(t,\xi,u\left(.\right)\right).
$$

Remark 2.1. 1) The dependence of the weighting matrices of the current time t, the dependence of the terminal cost on the current state ξ and the presence of quadratic terms of the expected controlled state process in the cost functional make the Problem (LQJ) time-inconsistent.

2) One way to get around the time-inconsistency issue is to consider only precommitted controls (i.e., the controls are optimal only when viewed at the initial time).

2.3. An example of time-inconsistent optimal control problem. We present a simple illustration of stochastic optimal control problem which is time-inconsistent. Our aim is to show that the classical SMP approach is not efficient in the study of this problem if it's viewed as time-consistent. For $n = d = 1$, consider the following controlled SDE starting from $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{cases} dX^{t,x}(s) = bu(s)\,ds + \sigma dW(s), \; s \in [t, T], \\ X^{t,x}(t) = x, \end{cases} \tag{2.3}$$

where b and σ are real constants. The cost functional is given by

$$J(t, x, u(.)) = \frac{1}{2}\mathbb{E}\left[\int_t^T |u(s)|^2\,ds + h(t)\left(X^{t,x}(T) - x\right)^2\right], \tag{2.4}$$

where $h(.) : [0, T] \to (0, \infty)$, is a general deterministic non-exponential discount function satisfying $h(0) = 1$, $h(s) \geq 0$ and $\int_0^T h(t)\,dt < \infty$. We want to address the following stochastic control problem.

Problem (E). *For any given initial pair $(t, x) \in [0, T] \times \mathbb{R}$, find a control $\bar{u}(.) \in \mathcal{L}^2_{\mathcal{F}}(t, T; \mathbb{R})$ such that*

$$J(t, x, \bar{u}(.)) = \inf_{u(.) \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R})} J(t, x, u(.)).$$

At a first stage, we consider the Problem (E) as a standard time consistent stochastic linear quadratic problem. Since $J(t, x, .)$ is convex and coercive, there exists then a unique optimal control for this problem for each fixed initial pair $(t, x) \in [0, T] \times \mathbb{R}$. Notice that the usual Hamiltonian associated to this problem is $\mathbb{H} : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ such that for every $(s, y, v, p, q) \in [0, T] \times \mathbb{R}^4$ we have

$$\mathbb{H}(s, y, v, p, q) = pbv + \sigma q - \frac{1}{2}v^2,$$

Let $u^{t,x}(.)$ be an admissible control for $(t, x) \in [0, T] \times \mathbb{R}$. Then the corresponding first order and second order adjoint equations are given respectively by

$$\begin{cases} dp^{t,x}(s) = q^{t,x}(s)\,dW(s), \; s \in [t,T], \\ p^{t,x}(T) = -h(t)\left(X^{t,x}(T) - x\right), \end{cases}$$

and

$$\begin{cases} dP^{t,x}(s) = Q^{t,x}(s)\,dW(s), \; s \in [t,T], \\ P^{t,x}(T) = -h(t), \end{cases}$$

the last equation has only the solution $\left(P^{t,x}(s), Q^{t,x}(s)\right) = (-h(t), 0)$, $\forall s \in [t,T]$.

Note that, the corresponding $\mathcal{H}$-function is given by

$$\mathcal{H}(s,y,v) = \mathbb{H}\left(s,y,v,p^{t,x}(s), q^{t,x}(s)\right) = p^{t,x}(s)\,bv + \sigma q^{t,x}(s) - \frac{1}{2}v^2,$$

which is a concave function of v. Then according to the sufficient condition of optimality, see e.g. Theorem 5.2 pp 138 in [16], for any fixed initial pair $(t,x) \in [0,T] \times \mathbb{R}$, Problem (E) is uniquely solvable with an optimal control $\bar{u}^{t,x}(.)$ having the representation

$$\bar{u}^{t,x}(s) = b\bar{p}^{t,x}(s), \; \forall s \in [t,T],$$

such that the process $\left(\bar{p}^{t,x}(.), \bar{q}^{t,x}(.)\right)$ is the unique adapted solution to the BSDE

$$\begin{cases} d\bar{p}^{t,x}(s) = \bar{q}^{t,x}(s)\,dW(s), \; s \in [t,T], \\ \bar{p}^{t,x}(T) = -h(t)\left(\bar{X}^{t,x}(s) - x\right). \end{cases}$$

By stadard arguments we can show that the processes $\left(\bar{p}^{t,x}(.), \bar{q}^{t,x}(.)\right)$ are explicitly given by

$$\begin{cases} \bar{p}^{t,x}(s) = -M^t(s)\left(\bar{X}^{t,x}(s) - x\right), \; s \in [t,T], \\ \bar{q}^{t,x}(s) = -\sigma M^t(s), \; s \in [t,T], \end{cases}$$

where $\bar{X}^{t,x}(.)$ is the solution of the state equation corresponding to $\bar{u}^{t,x}(.)$, given by

$$\begin{cases} d\bar{X}^{t,x}(s) = b^2\bar{p}^{t,x}(s)\,ds + \sigma dW(s), \; s \in [t,T], \\ \bar{X}^{t,x}(t) = x. \end{cases}$$

and

$$M^t(s) = \frac{h(t)}{b^2 h(t)(T-s) + 1}, \; \forall s \in [t,T].$$

A simple computation show that

$$\overline{u}^{t,x}(s) = -\frac{bh(t)}{b^2 h(t)(T-s) + 1}\left(\bar{X}^{t,x}(s) - x\right), \; \forall s \in [t,T],$$

clearly we have

$$\overline{u}^{t,x}(s) \neq 0, \ \forall s \in (t,T]. \tag{2.5}$$

In the next stage, we will prove that the Problem (E) is time-inconsistent, for this we first fix the initial data $(t,x) \in [0,T] \times \mathbb{R}$. Note that, if we assume that the Problem (E) is time-consistent, in the sense that for any $r \in [t,T]$ the restriction of $\bar{u}^{t,x}(.)$ on $[r,T]$ is optimal for Problem (E) with initial pair $\left(r, \bar{X}^{t,x}(r)\right)$, however as Problem (E) is uniquely solvable for any initial pair, we should have then $\forall r \in (t,T]$

$$\bar{u}^{t,x}(s) = \bar{u}^{r,\bar{X}^{t,x}(r)}(s) = -\frac{bh(r)}{b^2 h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s) - \bar{X}^{t,x}(r)\right), \quad \forall s \in [r,T],$$

where $\bar{X}^{r,\hat{X}^{t,x}(r)}(.)$ solves the SDE

$$\begin{cases} d\bar{X}^{r,\bar{X}^{t,x}(r)}(s) = b^2 \dfrac{h(r)}{b^2 h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s) - \bar{X}^{t,x}(r)\right) ds + \sigma dW(s), \quad \forall s \in [r,T], \\ \bar{X}^{r,\bar{X}^{t,x}(r)}(r) = \bar{X}^{t,x}(r). \end{cases}$$

In particular by the uniqueness of solution to the state SDE we should have

$$\bar{u}^{t,x}(r) = -\frac{bh(r)}{b^2 h(r)(T-r)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(r) - \bar{X}^{t,x}(r)\right) = 0,$$

is the only optimal solution of the Problem (E), this contradict (2.5). Therefore, the Problem (E) is not time-consistent, and more precisely, the solution obtained by the classical SMP is wrong and the problem is rather trivial since the only optimal solution equal to zero.

3. Characterization of equilibrium strategies. The purpose of this paper is to characterize open-loop Nash equilibriums instead of optimal controls. We use the game theoretic approach to handle the time inconsistency in the same perspective as Ekeland and Lazrak [6], Bjork and Murgoci [2]. Let us briefly describe the game perspective that we will consider, as follows.

- We consider a game with one player at each point t in $[0,T]$. This player represents the incarnation of the controller at time t and is referred to as "player t".

- The $t-th$ player can control the system only at time t by taking his/her strategy $u(t,.):\Omega\rightarrow\mathbb{R}^m$.

- A control process $u(.)$ is then viewed as a complete description of the chosen strategies of all players in the game.

- The reward to the player t is given by the functional $J(t,\xi,u(.))$, which depends only on the restriction of the control $u(.)$ to the time interval $[t,T]$.

In the above description, we have presented the concept of a "Nash equilibrium point" of the game. This is an admissible control process $\hat{u}(.)$ satisfying the following condition; Suppose that every player s, such that $s>t$, will use the strategy $\hat{u}(s)$. Then the optimal choice for player t is that, he/she also uses the strategy $\hat{u}(t)$.

Nevertheless, the problem with this "definition", is that the individual player t does not really influence the outcome of the game at all. He/she only chooses the control at the single point t, and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. Therefore, to characterize open-loop Nash equilibriums, which have not to be necessary feedback, we follow [9] who suggest the following formal definition inspired by [6] and [7].

