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ON THE ASYMPTOTIC BEHAVIOR OF THIRD-ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This work discusses asymptotic behavior of solutions of class of third-order non-linear delay differential equation with middle term. Our results in this paper extend and improve some the previous results, the sense that the results do not require delay function $(g_i(t))$ with monotonicity. As well, by using Riccati transformation technique, we establish some new oscillation criteria for third-order delay differential equation. Examples given in the study to clarify the new results.

Introduction. In this work, we consider new class of third order delay differential equations of the form

$$(r(t)\, x''(t))' + p(t)\, x'(t) + \sum_{i=1}^{n} q_i(t)\, f(x(g_i(t))) = 0, \tag{1.1}$$

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and

$$(1.2) \qquad \left(r\left(t\right)x''\left(t\right)\right)' + \phi\left(t, x'\left(t\right)\right) + \sum_{i=1}^{n} q_i\left(t\right) f\left(x\left(g_i\left(t\right)\right)\right) = 0,$$

where r, p, q_i and g_i are positive real-valued functions, $g_i\left(t\right) \leq t$, $\lim_{t\to\infty} g_i\left(t\right) = \infty$, $i = 1, 2, \ldots, n$, $\int_{t_0}^{\infty} r^{-1}\left(t\right) dt = \infty$ and

(A_1) $f \in C\left(\mathbb{R}, \mathbb{R}\right), \dfrac{f\left(u\right)}{u} \geq k > 0$ for $u \neq 0$,

(A_2) There exists a positive real function $p_*\left(t\right)$ such that $\phi\left(u, v\right) \geq p_*\left(u\right) v$ and $\phi\left(u, -v\right) = -\phi\left(u, v\right)$.

We intend to a solution of Eq. (1.1) or (1.2) a function $x(t) : [t_x, \infty) \to \mathbb{R}, t_x \geq t_0$ such that $r(t)x''(t)$ is continuously differentiable for all $t \in [t_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. Any solution of differential equation is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

Asymptotic properties of any solutions of the second/third order differential equation have been subject of intensive researching in the literature. This problem for differential equations with delay has received a great deal of attention in the last years, see for example ([1]–[14] and [16]–[18]).

This paper, in Section 2, we shall present some oscillation criteria for Eq. (1.1), which complement and extend the results in [14], [17] and [10]. In Section 3, we will establish some sufficient conditions which insure that any solution of Eq. (1.2) oscillates or converges to zero and also condition of Philos-type for oscillation. The results obtained essentially generalize and improve the earlier ones. Finally, examples are also presented to illustrate the relevance of the results.

The following lemmas due to Kiguradze [13] and Baculíková [3] will be useful in the rest of this paper.

Lemma 1.1 ([13]). *If the function y satisfies $y^{(i)} > 0$, $i = 0, 1, \ldots, n$, and $y^{(n+1)} < 0$, then*

$$\frac{y\left(t\right)}{t^n/n!} \geq \frac{y'\left(t\right)}{t^{n-1}/\left(n-1\right)!}.$$

Lemma 1.2 ([3]). *Assume that* $u(t) > 0, u'(t) \geq 0$ *and* $u''(t) \leq 0$ *on* $[t_0, \infty)$. *Then for each* $l \in (0,1)$ *there exists a* $T_l \geq t_0$ *such that*

$$\frac{u(g(t))}{g(t)} \geq l \frac{u(t)}{t} \quad \text{for } t \geq T_l.$$

2. Oscillation results for equation (1.1). In this section, we establish new oscillation criteria for solutions of equation (1.1) when $r'(t) > 0$. First, we show some lemmas that will be useful to establish our results.

Lemma 2.1. *Assume that*

(C_1) $\quad 2k \sum_{i=1}^{n} q_i(t) - p'(t) \geq 0$ *for* $t \geq t_0$ *and is not identically zero in any subinterval of* $[t_0, \infty)$.