Noting that, for brevity, in the rest of the paper, we suppress the subscript (s) for the coefficients $A(s), B(s), b(s), C_j(s), D_j(s), \sigma_j(s)$, and we use the notation $\varrho(z)$ instead of $\varrho(s,z)$ for $\varrho=E,F$ and c. In addition, sometimes we simply call $\hat{u}(.)$ an equilibrium control instead of open-loop Nash equilibrium control when there is no ambiguity.

Following [9], we first consider an equilibrium by local spike variation, given an admissible control $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. For any $t\in[0,T]$, $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$ and for any $\varepsilon>0$, define

$$u^{\varepsilon}(s)=\begin{cases}\hat{u}(s)+v, & \text{for } s\in[t,t+\varepsilon),\\ \hat{u}(s), & \text{for } s\in[t+\varepsilon,T],\end{cases}\tag{3.1}$$

we have the following definition.

Definition 3.1 (Open-loop Nash equilibrium)**.** *An admissible strategy $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ is an open-loop Nash equilibrium control for Problem (LQJ) if*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right)\right\}\geq 0,\tag{3.2}$$

for any $t \in [0,T]$, *and* $v \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$. *The corresponding equilibrium dynamics solves the following SDE with jumps*

$$\begin{cases} d\hat{X}(s) = \left\{ A\hat{X}(s) + B\hat{u}(s) + b \right\} ds + \sum_{j=1}^{d} \left\{ C_j \hat{X}(s) + D_j \hat{u}(s) + \sigma_j \right\} dW^j(s) \\ \qquad\qquad + \int_Z \left\{ E(z)\hat{X}(s-) + F(z)\hat{u}(s) + c(z) \right\} \tilde{N}(ds, dz), \; s \in [0,T], \\ \hat{X}_0 = x_0. \end{cases}$$

3.1. The flow of adjoint equations. We introduce the adjoint equations involved in the stochastic maximum principle which characterize the open-loop Nash equilibrium controls of Problem (LQJ). First, define the Hamiltonian $\mathbb{H} : \mathcal{D}[0,T] \times \mathbb{L}^1(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{L}^2(Z, \mathcal{B}(Z), \theta; \mathbb{R}^n) \to \mathbb{R}$ by

$$\begin{aligned} &\mathbb{H}(t,s,X,u,p,q,r(.)) \\ &= \langle p, AX + Bu + b \rangle + \sum_{j=1}^{d} \langle q_j, D_j X + C_j u + \sigma_j \rangle - \frac{1}{2} \langle R(t,s)u, u \rangle \\ &\quad + \int_Z \langle r(z), E(z)X + F(z)u + c(z) \rangle \theta(dz) \\ &\quad - \frac{1}{2} \left(\langle Q(t,s)X, X \rangle + \left\langle \bar{Q}(t,s)\mathbb{E}^t[X], \mathbb{E}^t[X] \right\rangle \right). \end{aligned} \tag{3.3}$$

Let $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ and denote by $\hat{X}(.)$ the corresponding controlled state process. For each $t \in [0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$, and satisfied by the triple of processes $(p(.;t), q(.;t), r(.,.;t))$ as follows

$$\begin{cases} dp(s;t) = -\Bigg\{ A^\top p(s;t) + \sum_{j=1}^{d} C_j^\top q_j(s;t) + \int_Z E(z)^\top r(s,z;t)\theta(dz) \\ \qquad\qquad -Q(t,s)\hat{X}(s) - \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right] \Bigg\} ds \\ \qquad\qquad + \sum_{j=1}^{d} q_j(s;t)\, dW^j(s) + \int_Z r(s-,z;t)\tilde{N}(ds,dz), \; s \in [t,T], \\ p(T;t) = -G(t)\hat{X}(T) - \bar{G}(t)\mathbb{E}^t\left[\hat{X}(T)\right] - \mu_1(t)\hat{X}(t) - \mu_2(t), \end{cases} \tag{3.4}$$

where $q(.;t) = (q_1(.;t), \ldots, q_d(.;t))$.

Similarly, we introduce the second order adjoint equation defined on the time interval $[t,T]$, and satisfied by the triple of processes $(P(.;t), \Lambda(.;t), \Gamma(.,.;t))$ as follows

$$(3.5)\quad \begin{cases} dP(s;t) = -\Big\{ A^{\top}P(s;t) + P(s;t)A + \sum_{j=1}^{d}\Big(C_j^{\top}P(s;t)C_j + \Lambda_j(s;t)C_j \\ \qquad + C_j^{\top}\Lambda_j(s;t)\Big) + \int_Z E(z)^{\top}(\Gamma(s,z;t) + P(s;t))E(z)\theta(dz) \\ \qquad + \int_Z \Gamma(s,z;t)E(z)\theta(dz) + \int_Z E(z)^{\top}\Gamma(s,z;t)\theta(dz) - Q(t,s)\Big\} ds \\ \quad + \sum_{j=1}^{d}\Lambda_j(s;t)\,dW_s^j + \int_Z \Gamma(s-,z;t)\tilde{N}(ds,dz),\ s\in[t,T], \\ P(T;t) = -G(t), \end{cases}$$

where $\Lambda(.;t) = (\Lambda_1(.;t), ..., \Lambda_d(.;t))$. Under **(H1)** the BSDE (3.4) is uniquely solvable in $\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{n\times d}\right) \times \mathcal{L}^{\theta,2}_{\mathcal{F}}([t,T]\times Z;\mathbb{R}^n)$, see e.g. [12]. Moreover there exists a constant $K>0$ such that

$$(3.6)\quad \mathbb{E}\left[\sup_{t\le s\le T}|p(s;t)|^2_{\mathbb{R}^n}\right] + \mathbb{E}\left[\int_t^T |q(s;t)|^2_{\mathbb{R}^{n\times d}}ds\right] + \mathbb{E}\left[\int_t^T\int_Z |r(s,z;t)|^2_{\mathbb{R}^n}\theta(dz)ds\right] \le K\left(1+|x_0|^2\right).$$

In an other hand, noting that the final data of the equation (3.5) is deterministic, it is straightforward to look at a deterministic solution. In addition we have the following representation

$$(3.7)\quad \begin{cases} dP(s;t) = -\Big\{ A^{\top}P(s;t) + P(s;t)A + \sum_{j=1}^{d}C_j^{\top}P(s;t)C_j \\ \qquad + \int_Z E(z)^{\top}P(s;t)E(z)\theta(dz) - Q(t,s)\Big\} ds,\ s\in[t,T], \\ P(T;t) = -G(t), \end{cases}$$

which is a uniquely solvable matrix-valued ordinary differential equation. Next, for each $t\in[0,T]$, associated with the 6-tuple $(\hat{u}(.), \hat{X}(.), p(.;t), q(.,t), r(.,.;t),$

$P(.;t))$ we define the $\mathcal{H}_t$-function as follows

$$\begin{aligned}\mathcal{H}_t(s,X,u) &= \mathbb{H}(t,s,X,\hat{u}(s)+u,p(s;t),q(s;t),r(s,.;t))\\ &+\frac{1}{2}u^\top\left\{\sum_{j=1}^{d}D_j^\top P(s;t)D_j+\int_Z F(z)^\top P(s;t)F(z)\theta(dz)\right\}u,\end{aligned}\tag{3.8}$$

where $(s,X,u)\in[t,T]\times\mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)\times\mathbb{R}^m$. In the rest of the paper, we will keep the following notation, for $(s,t)\in\mathcal{D}[0,T]$

$$\begin{aligned}\delta\mathbb{H}(t;s) &= \mathbb{H}\left(t,s,\hat{X}(s),\hat{u}(s)+u,p(s;t),q(s;t),r(s,.;t)\right)\\ &-\mathbb{H}\left(t,s,\hat{X}(s),\hat{u}(s),p(s;t),q(s;t),r(s,.;t)\right).\end{aligned}$$

3.2. A stochastic maximum principle for equilibrium controls. In this section, we present a version of Pontryagin's stochastic maximum principle which characterize the equilibrium controls of the Problem (LQJ). We derive the result by using the second order Taylor expansion in the special form spike variation (3.1). Here, we don't assume the non-negativity condition about the matrices Q, G and R as in [9] and [25].

The following theorem is the first main result of this work, it's providing a necessary and sufficient condition to characterize the open-loop Nash equilibrium controls for time-inconsistent Problem (LQJ).

Theorem 3.2 (Stochastic Maximum Principle For Equilibriums)**.** *Let* **(H1)** *holds. Then an admissible control $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ is an open-loop Nash equilibrium, if and only if, for any $t\in[0,T]$, there exist a unique triple of adapted processes $(p(.;t),q(.;t),r(.,.;t))$ which satisfy the BSDE* (3.4) *and a deterministic matrix-valued function $P(.;t)$ which satisfies the ODE* (3.7)*, such that the following condition holds, for all $u\in\mathbb{R}^m$*

$$\delta\mathbb{H}(t;t)+\frac{1}{2}u^\top\left\{\sum_{j=1}^{d}D_j^\top P(t;t)D_j+\int_Z F(z)^\top P(t;t)F(z)\theta(dz)\right\}u\leq 0,\quad \mathbb{P}-a.s.\tag{3.9}$$

Or equivalently, we have the following two conditions. The first order equilibrium condition

$$R(t,t)\hat{u}(t)-B^\top p(t;t)-\sum_{j=1}^{d}D_j^\top q_j(t;t)\tag{3.10}$$

$$- \int_Z F(z)^\top r(t,z;t)\,\theta(dz) = 0, \ \mathbb{P}\text{−}a.s,$$

and the second order equilibrium condition

$$R(t,t) - \sum_{j=1}^{d} D_j^\top P(t;t) D_j - \int_Z F(z)^\top P(t;t) F(z)\,\theta(dz) \geq 0. \tag{3.11}$$

Remark 3.3. Note that for each $t \in [0,T]$, (3.4) and (3.5) are backward stochastic differential equations. So, as we consider all t in $[0,T]$, all their corresponding adjoint equations form essentially a "flow" of BSDEs. Moreover, there is an additional constraint (3.9) which is equivalent to the conditions (3.10) and (3.11) that acts on the flow only when $s = t$, while the Pontryagin's stochastic maximum principle for optimal control involves only one system of forward-backward stochastic differential equation.

3.2.1. Proof of the Theorem 3.2. Our goal now, is to give a proof of the Theorem 3.2. The main idea is still based on the variational techniques in the same spirit of proving the stochastic Pontryagin's maximum principle [19].

Let $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ be an admissible control and $\hat{X}(.)$ the corresponding controlled process solution to the state equation. Consider the perturbed control $u^\varepsilon(.)$ defined by the spike variation (3.1) for some fixed arbitrary $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$ and $\varepsilon \in [0,T-t]$. Denote by $\hat{X}^\varepsilon(.)$ the solution of the state equation corresponding to $u^\varepsilon(.)$. Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, see e.g. [19], we have

$$\hat{X}^\varepsilon(s) - \hat{X}(s) = y^{\varepsilon,v}(s) + z^{\varepsilon,v}(s), \ s \in [t,T], \tag{3.12}$$

where $y^{\varepsilon,v}(.)$ and $z^{\varepsilon,v}(.)$ solve the following linear stochastic differential equations, respectively

$$\begin{cases} dy^{\varepsilon,v}(s) = Ay^{\varepsilon,v}(s)\,ds + \sum_{j=1}^{d} \left\{ C_j y^{\varepsilon,v}(s) + D_j v 1_{[t,t+\varepsilon)}(s) \right\} dW^j(s) \\ \qquad + \int_Z \left\{ E(z)\, y^{\varepsilon,v}(s-) + F(z)\, v 1_{[t,t+\varepsilon)}(s) \right\} \tilde{N}(ds,dz), \ s \in [t,T], \\ y^{\varepsilon,v}(t) = 0, \end{cases} \tag{3.13}$$

and

$$\begin{cases} dz^{\varepsilon,v}(s) = \left\{ Az^{\varepsilon,v}(s) + Bv1_{[t,t+\varepsilon)}(s) \right\} ds + \sum_{j=1}^{d} C_j z^{\varepsilon,v}(s)\, dW^j(s) \\ \qquad + \int_Z E(z)\, z^{\varepsilon,v}(s)\, \tilde{N}(ds,dz)\,,\ s \in [t,T]\,, \\ z^{\varepsilon,v}(t) = 0. \end{cases} \tag{3.14}$$

First, we present the following technical lemma needed later in this study, see the Appendix for its proof.

Lemma 3.4. *Under assumption* **(H1)**, *the following estimates hold*

$$\mathbb{E}^t\left[y^{\varepsilon}(s)\right] = 0,\ \ a.e.\ \ s \in [t,T]\ \ and\ \sup_{s\in[t,T]} \left|\mathbb{E}^t\left[z^{\varepsilon}(s)\right]\right|^2 = O\left(\varepsilon^2\right), \tag{3.15}$$

$$\mathbb{E}^t \sup_{s\in[t,T]} \left|y^{\varepsilon}(s)\right|^2 = O(\varepsilon)\ \ and\ \mathbb{E}^t \sup_{s\in[t,T]} \left|z^{\varepsilon}(s)\right|^2 = O\left(\varepsilon^2\right). \tag{3.16}$$

Moreover, we have the equality

$$\begin{aligned} & J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right) - J\left(t,\hat{X}(t),\hat{u}(.)\right) \\ & = -\mathbb{E}^t\left[\int_t^T \left\{ \delta\mathbb{H}(t;s) + \frac{1}{2}v^{\top}\left(\sum_{j=1}^{d} D_j^{\top} P(s;t)\, D_j \right.\right.\right. \\ & \qquad \left.\left.\left. + \int_Z F(z)^{\top} P(s;t)\, F(z)\, \theta(dz)\right) v \right\} 1_{[t,t+\varepsilon)}(s)\, ds \right] + o(\varepsilon)\,. \end{aligned} \tag{3.17}$$

Now, we are ready to give the proof of the Theorem 3.2.

Proof of Theorem 3.2. Given an open-loop Nash equilibrium $\hat{u}(.)$, then for any $t \in [0,T]$ and $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$, we have clearly

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left\{ J\left(t,\hat{X}(t),\hat{u}(.)\right) - J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)\right\} \leq 0,$$

which leads from (3.17) to

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}^t & \left[\int_t^T \left\{ \delta\mathbb{H}(t;s) + \frac{1}{2}v^{\top}\sum_{j=1}^{d} D_j^{\top} P(s;t)\, D_j \right.\right. \\ & \left.\left. + \frac{1}{2}v^{\top}\int_Z F(z)^{\top} P(s;t)\, F(z)\, \theta(dz)\, v \right\} 1_{[t,t+\varepsilon)}(s)\, ds \right] \leq 0, \end{aligned}$$

from which we deduce

$$\delta\mathbb{H}(t;t)+\frac{1}{2}v^{\top}\left(\sum_{j=1}^{d}D_j^{\top}P(t;t)D_j+\int_Z F(z)^{\top}P(t;t)F(z)\theta(dz)\right)v\leq 0,\ \mathbb{P}-a.s,$$

Therefore, the inequality (3.9) is ensured by stetting $v\equiv u$ for an arbitrarily $u\in\mathbb{R}^m$.

Conversely, given an admissible control $\hat{u}(.)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. Suppose that for any $t\in[0,T]$, the variational inequality (3.9) holds. Then for any $v\in\mathbb{L}^2(\Omega,\mathcal{F}(t),\mathbb{P};\mathbb{R}^m)$ it yields

$$\delta\mathbb{H}(t;t)+\frac{1}{2}v^{\top}\left(\sum_{j=1}^{d}D_j^{\top}P(t;t)D_j+\int_Z F(z)^{\top}P(t;t)F(z)\theta(dz)\right)v\leq 0,\ \mathbb{P}-a.s,$$

consequently

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{\delta\mathbb{H}(t;s)+\frac{1}{2}v^{\top}\left(\sum_{j=1}^{d}D_j^{\top}P(s;t)D_j\right.\right.\right.$$
$$\left.\left.\left.+\int_Z F(z)^{\top}P(s;t)F(z)\theta(dz)\right)v\right\}ds\right]\leq 0.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left\{J\left(t,\hat{X}(t),\hat{u}(.)\right)-J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)\right\}\leq 0.$$

Thus $\hat{u}(.)$ is an equilibrium control.

Easy manipulations show that the variational inequality (3.9) is equivalent to

$$\mathcal{H}_t\left(t,\hat{X}(t),0\right)=\max_{u\in\mathbb{R}^m}\mathcal{H}_t\left(t,\hat{X}(t),u\right),$$

then (3.10) and (3.11) follow respectively from the following first order and second order conditions at the maximum point $u=0$ for the quadratic function $\mathcal{H}_t\left(t,\hat{X}(t),u\right)$

$$\mathcal{D}_u\mathcal{H}_t\left(t,\hat{X}(t),0\right)=0\text{ and }\mathcal{D}_u^2\mathcal{H}_t\left(t,\hat{X}(t),u\right)\leq 0,$$

where we denote by $\mathcal{D}_u\mathcal{H}_t$ (resp. $\mathcal{D}_u^2\mathcal{H}_t$) the gradient (resp. the Hessian) of $\mathcal{H}_t$ with respect to the variable u. Then, the required result is directly follows. □

In Theorem 3.2, in view of condition (3.9), as long as the term

$$-\sum_{j=1}^{d} D_j^{\top} P\left(t;t\right) D_j - \int_Z F\left(z\right)^{\top} P\left(t;t\right) F\left(z\right) \theta\left(dz\right),$$

for each $t \in [0,T]$ is sufficiently positive definite, the necessary and sufficient condition for equilibriums might still be satisfied even if $R\left(t,t\right)$ is negative. This is different from [9] and [25] where the authors have assumed the non-negativity of the matrices Q, G and R in order to state their stochastic maximum principle for open-loop Nash equilibriums. Moreover, in the case where $Q\left(t,s\right) \geq 0$ for every $s \in [t,T]$, and $G\left(t\right) \geq 0$, it follows that the solution of the second order adjoint equation satisfies $P\left(t;t\right) \leq 0$, then if further we have $R\left(t,t\right) \geq 0$, Thus the condition that

$$R\left(t,t\right) - \sum_{j=1}^{d} D_j^{\top} P\left(t;t\right) D_j - \int_Z F\left(z\right)^{\top} P\left(t;t\right) F\left(z\right) \theta\left(dz\right) \geq 0,$$

is obviously satisfied. Therefore, we summarize the main theorem into the following Corollary.