Let $x(t)$ *be a nonoscillatory solution of* (1.1) *that is eventually positive with*

$$G[x(t_1)] = r(t_1)\left(x'(t_1)\right)^2 - 2r(t_1)x(t_1)x''(t_1) - p(t_1)x^2(t_1) \geq 0, \tag{2.1}$$

for some $t_1 \in [t_0, \infty)$. *Then there exists* $t_2 \geq t_1$ *such that*

$$x(t) > 0,\ x'(t) > 0,\ x''(t) > 0 \text{ and } x'''(t) < 0, \tag{2.2}$$

for $t \geq t_2$.

P r o o f. Let x be a nonoscillatory solution of Eq. (1.1) that is eventually positive with the condition $G[x(t_1)] \geq 0$ for some $t_1 \in [t_0, \infty)$. Then there exists a $t_2 \geq t_1$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geq t_2$ and $i = 1, 2, \ldots, n$. From (1.1) and (2.1), we get

$$G'[x(t)] = r'(t)\left(x'(t)\right)^2 + 2x(t)\sum_{i=1}^{n} q_i(t) f(x(g_i(t))) - p'(t)x^2(t$$

Thus, from (A_1) and (C_1), we obtain $G'[x(t)] \geq 0$ for $t \geq t_2$. So there exists a point $t_3 \geq t_2$ such that $G[x(t)]$ is nonnegative and strictly increasing for $t \geq t_3$. Since $G[x(t)] \geq 0$ for $t \geq t_3$, we have

$$2r(t)\frac{d}{dt}\left(\frac{x'(t)}{x(t)}\right) = x^{-2}(t)\left(2r(t)x(t)x''(t) - 2r(t)\left(x'(t)\right)^2\right)$$

$$\leq \quad -x^2(t)\left(p(t)\,x^2(t)+r(t)\left(x'(t)\right)^2\right)<0.$$

Hence, the function x'/x is decreasing on $[t_3,\infty)$. This means that $x(t) > 0, x'(t) \neq 0$ for $t \geq t_3$.

The rest of the proof is the same in [14, Lemmas 3.1], and hence, is omitted. □

Lemma 2.2. *Assume that* (C_1) *holds. Let* $x(t)$ *be solution of Eq.* (1.1) *satisfying* (2.1) *for some* $t_1 \in [t_0,\infty)$. *Then, there exist a* $t_2 \geq t_0$ *and constant* M *such that*

$$x(g_i(t)) \geq \frac{g_i^2(t)}{t}\frac{M}{2}, \tag{2.3}$$

for $t \geq t_2$ *and* $i = 1,2,\ldots,n$.

Proof. Let $x(t)$ be solution of Eq. (1.1) satisfying (2.1) for some $t_1 \in [t_0,\infty)$. Then, by Lemma 2.1, there exists a $t_2 \geq t_1$ such that (2.2) holds for $t \geq t_2$. Thus, from Lemma 1.1, we have

$$\frac{x'(t)}{x(t)} \leq \frac{2}{t},$$

for $t \geq t_2$. Integrating this inequality from $g_i(t)$ to t, we see that

$$\frac{x(t)}{x(g_i(t))} \leq \frac{t^2}{g_i^2(t)}. \tag{2.4}$$

Since, $x'(t) > 0$ for $t \geq t_2$ and increasing, we get $x'(t) > M > 0$ for $t \geq t_2$. Thus, by using $x(t_2) > 0$, we obtain

$$x(t) \geq x(t_2) + M(t-t_2) \geq \frac{M}{2}t.$$

Hence, from (2.4), we have

$$x(g_i(t)) \geq \frac{g_i^2(t)}{t^2}x(t) \geq \frac{g_i^2(t)}{t}\frac{M}{2}.$$

The proof is complete. □

In the following theorems, we establish some oscillation criteria for Eq. (1.1) when $p(t)$ with monotonicity.