Corollary 3.5. *Let* **(H1)–(H2)** *hold. Then an admissible control* $\hat{u}\left(.\right) \in \mathcal{L}_{\mathcal{F}}^2\left(0,T;\mathbb{R}^m\right)$ *is an equilibrium control, if and only if, for any* $t \in [0,T]$*, there exists a triple of adapted processes* $\left(p\left(.;t\right), q\left(.;t\right), r\left(.,.;t\right)\right)$ *which satisfies the BSDE* (3.4)*, with only the first order condition* (3.10) *holds.*

4. Linear feedback stochastic equilibrium control. In this section, we consider only the case where the Brownian motion is one-dimensional ($d = 1$) for simplicity of presentation. There is no essential difficulty with the multidimensional Brownian motions. All the indices j will then be dropped. Our goal is to obtain a state feedback representation of an equilibrium control for Problem (LQJ) via some class of ordinary differential equations.

We first consider the following system of coupled generalized Riccati equations, for $(t,s) \in \mathcal{D}\left[0,T\right]$

$$
(4.1)\quad \begin{cases}
0 = \dfrac{\partial M}{\partial s}(t,s) + M(t,s)A + A^{\top}M(t,s) + C^{\top}M(t,s)C \\
\quad +\displaystyle\int_Z E(z)^{\top} M(t,s)E(z)\theta(dz) \\
\quad -\left(M(t,s)B + C^{\top}M(t,s)D + \displaystyle\int_Z E(z)^{\top} M(t,s)F(z)\theta(dz)\right)\Psi(s) \\
\quad +Q(t,s), \\
0 = \dfrac{\partial \bar{M}}{\partial s}(t,s) + \bar{M}(t,s)A + A^{\top}\bar{M}(t,s) - \bar{M}(t,s)B\Psi(s) + \bar{Q}(t,s), \\
0 = \dfrac{\partial \Upsilon}{\partial s}(t,s) + A^{\top}\Upsilon(t,s), \\
0 = \dfrac{\partial \varphi}{\partial s}(t,s) + \left(M(t,s) + \bar{M}(t,s)\right)(b - B\psi(s)) + A^{\top}\varphi(t,s) \\
\quad +C^{\top}M(t,s)(\sigma - D\psi(s)) \\
\quad +\displaystyle\int_Z E(z)^{\top} M(t,s)(c(z) - F(z)\psi(s))\theta(dz), \\
M(t,T) = G(t),\ \bar{M}(t,T) = \bar{G}(t),\ \Upsilon(t,T) = \mu_1(t), \\
\quad \varphi(t,T) = \mu_2(t), t \in [0,T],
\end{cases}
$$

where

$$
\det\left(R(t,t) + D^{\top}M(t,t)D + \int_Z F(z)^{\top} M(t,t)F(z)\theta(dz)\right) \neq 0,\ \forall t \in [0,T],
$$

The maps $\Theta(.)$, $\Psi(.)$ and $\psi(.)$ are given for $t \in [0,T]$ by

$$
(4.2)\quad \begin{cases}
\Theta(t) = \left(R(t,t) + D^{\top}M(t,t)D + \displaystyle\int_Z F(z)^{\top} M(t,t)F(z)\theta(dz)\right)^{-1}, \\
\Psi(t) = \Theta(t)\left\{B^{\top}\left(M(t,t) + \bar{M}(t,t) + \Upsilon(t,t)\right) + D^{\top}M(t,t)C \right. \\
\qquad\qquad \left. +\displaystyle\int_Z F(z)^{\top} M(t,t)E(z)\theta(dz)\right\}, \\
\psi(t) = \Theta(t)\left\{B^{\top}\varphi(t,t) + D^{\top}M(t,t)\sigma \right. \\
\qquad\qquad \left. +\displaystyle\int_Z F(z)^{\top} M(t,t)c(z)\theta(dz)\right\}.
\end{cases}
$$

Theorem 4.1. *Let* **(H1)–(H2)** *hold. If there exists a solution to the system* (4.1)*. Then the stochastic control problem* (2.2) *subject to the SDE* (2.1)*, has a feedback Nash equilibrium solution*

$$\hat{u}(t) = -\Psi(t)\hat{X}(t) - \psi(t), \ \forall t \in [0,T]. \tag{4.3}$$

Proof. Suppose that $\hat{u}(.)$ is an equilibrium control and denote by $\hat{X}(.)$ the corresponding controlled process. Then in view of Theorem 3.2, there exists an adapted process $\left(\hat{X}(.), (p(.;t), q(.;t), r(.,.;t))_{t\in[0,T)}\right)$ solution to the following forward-backward SDE with jumps, parametrized by $t \in [0,T]$

$$\begin{cases} d\hat{X}(s) = \left\{A\hat{X}(s) + B\hat{u}(s) + b\right\}ds + \left\{C\hat{X}(s) + D\hat{u}(s) + \sigma\right\}dW(s) \\ \qquad + \int_Z \left\{E(z)\hat{X}(s-) + F(z)\hat{u}(s) + c(z)\right\}\tilde{N}(ds,dz), \ s \in [0,T], \\ dp(s;t) = -\left\{A^\top p(s;t) + C^\top q(s;t) + \int_Z E(z)^\top r(s,z;t)\,\theta(dz) \right. \\ \qquad \left. -Q(t,s)\hat{X}(s) - \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right]\right\}ds \\ \qquad + q(s;t)\,dW(s) + \int_Z r(s-,z;t)\,\tilde{N}(ds,dz), \ s \in [t,T], \\ \hat{X}_0 = x_0, \ p(T;t) = -G(t)\hat{X}(T) - \bar{G}(t)\mathbb{E}^t\left[\hat{X}(T)\right] \\ \qquad - \mu_1(t)\hat{X}(t) - \mu_2(t), \ t \in [0,T], \end{cases} \tag{4.4}$$

such that the following condition holds

$$R(t,t)\hat{u}(t) - B^\top p(t;t) - D^\top q(t;t) - \int_Z F(z)^\top r(t,z;t)\,\theta(dz) = 0, \quad \mathbb{P}-a.s, \ \forall t \in [0,T]. \tag{4.5}$$

Now, to solve the above stochastic system, we conjecture that $\hat{X}(.)$ and $p(.;t)$ for $t \in [0,T)$ are related by the following relation

$$p(s;t) = -M(t,s)\hat{X}(s) - \bar{M}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right] - \Upsilon(t,s)\hat{X}(t) - \varphi(t,s), \ \forall (t,s) \in \mathcal{D}[0,T], \tag{4.6}$$

for some deterministic functions $M(.,.), \bar{M}(.,.), \Upsilon(.,.) \in C^{0,1}\left(\mathcal{D}[0,T], \mathbb{R}^{n\times n}\right)$ and $\varphi(.,.) \in C^{0,1}\left(\mathcal{D}[0,T], \mathbb{R}^n\right)$ such that

$$M(t,T) = G(t), \ \bar{M}(t,T) = \bar{G}(t), \ \Upsilon(t,T) = \mu_1(t), \ \varphi(t,T) = \mu_2(t), \ \forall t \in [0,T]. \tag{4.7}$$

Applying Itô's formula to (4.6) and using (4.4), it yields

$$
\begin{aligned}
dp(s;t) = &\Big\{ -\frac{\partial M}{\partial s}(t,s)\hat{X}(s) - \frac{\partial \bar{M}}{\partial s}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big] - \frac{\partial \Upsilon}{\partial s}(t,s)\hat{X}(t) \\
&- \frac{\partial \varphi}{\partial s}(t,s) - M(t,s)\Big(A\hat{X}(s) + Bu(s) + b\Big) \\
&- \bar{M}(t,s)\Big(A\mathbb{E}^t\Big[\hat{X}(s)\Big] + B\mathbb{E}^t[u(s)] + b\Big)\Big\} ds \\
&- M(t,s)\Big(C\hat{X}(s) + D\hat{u}(s) + \sigma\Big) dW(s) \\
&- \int_Z M(t,s)\Big(E(z)\hat{X}(s-) + F(z)\hat{u}(s) + c(z)\Big)\tilde{N}(ds,dz), \\
= &-\Big\{A^\top p(s;t) + C^\top q(s;t) + \int_Z E(z)^\top r(s,z;t)\theta(dz) \\
&\quad -Q(t,s)\hat{X}(s) - \bar{Q}(t,s)\mathbb{E}^t\Big[\hat{X}(s)\Big]\Big\} ds + q(s;t)\,dW(s) \\
&\quad + \int_Z r(s-,z;t)\tilde{N}(ds,dz),\ \forall s \in [t,T],
\end{aligned} \tag{4.8}
$$

from which we deduce

$$
q(s;t) = -M(t,s)\Big(C\hat{X}(s) + D\hat{u}(s) + \sigma\Big), \tag{4.9}
$$

$$
r(s,z;t) = -M(t,s)\Big(E(z)\hat{X}(s) + F(z)\hat{u}(s) + c(z)\Big). \tag{4.10}
$$

We put the above expressions of $q(s;t)$ and $r(s,z;t)$ into (4.5), then

$$
\begin{aligned}
0 = &R(t,t)\hat{u}(t) + B^\top\Big(\big(M(t,t) + \bar{M}(t,t) + \Upsilon(t,t)\big)\hat{X}(t) + \varphi(t,t)\Big) \\
&+ D^\top M(t,t)\Big(C\hat{X}(t) + D\hat{u}(t) + \sigma\Big) \\
&+ \int_Z F(z)^\top M(t,t)\Big(E(z)\hat{X}(t) + F(z)\hat{u}(t) + c(z)\Big)\theta(dz),
\end{aligned}
$$

Subsequently, we obtain with the above notations

$$
\Theta(t)^{-1}\Big(\hat{u}(t) + \Psi(t)\hat{X}(t) + \psi(t)\Big) = 0,\ \forall t \in [0,T].
$$

Hence (4.3) holds, and for any $(t,s) \in \mathcal{D}[0,T]$, we have

$$
\mathbb{E}^t[\hat{u}(s)] = -\Psi(s)\mathbb{E}^t\Big[\hat{X}(s)\Big] - \psi(s). \tag{4.11}
$$

Next, comparing the ds term in (4.8) by the one in the BSDE from (4.4), then by using the expressions (4.3) and (4.11), we obtain

$$\begin{aligned}
0 = &\left\{ \frac{\partial M}{\partial s}(t,s) + M(t,s)A + A^\top M(t,s) + C^\top M(t,s)C \right.\\
&+ \int_Z E(z)^\top M(t,s)E(z)\theta(dz)\\
&- \Big(M(t,s)B + C^\top M(t,s)D\\
&\left. + \int_Z E(z)^\top M(t,s)F(z)\theta(dz) \Big) \Psi(s) + Q(t,s) \right\} \hat{X}(s)\\
&+ \left\{ \frac{\partial \bar{M}}{\partial s}(t,s) + \bar{M}(t,s)A + A^\top \bar{M}(t,s) \right.\\
&\left. - \bar{M}(t,s)B\Psi(s) + \bar{Q}(t,s) \right\} \mathbb{E}^t\left[\hat{X}(s)\right]\\
&+ \left\{ \frac{\partial \Upsilon}{\partial s}(t,s) + A^\top \Upsilon(t,s) \right\} \hat{X}(t)\\
&+ \frac{\partial \varphi}{\partial s}(t,s) + \big(M(t,s) + \bar{M}(t,s) \big)(b - B\psi(s)) + A^\top \varphi(t,s)\\
&+ C^\top M(t,s)(\sigma - D\psi(s)) + \int_Z E(z)^\top M(t,s)(c(z) - F(z)\psi(s))\theta(dz).
\end{aligned}$$

This suggests that the functions $M(.,.)$, $\bar{M}(.,.)$, $\Upsilon(.,.)$ and $\varphi(.,.)$ solve the system (4.1).

Note that, we can check that $\Psi(.)$ and $\psi(.)$ in (4.2) are both uniformly bounded. Then the following linear SDE, for $s \in [0,T]$

$$\begin{cases}
d\hat{X}(s) = \left\{ (A - B\Psi(s))\hat{X}(s) + b - B\psi(s) \right\} ds\\
\qquad + \left\{ (C - D\Psi(s))\hat{X}(s) + \sigma - D\psi(s) \right\} dW(s)\\
\qquad + \int_Z \left\{ (E(z) - F(z)\Psi(s))\hat{X}(s-) + c(z) - F(z)\psi(s) \right\} \widetilde{N}(ds,dz),\\
\hat{X}(0) = x_0,
\end{cases}$$

is uniquely solvable, and the following estimate holds

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left| \hat{X}(s) \right|^2 \right] \leq K\left(1 + x_0^2\right).$$

So the control $\hat{u}(.)$ defined by (4.3) is admissible. □

Remark 4.2. Note that, the verification theorem (Theorem 4.1) assumes the existence of a solution to the system (4.1). However, since the ODEs which should be solved by $M(.,.)$ and $\bar{M}(.,.)$ do not have a symmetry structure. The general solvability for this type of ODEs when $(n > 1)$ remains an outstanding open problem. We will see in the next section two examples in the case when $n = 1$, this case is important, especially in financial applications as will be confirmed by the mean–variance portfolio selection model. Also, we remark that a special feature of the case when $n = 1$ is that the state $X(.)$ is one-dimensional, so are the unknown $M(.,.), \bar{M}(.,.), \Upsilon(.,.)$ and $\varphi(.,.)$ of the system (4.1). This makes it easier to solve (4.1).

5. Some applications.

5.1. Mean-variance portfolio selection problem. In this subsection, we discuss the continuous-time Markowitz's mean-variance portfolio selection problem. We apply Theorem 4.1 to obtain a state feedback representation of an equilibrium control for the problem. In the absence of Poisson random jumps this problem is discussed in [9].

The problem is formulated as follows: We consider a financial market, in which two securities are traded continuously. One of them is a bond, with price $S^0(s)$ at time $s \in [0,T]$ governed by

$$dS^0(s) = S^0(s)\, r(s)\, ds,\ \ S^0(0) = s_0 > 0. \tag{5.1}$$

There is also a stock with unit price $S^1(s)$ at time $s \in [0,T]$ governed by

$$dS^1(s) = S^1(s-)\left(\alpha(s)\, ds + \beta(s)\, dW(s) + \int_{\mathbb{R}^*} \gamma(s,z)\, \widetilde{N}(ds,dz)\right), \qquad S^1(0) = s^1 > 0. \tag{5.2}$$

where $r : [0,T] \to (0,\infty)$, $\alpha, \beta : [0,T] \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R}^* \to \mathbb{R}$ are assumed to be deterministic and continuous, we also assume a uniform ellipticity condition as follow $\sigma(t)^2 + \int_Z \gamma(t,z)^2\, \theta(dz) \geq \delta$, *a.e*, for some $\delta > 0$. For an investor, a portfolio $\pi(.)$ is a process represents the amount of money invested in the stock. The wealth process $X^{x_0,\pi(.)}(.)$ corresponding to initial capital $x_0 > 0$, and portfolio $\pi(.)$, satisfies then the equation

$$\begin{cases} dX(s) = \left(r(s)\, X(s) + \pi(s)\left(\alpha(s) - r(s)\right)\right) ds + \pi(s)\, \beta(s)\, dW(s) \\ \qquad\qquad + \pi(s) \int_{\mathbb{R}^*} \gamma(s,z)\, \widetilde{N}(ds,dz),\ \text{ for } t \in [0,T], \\ X(0) = x_0. \end{cases} \tag{5.3}$$

As time evolves, we need to consider the controlled stochastic differential equation parametrized by $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$ and satisfiied by $X(.)$

$$
(5.4)\quad \begin{cases} dX(s) = (r(s)X(s) + \pi(s)(\alpha(s) - r(s)))\,ds + \pi(s)\beta(s)\,dW(s) \\ \qquad\qquad + \pi(s)\displaystyle\int_{\mathbb{R}^*}\gamma(s,z)\widetilde{N}(ds,dz),\ \text{for } s \in [t,T], \\ X(t) = \xi. \end{cases}
$$

The objective is to maximize the conditional expectation of terminal wealth $\mathbb{E}^t[X(T)]$, and at the same time to minimize the conditional variance of the terminal wealth $\mathrm{Var}^t[X(T)]$, over controls $\pi(.)$ valued in $\mathbb{R}$. Then, the mean-variance portfolio optimization problem is denoted as: minimizing the cost $J(t,\xi,.)$, given by

$$
(5.5)\qquad J(t,\xi,\pi(.)) = \frac{1}{2}\mathrm{Var}^t[X(T)] - (\mu_1(t)\xi + \mu_2(t))\,\mathbb{E}^t[X(T)],
$$

subject to (5.4). Here $\mu_1, \mu_2 : [0,T] \to (0,\infty)$, are some deterministic non constant, continuous and bounded functions. The above model cover the one in [9], since, in our case, the weight between the conditional variance and the conditional expectation depends on the current wealth level, as well as, the current time, while in [9] the weight depends on the current wealth level only. Hence, in the above model, there are three different sources of time-inconsistency. Moreover, the above model is mathematically a special case of the general LQ problem formulated earlier in this paper, with $n = d = m = 1$. Then we can apply Theorem 4.1 to obtain a Nash equilibrium control. We recall that, the definition of equilibrium controls is in the sense of open-loop, which is different from the feedback one in [3], [4] and [27].