Theorem 2.1. *Assume that* (C_1) *holds,* $p'(t) \le 0$. *Let* $x(t)$ *be solution of Eq.* (1.1) *satisfying* (2.1) *for some* $t_1 \in [t_0, \infty)$. *If*

$$\int_{t_0}^{\infty} \frac{1}{s} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\right) g_i^2(s)\, ds = \infty, \tag{2.5}$$

then $x(t)$ *is oscillatory.*

Proof. Let x be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \ge t_1, i = 1, 2, \ldots, n$ and $x(t)$ satisfying (2.1) for some $t_1 \in [t_0, \infty)$. From Lemma 2.1, there exists a $t_2 \ge t_1$ such that (2.2) holds for $t \ge t_2$. Now, by integrating Eq. (1.1) from t_2 to t and using (A_1), we obtain

$$r(t)\, x''(t) - r(t_2)\, x''(t_2) \le -\int_{t_2}^{t} p(s)\, x'(s)\, ds - \int_{t_2}^{t} k \sum_{i=1}^{n} q_i(s)\, x(g_i(s))\, ds.$$

Integrating by parts, we get

$$\begin{aligned} r(t_2)\, x''(t_2) + p(t_2)\, x(t_2) \;\ge\;& r(t)\, x''(t) + p(t)\, x(t) \\ &+ \int_{t_2}^{t} \sum_{i=1}^{n} \left(kq_i(s) - p'(s) \frac{x(s)}{x(g_i(s))}\right) x(g_i(s))\, ds, \end{aligned}$$

and so

$$r(t_2)\, x''(t_2) + p(t_2)\, x(t_2) \ge \int_{t_2}^{t} \sum_{i=1}^{n} \left(kq_i(s) - p'(s) \frac{x(s)}{x(g_i(s))}\right) x(g_i(s))\, ds. \tag{2.6}$$

Since $p'(t) \le 0, x'(t) > 0$ and $g(t) \le t$, (2.6) yields

$$r(t_2)\, x''(t_2) + p(t_2)\, x(t_2) \ge \int_{t_2}^{t} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\right) x(g_i(s))\, ds.$$

From Lemma 2.2, we have

$$\left(r(t_2)\, x''(t_2) + p(t_2)\, x(t_2)\right) \ge \frac{M}{2} \int_{t_2}^{t} \frac{1}{s} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\right) g_i^2(s)\, ds.$$

Taking the limit of both sides as $t \to \infty$, we get that

$$\int_{t_2}^{\infty} \frac{1}{s} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\right) g_i^2(s)\, ds < \infty,$$

which contradicts assumption (2.5). This completes the proof. □

Theorem 2.2. *Assume that* $p'(t) \geq 0$ *and*

$$(C_2) \qquad 2k\sum_{i=1}^{n} q_i(t) - p'(t)\frac{t^2}{g_i^2(t)} \geq 0 \quad \text{for } t \geq t_0 \text{ and } i = 1, 2, \ldots, n.$$

Let $x(t)$ *be solution of Eq.* (1.1) *satisfying* (2.1) *for some* $t_1 \in [t_0, \infty)$*. If*

$$\int_{t_0}^{\infty} \frac{1}{s}\left(k\sum_{i=1}^{n} q_i(s)\, g_i^2(s) - p'(s)\, s^2\right) ds = \infty, \tag{2.7}$$

then $x(t)$ *is oscillatory.*

Proof. We note that the condition (C_2) lead to (C_1). Therefore, from Lemma 2.1 $x(t)$ satisfies (2.2). Proceeding as in the proof of Theorem 2.1, we see that (2.6) holds. By Lemma 2.2, we have (2.3) and (2.4) hold and hence

$$r(t_2)\, x''(t_2) + p(t_2)\, x(t_2) \geq \int_{t_2}^{t} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\frac{s^2}{g_i^2(s)}\right) x(g_i(s))\, ds.$$

From (C_2), we obtain

$$\big(r(t_2)\, x''(t_2) + p(t_2)\, x(t_2)\big) \geq \frac{M}{2}\int_{t_2}^{t} \sum_{i=1}^{n} \left(kq_i(s) - p'(s)\frac{s^2}{g_i^2(s)}\right) \frac{g_i^2(s)}{s} ds,$$

so,

$$\int_{t_2}^{\infty} \frac{1}{s}\left(\sum_{i=1}^{n} kq_i(s)\, g_i^2(s) - p'(s)\, s^2\right) ds < \infty,$$

which contradicts (2.7). This completes the proof. □

In the following theorem, we extend the results of Lazer [14].

Theorem 2.3. *Assume that* (C_1) *hold. Let* $x(t)$ *be solution of Eq.* (1.1) *satisfying* (2.1) *for some* $t_1 \in [t_0, \infty)$*. If for some* $m < 1/2$*, the second-order differential equation*

$$\big(r(t)\, u'(t)\big)' + \left(p(t) + \frac{km}{t}\sum_{i=1}^{n} q_i(t)\, g_i^2(t)\right) u(t) = 0, \tag{2.8}$$

is oscillatory, then $x(t)$ *is oscillatory.*

P r o o f. Let x be a nonoscillatory solution of Eq. (1.1) with (2.1) for some $t_1 \in [t_0, \infty)$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geq t_1, i = 1, 2, \ldots, n$. From Lemma 2.1, there exists a $t_2 \geq t_1$such that (2.2) holds for $t \geq t_2$. Next, We can write equation (1.1) as the system

$$\begin{aligned} u(t) &= x'(t) \\ u'(t) &= x''(t) \\ \left(r(t)\,u'(t)\right)' &= -p(t)\,u(t) - \sum_{i=1}^{n} q_i(t)\, f(x(g_i(t))). \end{aligned}$$

The last equation can be written as

$$\left(r(t)\,u'(t)\right)' + \left(p(t) + \sum_{i=1}^{n} q_i(t)\,\frac{f(x(g_i(t)))}{u(t)}\right) u(t) = 0. \tag{2.9}$$

From Lemma (2.2) and (A_1), we see that

$$\begin{aligned} p(t) + \sum_{i=1}^{n} q_i(t)\,\frac{f(x(g_i(t)))}{u(t)} &\geq p(t) + k\sum_{i=1}^{n} q_i(t)\,\frac{x(g_i(t))}{x'(t)} \\ &\geq p(t) + k\sum_{i=1}^{n} q_i(t)\,\frac{g_i^2(t)}{t^2}\,\frac{x(t)}{x'(t)}. \end{aligned} \tag{2.10}$$

By Lemma (1.1), we have $\dfrac{x(t)}{x'(t)} \geq \dfrac{t}{2}$ for $t \geq t_2$. Since $m < 1/2$, there esists a $t_3 \geq t_2$ such that $\dfrac{x(t)}{x'(t)} \geq mt$ for $t \geq t_3$. Hence, (2.8) yields

$$p(t) + \sum_{i=1}^{n} q_i(t)\,\frac{f(x(g_i(t)))}{u(t)} \geq p(t) + km\sum_{i=1}^{n} q_i(t)\,\frac{g_i^2(t)}{t}.$$

Since (2.8) is oscillatory, from the Sturm Comparison Theorem, every nontrivial solution of (2.9) is oscillatory. This contradicts the fact that $u(t) = x'(t) > 0$. This completes the proof. □