The optimal control problem associated with (5.4) and (5.5) is equivalent to minimize

$$
J(t,\xi,u(.)) = \frac{1}{2}\left(\mathbb{E}^t\left[X(T)^2\right] - \mathbb{E}^t[X(T)]^2\right) - (\mu_1(t)\xi + \mu_2(t))\,\mathbb{E}^t[X(T)]
$$

subject to (5.4). Denote

$$
\rho(t) = \frac{(\alpha(s) - r(s))^2}{\beta(t)^2 + \int_{\mathbb{R}^*}\gamma(t,z)^2\,\theta(dz)}.
$$

Thus, the system (4.1) reduces for $(t,s) \in \mathcal{D}[0,T]$ to the following system

$$
(5.6)\quad \begin{cases}
\frac{\partial M}{\partial s}(t,s) \\
\qquad + \left\{ 2r(s) - \frac{\rho(s)}{M(s,s)} \left(M(s,s) + \bar{M}(s,s) + \Upsilon(s,s) \right) \right\} M(t,s) = 0, \\
\frac{\partial \bar{M}}{\partial s}(t,s) \\
\qquad + \left\{ 2r(s) - \frac{\rho(s)}{M(s,s)} \left(M(s,s) + \bar{M}(s,s) + \Upsilon(s,s) \right) \right\} \bar{M}(t,s) = 0, \\
\frac{\partial \Upsilon}{\partial s}(t,s) + r(s)\Upsilon(t,s) = 0, \\
\frac{\partial \varphi}{\partial s}(t,s) + r(s)\varphi(t,s) = 0, \\
M(t,T) = 1,\ \bar{M}(t,T) = -1,\ \Upsilon(t,T) = -\mu_1(t), \\
\varphi(t,T) = -\mu_2(t),\ \forall t \in [0,T].
\end{cases}
$$

Clearly, if $M(.,.)$ and $\bar{M}(.,.)$ are solutions to the first and the second equations, respectively, in (5.6), then $\tilde{M}(.,.) = \left(\bar{M} + M\right)(.,.)$ solves the following ODE, $\forall (t,s) \in \mathcal{D}[0,T]$

$$
(5.7)\quad \begin{cases}
\frac{\partial \tilde{M}}{\partial s}(t,s) + \left\{ 2r(s) - \frac{\rho(s)}{M(s,s)} \left(\tilde{M}(s,s) + \Upsilon(s,s) \right) \right\} \tilde{M}(t,s) = 0, \\
\tilde{M}(t,T) = 0,\ t \in [0,T],
\end{cases}
$$

which is equivalent to

$$
\tilde{M}(t,s) = \tilde{M}(t,T)\, e^{\int_s^T \left(2r(\tau) - \frac{\rho(\tau)}{M(\tau,\tau)} \left(\tilde{M}(\tau,\tau) + \Upsilon(\tau,\tau) \right) \right) d\tau},
$$

from the boundary condition in (5.7), it yields

$$
\bar{M}(t,s) + M(t,s) = \tilde{M}(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T].
$$

Moreover, we remark that all data of the ODEs which should be solved by $M(.,.)$ and $\bar{M}(.,.)$ are not influenced by t, thus (5.6) reduces to

$$
(5.8)\quad \begin{cases}
\frac{dM}{ds}(s) + 2r(s)M(s) - \rho(s)\Upsilon(s,s) = 0,\ \forall s \in [0,T], \\
\bar{M}(s) = -M(s),\ \forall s \in [0,T], \\
\frac{\partial \Upsilon}{\partial s}(t,s) + r(s)\Upsilon(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T], \\
\frac{\partial \varphi}{\partial s}(t,s) + r(s)\varphi(t,s) = 0,\ \forall (t,s) \in \mathcal{D}[0,T], \\
M(T) = 1,\ \Upsilon(t,T) = -\mu_1(t),\ \varphi(t,T) = -\mu_2(t),\ \forall t \in [0,T].
\end{cases}
$$

which is explicitly solved by

$$
(5.9)\quad \begin{cases} M(s) = e^{2\int_s^T r(\tau)d\tau}\left\{1+\int_s^T e^{-\int_\tau^T r(l)dl}\mu_1(\tau)\rho(\tau)\,d\tau\right\},\ \forall s\in[0,T], \\ \bar{M}(s) = -e^{2\int_s^T r(\tau)d\tau}\left\{1+\int_s^T e^{-\int_\tau^T r(l)dl}\mu_1(\tau)\rho(\tau)\,d\tau\right\},\ \forall s\in[0,T], \\ \Upsilon(t,s) = -\mu_1(t)\,e^{\int_s^T r(\tau)d\tau},\ \forall(t,s)\in\mathcal{D}[0,T], \\ \varphi(t,s) = -\mu_2(t)\,e^{\int_s^T r(\tau)d\tau},\ \forall(t,s)\in\mathcal{D}[0,T]. \end{cases}
$$

In view of Theorem 4.1, the representation of the Nash equilibrium control (4.3) then gives

$$
(5.10)\qquad \hat{\pi}(s) = -\Psi(s)\hat{X}(s) - \psi(s),\forall s\in[0,T],
$$

where, $\forall s\in[0,T]$

$$
\Psi(s) = \frac{\rho(s)}{(\alpha(s)-r(s))}\frac{\Upsilon(s,s)}{M(s)} \text{ and } \psi(s) = \frac{\rho(s)}{(\alpha(s)-r(s))}\frac{\varphi(s,s)}{M(s)}.
$$

The corresponding equilibrium dynamics solves, for $s\in[0,T]$, the folowing SDEJ

$$
\begin{cases} d\hat{X}(s) = \Big\{(r(s)-\Psi(s)(\alpha(s)-r(s)))\hat{X}(s)-\psi(s)(\alpha(s)-r(s))\Big\}ds \\ \qquad\qquad -\Big(\Psi(s)\hat{X}(s)+\psi(s)\Big)\Big\{\beta(s)\,dW(s)+\int_{\mathbb{R}^*}\gamma(s,z)\widetilde{N}(ds,dz)\Big\}, \\ \hat{X}(0) = x_0. \end{cases}
$$

5.1.1. Special cases and relationship to other works. Equilibrium investment strategies for mean–variance models have been studied in [1], [2] and [9] among others in different frameworks. In this paragraph, we will compare our results with some existing ones in literature. First, suppose that the price process of the risky asst do not have jumps, i.e $\gamma(s,z)=0$ a.e.

Special case 1. When $\mu_1(t)\equiv 0$ and $\mu_2(t)\equiv\mu_2>0$. In this case the objective is equivalent to Basak and Chabakauri [1] and Bjork and Murguci [2] in which the equilibrium is defined within the class of feedback controls. Moreover the equilibrium strategy $\hat{\pi}(.)$ given in our study by (5.10) change to

$$
\hat{\pi}(s) = \mu_2\frac{(\alpha(s)-r(s))}{\beta(t)^2}e^{-\int_s^T r(\tau)d\tau},\ s\in[0,T].
$$

It is worth pointing out that the above equilibrium solution is the same form as that obtained in Bjork and Murguci [2] by solving the extended HJB equations.

Special case 2. Suppose that $\mu_1(t) \equiv \mu_2 > 0$ and $\mu_2(t) \equiv \mu_2 \equiv 0$. In this case, the equilibrium strategy $\hat{\pi}(.)$ given by expressions (5.10) change to

$$\hat{\pi}(s) = \frac{\mu_1(\alpha(s) - r(s))}{\left(1 + \mu_1 \int_s^T e^{-\int_\tau^T r(l)dl} \rho(\tau) d\tau\right) \beta(s)^2} e^{-\int_s^T r(\tau)d\tau} \hat{X}(s),$$

which is the same as the solution obtained in Hu et al [9] with one risky asset.

5.2. General discounting LQ regulator. In this subsection, we consider an example of a general discounting time-inconsistent LQ model. The objective is to minimize the expected cost functional, that is earned during a finite time horizon

$$J(t, \xi, u(.)) = \frac{1}{2}\mathbb{E}^t \left[\int_t^T |u(s)|^2 ds + h(t) |X(T) - \xi|^2\right] \tag{5.11}$$

where $h(.) : [0,T] \to (0, \infty)$, is a general deterministic non-exponential discount function satisfying $h(0) = 1$, $h(s) \geq 0$ and $\int_0^T h(t) dt < \infty$. Subject to a controlled one dimontional SDE parametrized by $(t, \xi) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$

$$\begin{cases} dX(s) = \{aX(s) + bu(s)\} ds + \sigma dW(s) + c\int_{\mathbb{R}^*} \tilde{N}(ds, dz), \ s \in [0,T], \\ X(t) = \xi, \end{cases} \tag{5.12}$$

where a, b and c are real constant. As mentioned in [2], this is a time-inconsistent version of the classical linear quadratic regulator, we want control the system so that the final sate $X(T)$ is close to ξ while at the same time we keep the control energy (formalized by the running cost) small. Note that, here the time-inconsistency is due to the fact that the terminal cost depends explicitly on the current state ξ as well as the current time t. Hence there are two different sources of time-inconsistency. For this example, the system (4.1) reduces, $\forall (t, s) \in \mathcal{D}[0,T]$, to

$$
(5.13)\qquad \begin{cases} \dfrac{\partial M}{\partial s}(t,s)+2aM(t,s) \\ \qquad -b^2 M(t,s)\left\{M(s,s)+\bar{M}(s,s)+\Upsilon(s,s)\right\}=0, \\ \dfrac{\partial \bar{M}}{\partial s}(t,s)+2a\bar{M}(t,s) \\ \qquad -b^2 \bar{M}(t,s)\left\{M(s,s)+\bar{M}(s,s)+\Upsilon(s,s)\right\}=0, \\ \dfrac{\partial \Upsilon}{\partial s}(t,s)+a\Upsilon(t,s)=0, \\ \dfrac{\partial \varphi}{\partial s}(t,s)+a\varphi(t,s)-b^2\left\{M(t,s)+\bar{M}(t,s)\right\}\varphi(s,s)=0, \\ M(t,T)=h(t),\ \bar{M}(t,T)=0,\ \Upsilon(t,T)=h(t), \\ \varphi(t,T)=0,\ \forall t\in[0,T], \end{cases}
$$

obviously, $\Upsilon(.,.)$ is explicitely given by

$$
(5.14)\qquad \Upsilon(t,s)=h(t)\exp\{a(T-s)\},\ \forall(t,s)\in\mathcal{D}[0,T].
$$

Moreover, we can check that $M(.,.),\bar{M}(.,.)$ and $\varphi(.,.)$ solve (5.13), if and only if, they solve the following system of integral equations, for all $(t,s)\in\mathcal{D}[0,T]$

$$
(5.15)\quad \begin{cases} M(t,s)=M(t,T)\,e^{\int_s^T\left\{2a-b^2\left(M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau)\right)\right\}d\tau},. \\ \bar{M}(t,s)=\bar{M}(t,T)\,e^{\int_s^T\left\{2a-b^2\left(M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau)\right)\right\}d\tau}, \\ \varphi(t,s)=\varphi(t,T)\,e^{a(T-s)}-b^2\displaystyle\int_s^T e^{a(\tau-s)}\left(M(t,\tau)+\bar{M}(t,\tau)\right)\varphi(\tau,\tau)\,d\tau. \end{cases}
$$

on the other hand, we have $\bar{M}(t,T)=\varphi(t,T)=0$, then (5.15) reduces $\forall(t,s)\in\mathcal{D}[0,T]$, to

$$
(5.16)\qquad \begin{cases} M(t,s)=M(t,T)\,e^{\int_s^T\left\{2a-b^2(M(r,r)+\Upsilon(r,r))\right\}dr}, \\ \bar{M}(t,s)=0, \\ \varphi(t,s)=-b^2\displaystyle\int_s^T e^{a(\tau-s)}M(t,\tau)\,\varphi(\tau,\tau)\,d\tau. \end{cases}
$$

It is clear that if $M(.,.)$ is the solution of the first equation in (5.16), then

$$
\varphi(s,s)=-b^2\int_s^T e^{a(\tau-s)}M(s,\tau)\,\varphi(\tau,\tau)\,d\tau,\ \forall s\in[0,T],
$$

thus, there exists some constant $L > 0$ such that $|\varphi(s,s)| \leq L \int_s^T |\varphi(\tau,\tau)| \, d\tau$, then by Gronwall Lemma, we conclude that $\varphi(s,s) = 0, \forall s \in [0,T]$. Therefore $\varphi(t,s) = 0$, $\forall (t,s) \in \mathcal{D}[0,T]$, is the unique solution to the last equation in the system (5.16).

Now, it's remains to solve the first equation in the system (5.16). Now, it is easy to check that the first equation in the system (5.16) is equivalent to

$$(5.17) \qquad \begin{cases} \dfrac{\partial M}{\partial s}(t,s) + 2aM(t,s) - b^2 M(t,s)\{M(s,s) + \Upsilon(s,s)\} = 0, \\ \hfill \forall (t,s) \in \mathcal{D}[0,T], \\ M(t,T) = h(t). \end{cases}$$

We try a solution of the form $M(t,s) = h(t)N(s)$, we finde that $N(.)$ should solve the following ODE

$$(5.18) \qquad \begin{cases} \dfrac{dN}{ds}(s) + \left(2a + b^2\Upsilon(s,s)\right)N(s) - b^2 h(s) N(s)^2 = 0, s \in [0,T] \\ N(T) = 1, \end{cases}$$

We put $N(s) = \dfrac{1}{y(s)}$, the equation (5.18) leads to

$$\begin{cases} \dfrac{dy}{ds}(s) - \left(2a + b^2\Upsilon(s,s)\right)y(s) + b^2 h(s) = 0, \; s \in [0,T] \\ y(T) = 1, \end{cases}$$

which is explicitly solvable by

$$y(s) = e^{-\int_s^T \left\{2a + b^2\Upsilon(\tau,\tau)\right\} d\tau}\left(1 + b^2 \int_s^T e^{\int_\tau^T \left\{2a + b^2\Upsilon(l,l)\right\} dl} h(\tau)\, d\tau\right), \; s \in [0,T],$$

thus

$$M(t,s) = h(t) \frac{e^{\int_s^T \left\{2a + b^2\Upsilon(\tau,\tau)\right\} d\tau}}{1 + b^2 \int_s^T e^{\int_\tau^T \{2a + b^2\Upsilon(l,l)\} dl} h(\tau)}, \; (t,s) \in \mathcal{D}[0,T]$$

In view of Theorem 4.1, the representation (4.3) of the Nash equilibrium control, then gives

$$(5.19) \qquad \hat{u}(s) = -b\{\Upsilon(s,s) + M(s,s)\}\hat{X}(s), \; \forall s \in [0,T],$$

and the corresponding equilibrium dynamics solves the SDEJ

$$(5.20)\qquad \begin{cases} d\hat{X}(s) = \left\{a - b^2\left(\Upsilon(s,s) + M(s,s)\right)\right\}\hat{X}(s)\,ds + \sigma dW(s) \\ \qquad\qquad + c\int_{\mathbb{R}^*}\tilde{N}(ds,dz),\ s \in [0,T], \\ X(0) = x_0. \end{cases}$$

To conclude this section let us present the following remark.

Remark 5.1. The Problem (E) given by the subsection 2.3, is in fact shown to be a particular case of the general discounting LQ regulator model, formulated earlier in this paragraph, in the case when $a = c = 0$, and the final data $\xi = x$, this leads to the following representation of the Nash equilibrium control of this problem

$$\hat{u}(s) = -b\left(h(s) + M(s,s)\right)\hat{X}(s), \forall s \in [0,T],$$

where $M(t,s)$ solves

$$M(t,s) = h(t)\, e^{\int_s^T -b^2(M(\tau,\tau)+h(\tau))d\tau},\ \text{for}\ (t,s) \in \mathcal{D}[0,T],$$

and the corresponding equilibrium dynamics solves the SDE

$$\begin{cases} d\hat{X}(s) = -b^2\left\{h(s) + M(s,s)\right\}\hat{X}(s)\,ds + \sigma dW(s),\ s \in [0,T], \\ X(0) = x_0. \end{cases}$$

This in fact, the correct solution of the Problem (E).