Example 2.1. Consider the third order delay differential equation

$$x'''(t) + \frac{a}{t^2}x'(t) + \sum_{i=1}^{n} \frac{b}{t g_i^2(t)}\,x(g_i(t)) = 0, \;\; t \geq 1, \tag{2.11}$$

where g_i are positive delay functions, $g_i(t) \leq t$ for $i = 1, 2, \ldots, n$ and a, b are positive constants such that $a < 1/4$. To apply Theorem 2.3, we note that $k = 1$ and the equation (2.8) becomes

$$u''(t) + \left(\frac{a}{t^2} + n\frac{bm}{t^2}\right) u'(t) = 0. \tag{2.12}$$

Applying the Hille-Kneser criterion, we see that equation (2.12) is oscillatory if $a + nbm > 1/4$ for some $m < 1/2$. That is, $2a + nb > 1/2$. By Theorem 2.3, if $2a + nb > 1/2$, then we have every solution of Eq. (2.11) satisfying (2.1) is oscillatory.

Example 2.2. Consider the third order delay differential equation

$$\left(tx''(t)\right)' + x'(t) + \frac{2}{t}x\left(\frac{t}{2}\right)\left(x^2\left(\frac{t}{2}\right) + 2\right) = 0, \quad t \geq 1, \tag{2.13}$$

We note that $n = 1$, $f(x) = x\left(x^2 + 2\right)$ with $k = 2$ and the equation (2.8) becomes

$$\left(tu'(t)\right)' + (m+1)\, u(t) = 0. \tag{2.14}$$

By [6], we see that equation (2.14) is oscillatory. Then, by Theorem 2.3, every solution of Eq. (2.13) satisfying (2.1) is oscillatory.

3. Oscillation results for equation (1.2). In this section, we establish some new oscillatory criteria for Eq. (1.2). First, we state and prove some useful lemmas, which we will use in the proof of our main results.

Lemma 3.1. *Suppose that the second-order differential equation*

$$\left(r(t)\, v'(t)\right)' + p_*(t)\, v(t) = 0 \tag{3.1}$$

is nonoscillatory. If x is a nonoscillatory solution of Eq. (1.2)*, then there exists a $t_1 \geq t_0$ such that either $x(t)\, x'(t) > 0$ or $x(t)\, x'(t) < 0$ for $t \geq t_1$.*

Proof. Suppose that x is a nonoscillatory solution of (1.2). Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, \ldots, n$. We note that $w(t) = -x'(t)$ is a solution of the second order nonhomogeneous delay differential equation

$$\left(r(t)\, w'(t)\right)' + \phi(t, w(t)) = \sum_{i=1}^{n} q_i(t)\, f(x(g_i(t))). \tag{3.2}$$

Now, we shall prove that all solutions of (3.2) are nonoscillatory. If possible, let w be an oscillatory solution of (3.2) with consecutive zeros at b and c such that $t_1 < b < c$, $w'(b) \geq 0$ and $w'(c) \leq 0$. Let $v(t)$ be a solution of (3.1) and $v(t) > 0$ for $t \geq t_1$. The case when $v(t)$ is ultimately negative can similarly be deal with. From (1.2), (3.1) and (A_2) we obtain

$$\begin{aligned} v(t)\sum_{i=1}^{n} q_i(t) f(x(g_i(t))) &= v(t)\left[-\left(r(t)x''(t)\right)' - \phi\left(t, x'(t)\right)\right] \\ &\leq \left(r(t)w'(t)\right)' v(t) + p_*(t) w(t) v(t) \\ &= \left(r(t)w'(t)\right)' v(t) - \left(r(t)v'(t)\right)' w(t) \\ &= \left[r(t)w'(t)v(t) - r(t)v'(t)w(t)\right]'. \end{aligned} \tag{3.3}$$

By integrating (3.3) from b to c, we get a contradiction. This contradiction completes the proof. □

Lemma 3.2. *Let $x(t)$ be an eventually positive solution of the equation (1.2) such that $x(t)x'(t) > 0$ eventually. Then there exists a $t_1 \geq t_0$ such that*

$$x'(t) > 0, x''(t) > 0 \text{ and } \left(r(t)x''(t)\right)' \leq 0, \tag{3.4}$$

for $t \geq t_1$.