Conclusion and future work. *In this paper, we have studied a class of dynamic decision problems of a general time-inconsistent type. We have used the game theoretic approach to handle the time inconsistency. During this study open-loop Nash equilibrium controls are constructed as an alternative of optimal controls. This has been accomplished through stochastic maximum principle that includes a flow of forward-backward stochastic differential equations under maximum condition. The inclusion of concrete examples confirms the validity of our proposed study. The work can be extended in several ways. For example, this approach can be extended to a mean field game to construct decentralized strategies and obtain an estimate of their performance. The reserch on this topic is in progress and will appear in our forthcoming paper.*

6. Appendix: Additional proofs.

Proof of Lemma 3.3. Let $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$ and $\varepsilon \in [0, T-t]$. Since $\mathbb{E}^t[y^{\varepsilon,v}(.)]$ and $\mathbb{E}^t[z^{\varepsilon,v}(.)]$ solve the following ODEs, respectively

$$\begin{cases} d\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right] = A\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right]ds, \ s\in[t,T], \\ \mathbb{E}^t\left[y^{\varepsilon,v}(t)\right] = 0, \end{cases}$$

and

$$\begin{cases} d\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right] = \left\{A\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right] + B\mathbb{E}^t\left[v\right]1_{[t,t+\varepsilon)}(s)\right\}ds, \ s\in[t,T], \\ \mathbb{E}^t\left[z^{\varepsilon,v}(t)\right] = 0. \end{cases}$$

Thus, it is clear that $\mathbb{E}^t\left[y^{\varepsilon,v}(s)\right] = 0$, *a.e.* $s\in[t,T]$. According to Gronwall's inequality there exists a positive constant K such that $\sup\limits_{s\in[t,T]}\left|\mathbb{E}^t\left[z^{\varepsilon,v}(s)\right]\right|^2 \leq K\varepsilon^2$. Moreover, by Lemma 2.1. in [19], we obtain (3.16).

By these estimates, we can calculate the differencewe consider the difference

$$\begin{aligned} &J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right) - J\left(t,\hat{X}(t),\hat{u}(.)\right) \\ &= \mathbb{E}^t\Bigg[\int_t^T\Bigg\{\left\langle Q(t,s)\hat{X}(s) + \bar{Q}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right], y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle \\ &\qquad + \frac{1}{2}\left\langle Q(t,s)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right), y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle \\ &\qquad + \frac{1}{2}\left\langle \bar{Q}(t,s)\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right], \mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right]\right\rangle \\ &\qquad + \left\langle R(t,s)\hat{u}(s), v\right\rangle 1_{[t,t+\varepsilon)}(s) + \frac{1}{2}\left\langle R(t,s)v,v\right\rangle 1_{[t,t+\varepsilon)}(s)\Bigg\}ds \\ &+ \frac{1}{2}\left\langle G(t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle \\ &+ \left\langle G(t)\hat{X}(T) + \bar{G}(t)\mathbb{E}^t\left[\hat{X}(T)\right] + \mu_1(t)\hat{X}(t) + \mu_2(t), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle \\ &+ \frac{1}{2}\left\langle \bar{G}(t)\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right], \mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]\right\rangle\Bigg]. \end{aligned} \tag{A.1}$$

In the other hand, from **(H1)** and (3.15) – (3.16) the following estimate holds

$$\begin{aligned} &\mathbb{E}^t\Bigg[\int_t^T\frac{1}{2}\left\langle \bar{Q}(t,s)\mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right], \mathbb{E}^t\left[y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right]\right\rangle ds \\ &\qquad + \frac{1}{2}\left\langle \bar{G}(t)\mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right], \mathbb{E}^t\left[y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right]\right\rangle\Bigg] = o(\varepsilon). \end{aligned}$$

Then, from the terminal conditions in the adjoint equations, it follows that

$$\begin{aligned}
&J\left(t,\hat{X}(t),u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right)\\
&=\mathbb{E}^{t}\Bigg[\int_{t}^{T}\Big\{\Big\langle Q(t,s)\hat{X}(s)+\bar{Q}(t,s)\mathbb{E}^{t}\left[\hat{X}(s)\right],y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\Big\rangle\\
&\qquad+\frac{1}{2}\left\langle Q(t,s)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right),y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle\\
&\qquad+\left\langle R(t,s)\hat{u}(s),v\right\rangle 1_{[t,t+\varepsilon)}(s)+\frac{1}{2}\left\langle R(t,s)v,v\right\rangle 1_{[t,t+\varepsilon)}(s)\Big\}ds\\
&\quad-\left\langle p(T;t),y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\\
&\quad-\frac{1}{2}\left\langle P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right),y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\Bigg]\\
&\quad+o(\varepsilon).
\end{aligned}\tag{A.2}$$

Now, by applying Ito's formula to $s\mapsto\left\langle p(s;t),y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle$ on $[t,T]$ and by taking the conditional expectation, we get

$$\begin{aligned}
\mathbb{E}^{t}\left[\left\langle p(T;t),y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\right]&=\mathbb{E}^{t}\Bigg[\int_{t}^{T}\Big\{v^{\top}B^{\top}p(s;t)1_{[t,t+\varepsilon)}(s)\\
&\quad+\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^{\top}\left(Q(t,s)\hat{X}(s)+\bar{Q}(t,s)\mathbb{E}^{t}\left[\hat{X}(s)\right]\right)\\
&\quad+\sum_{j=1}^{d}v^{\top}D_{j}^{\top}q_{j}(s;t)1_{[t,t+\varepsilon)}(s)\\
&\quad+\int_{Z}v^{\top}F(z)^{\top}r(s,z;t)1_{[t,t+\varepsilon)}(s)\theta(dz)\Big\}ds\Bigg].
\end{aligned}\tag{A.3}$$

Again, by applying Ito's formula to

$$s\mapsto\left\langle P(s;t)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right),y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right\rangle$$

on $[t,T]$, we get by taking the conditional expectation

$$\begin{aligned}
&\mathbb{E}^t\left[\left\langle P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\right] \\
&=\mathbb{E}^t\Bigg[\int_t^T\Big\{2\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top P(s;t)\,Bv1_{[t,t+\varepsilon)}(s) \\
&\qquad+\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top Q(t,s)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right) \\
&\qquad+\sum_{j=1}^d\Big\{2\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top C_j^\top+v^\top D_j^\top\Big\}P(s;t)\,D_jv1_{[t,t+\varepsilon)}(s) \\
&\qquad+\int_Z\Big\{2\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top E(z)^\top \\
&\qquad\qquad+v^\top F(z)^\top\Big\}P(s;t)\,F(z)\,v1_{[t,t+\varepsilon)}(s)\,\theta(dz)\Big\}\,ds\Bigg]
\end{aligned}\tag{A.4}$$

Moreover, we conclude from **(H1)** together with (3.15) – (3.16) that

$$\begin{aligned}
&\mathbb{E}^t\left[\int_t^T\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top P(s;t)\,Bv1_{[t,t+\varepsilon)}(s)\,ds\right]=o(\varepsilon), \\
&\mathbb{E}^t\left[\int_t^T\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top C_j^\top P(s;t)\,D_jv1_{[t,t+\varepsilon)}(s)\,ds\right]=o(\varepsilon), \\
&\mathbb{E}^t\left[\int_t^T\int_Z\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top E(z)^\top P(s;t)\,F(z)\,v1_{[t,t+\varepsilon)}(s)\,\theta(dz)\,ds\right]=o(\varepsilon).
\end{aligned}\tag{A.5}$$

Then by invoking $(A.5)$ it holds

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}^t\left[\left\langle P(T;t)\left(y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right), y^{\varepsilon,v}(T)+z^{\varepsilon,v}(T)\right\rangle\right] \\
&=\frac{1}{2}\mathbb{E}^t\Bigg[\int_t^T\Big\{\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right)^\top Q(t,s)\left(y^{\varepsilon,v}(s)+z^{\varepsilon,v}(s)\right) \\
&\qquad+\sum_{j=1}^d v^\top D_j^\top P(s;t)\,D_jv1_{[t,t+\varepsilon)}(s) \\
&\qquad+\int_Z v^\top F(z)^\top P(s;t)\,F(z)\,v1_{[t,t+\varepsilon)}(s)\,\theta(dz)\Big\}\,ds\Bigg]+o(\varepsilon).
\end{aligned}\tag{A.6}$$

By taking $(A.3)$ and $(A.6)$ in $(A.2)$, it follows that

$$\begin{aligned}
&J\left(t,\hat{X}(t),u^\varepsilon(.)\right)-J\left(t,\hat{X}(t),\hat{u}(.)\right) \\
&=-\mathbb{E}^t\Bigg[\int_t^T\Big\{v^\top B^\top p(s;t)+\sum_{j=1}^d v^\top D_j^\top q_j(s,t)+\frac{1}{2}\sum_{j=1}^d v^\top D_j^\top P(s;t)\,D_jv
\end{aligned}$$

$$
-v^{\top} R(t,s)\,\hat{u}(s) - \frac{1}{2} v^{\top} R(t,s)\, v
+ \int_Z \Big(r(s,z;t)^{\top} F(z)\, v
+ \frac{1}{2} v^{\top} F(z)^{\top} P(s;t)\, F(z)\, v \Big)\, \theta(dz) \Bigg\} 1_{[t,t+\varepsilon)}(s)\, ds \Bigg] + o(\varepsilon),
$$

which is equivalent to (3.17). □

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