The proof of this lemma is similar to that of the proof of Lemma 1 of Skerlik [16], and hence is omitted.

Lemma 3.3. *Let $x(t)$ be an eventually positive solution of the equation (1.2) such that $x(t)x'(t) < 0$ eventually. If*

(C_3) $\quad k\sum_{i=1}^{n} q_i(s) - p_*'(s) \geq 0$ *for $t \geq t_0$ and is not identically zero in any subinterval of $[t_0, \infty)$, and*

$$\int_{t_0}^{\infty}\left(k\sum_{i=1}^{n} q_i(s) - p_*'(s)\right) ds = \infty,$$

then $x(t)$ is converges to zero as $t \to \infty$.

Proof. Assume that $x(t)$ be an eventually positive solution of the equation (1.2) such that $x'(t) < 0$ eventually. Thus, there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x'(t) < 0$ for $t \geq t_1$, and hence,

$$\lim_{t\to\infty} x(t) = \sigma \geq 0.$$

Now. We shall prove that $\sigma = 0$. If $\sigma > 0$, then we have $x(t) \geq \sigma$ for t enough large. By integrating Eq. (1.2) from t_1 to t and using (A_2), we obtain

$$r(t)\,x''(t) \leq M - p_*(t)\,x(t) + \int_{t_1}^{t} p_*'(s)\,x(s)\,ds - \int_{t_1}^{t} \sum_{i=1}^{n} q_i(s)\,f(x(g_i(s)))\,ds,$$

where $M = r(t_1)\,x''(t_1) + p_*(t_1)\,x(t_1)$. From (A_1), we get

$$r(t)\,x''(t) \leq M - \int_{t_1}^{t} \left(k \sum_{i=1}^{n} q_i(s) \frac{x(g_i(s))}{x(s)} - p_*'(s) \right) x(s)\,ds.$$

Since $x'(t) < 0$ and $g(t) < t$, we see that

$$r(t)\,x''(t) \leq M - \sigma \int_{t_1}^{t} \left(k \sum_{i=1}^{n} q_i(s) - p_*'(s) \right) ds.$$

From (C_3), we have $\lim_{t\to\infty} r(t)\,x''(t) = -\infty$. Hence, there exists $\delta < 0$ such that $r(t)\,x''(t) \leq \delta$ for large t and so $x''(t) < 0$ for large t. But $x''(t) < 0$ and $x'(t) < 0$ eventually imply $x(t) < 0$ for $t \geq t_1$. This contradiction completes the proof. □

The next theorems is obtained by using Riccati transformation technique.

Theorem 3.1. *Assume that $r'(t) > 0$, (C_3) holds and the second-order differential equation (3.1) is nonoscillatory. If there exists a positive function $\rho(t)$ such that for $T > t_0$*

$$\int_{T}^{\infty} \left(k \frac{\rho(s)}{s^2} \sum_{i=1}^{n} q_i(s)\,g_i^2(s) - \frac{\left[\rho'(s)\,r(s) - (s-t_0)\,\rho(s)\,p_*(s)\right]^2}{4(s-t_0)\,\rho(s)\,r(s)} \right) ds = \infty, \tag{3.5}$$

then every solution of Eq. (1.2) is either oscillatory or tends to zero as $t \to \infty$.

P r o o f. Let x be a nonoscillatory solution of Eq. (1.2). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geq t_1$, $i = 1, 2, \ldots, n$. Hence, from Lemma 3.1, there exists a $t_2 \geq t_1$ such that $x'(t) > 0$ or $x'(t) < 0$ for $t \geq t_2$. If $x'(t) < 0$, by Lemma 2.5, we get that $\lim_{t\to\infty} x(t) = 0$.

Next, Let $x'(t) > 0$ for $t \geq t_2$. By Lemma 3.2, we see that (3.4) holds for $t \geq t_2$. Since $r'(t) > 0$, we have

$$x'''(t) < 0, \tag{3.6}$$

and so,

$$x'(t) \geq \int_{t_2}^{t} x''(s)\,ds \geq x''(t)(t-t_2). \tag{3.7}$$

We define

$$\omega(t) = \rho(t)\frac{r(t)x''(t)}{x(t)}.$$

Then $\omega(t) > 0$, and

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)x''(t))'}{x(t)} - \rho(t)\frac{r(t)x''(t)}{x^2(t)}x'(t).$$

By using (1.2), (A_1) and (A_2), we obtain

$$\begin{aligned}\omega'(t) \leq{}& \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)p_*(t)\frac{x'(t)}{x(t)} - k\rho(t)\sum_{i=1}^{n} q_i(t)\frac{x(g_i(t))}{x(t)}\\ &-\rho(t)\frac{r(t)x''(t)}{x^2(t)}x'(t).\end{aligned} \tag{3.8}$$

Now, from Lemma 1.1, (3.4), we have

$$\frac{x'(t)}{x(t)} \leq \frac{2}{t},$$

for $t \geq t_2$. Integrating this inequality from $g_i(t)$ to t, we see that

$$\frac{x(g_i(t))}{x(t)} \geq 0,$$

which with (3.7) and (3.8) gives

$$\begin{aligned}\omega'(t) \leq{}& -k\frac{\rho(t)}{t^2}\sum_{i=1}^{n} q_i(t)g_i^2(t) + \left(\frac{\rho'(t)}{\rho(t)} - (t-t_2)\frac{p_*(t)}{r(t)}\right)\omega(t)\\ &-\frac{(t-t_2)}{\rho(t)r(t)}\omega^2(t).\end{aligned}$$

This implies that

$$\omega'(t) \leq -k\frac{\rho(t)}{t^2}\sum_{i=1}^{n} q_i(t)g_i^2(t) + \frac{[\rho'(t)r(t) - (t-t_2)\rho(t)p_*(t)]^2}{4(t-t_2)\rho(t)r(t)}. \tag{3.9}$$

Integrating (3.9) from t_3 to t, we have,

$$\int_{t_3}^{t}\left(k\frac{\rho(s)}{s^2}\sum_{i=1}^{n}q_i(s)\,g_i^2(s)-\frac{\left[\rho'(s)\,r(s)-(s-t_2)\,\rho(s)\,p_*(s)\right]^2}{4\,(s-t_2)\,\rho(s)\,r(s)}\right)ds<\omega(t_2),$$

where $t_3>t_2$, which contradicts (3.5). This completes the proof. □

Theorem 3.2. *Assume that $r'(t)>0$, (C_3) holds and the second-order differential equation* (3.1) *is nonoscillatory. If there exists a positive function $\rho(t)$ such that*

$$\int_{t_0}^{\infty}\left(\rho(s)\left(p_*(s)+kl\frac{g^2(s)}{2s}\sum_{i=1}^{n}q_i(s)\right)-\frac{r(s)\left(\rho'(s)\right)^2}{4\rho(s)}\right)ds=\infty, \tag{3.10}$$

where $g(t)=\min\{g_i(t):i=1,2,\ldots,n\}$ and $l\in(0,1)$ arbitrarily chosen, then every solution of Eq. (1.2) *is either oscillatory or tends to zero as $t\to\infty$.*

Proof. Proceeding as in the proof of Theorem 3.1, we see that (3.6) holds for $t\geq t_2$. Thus, by Lemma 1.2 with $u(t)=x'(t)$, we have for $l\in(0,1)$

$$\frac{1}{x'(t)}\geq l\frac{g(t)}{t}\frac{1}{x'(g(t))}. \tag{3.11}$$

Using Lemma 1.1, (3.4) and (3.11), we obtain

$$\frac{x(g_i(t))}{x'(t)}\geq\frac{x(g(t))}{x'(t)}\geq l\frac{g^2(t)}{2t}.$$

Next, we define

$$\omega(t)=\rho(t)\,\frac{r(t)\,x''(t)}{x'(t)}.$$

The rest of the proof runs as in Theorem 3.1. The proof is complete. □

Theorem 3.3. *Assume that (C_3) holds and the second-order differential equation* (3.1) *is nonoscillatory. If there exists a positive function $\rho(t)$ such that*

$$\int_{t_0}^{\infty}\left(k\rho(s)\sum_{i=1}^{n}q_i(s)-\frac{\left[\rho'(s)-\rho(s)\,p_*(s)\,R_{t_0}(g(s))\right]^2}{4\rho(s)\,R_{t_0}(g(s))\,g'(s)}\right)ds=\infty, \tag{3.12}$$

where $R_u(t)=\int_u^t r^{-1}(s)\,ds$, $u\geq t_0$ and $g(t)=\min\{g_i(t):i=1,2,\ldots,n\}$, then every solution of Eq. (1.2) *is either oscillatory or tends to zero as $t\to\infty$.*

P r o o f. Let x be a nonoscillatory solution of Eq. (1.2). Let, without loss of generality, that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. By Lemma 3.1, there exists a $t_2 \geq t_1$ such that $x'(t) > 0$ or $x'(t) < 0$ for $t \geq t_2$. If $x'(t) < 0$, from Lemma 2.5, we get that $\lim_{t\to\infty} x(t) = 0$. Next, assume that $x'(t) > 0$. Then, by Lemma 3.2, we see that (3.4) holds for $t \geq t_2$. Thus, we get

$$\begin{aligned} x'(t) &= x'(t_2) + \int_{t_2}^{t} \frac{r(s)\, x''(s)}{r(s)} ds \\ &\geq \left[r(t)\, x''(t)\right] R_{t_2}(t). \end{aligned}$$

Since $g(t) \leq g_i(t) \leq t$ and $\left(r(t)\, x''(t)\right)' \leq 0$ for $i = 1, 2, \ldots, n$, $t \geq t_2$, we obtain

$$x'(t) \geq x'(g(t)) \geq \left[r(t)\, x''(t)\right] R_{t_2}(g(t)). \tag{3.13}$$

Now, we define

$$\omega(t) = \rho(t) \frac{r(t)\, x''(t)}{x(g(t))}.$$

By using Eq. (1.2) and (3.13), we have

$$\begin{aligned} \omega'(t) \leq\; & -k\rho(t) \sum_{i=1}^{n} q_i(t) + \left(\frac{\rho'(t)}{\rho(t)} - p_*(t)\, R_{t_2}(g(t))\right) \omega(t) \\ & - \frac{R_{t_2}(g(t))\, g'(t)}{\rho(t)} \omega^2(t). \end{aligned} \tag{3.14}$$

The rest of the proof is the same in Theorem 3.1, and hence, is omitted. □

In the following, we present some new oscillation results for Eq. (1.2), by using an integral averages condition of Philos-type [15]. First, we introduce a class of functions $\Im$. Let

$$D_0 = \{(t, s) : t > s \geq t_0\} \text{ and } D = \{(t, s) : t \geq s \geq t_0\}.$$

A kernel function $H \in C(D, \mathbb{R})$ is said to belong to the function class $\Im$, written by $H \in \Im$, if

(i) $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ on D_0.

(ii) $H(t, s)$ has a continuous and nonpositive partial derivative $\partial H / \partial s$ on D_0 such that the condition

$$-\frac{\partial H(t, s)}{\partial s} = h(t, s) \sqrt{H(t, s)} \text{ for } (t, s) \in D_0$$

is satisfied for some $h \in C(D, \mathbb{R})$.

Theorem 3.4. *Assume that* (C_3) *holds and the second-order differential equation* (3.1) *is nonoscillatory. If there exist functions* $H \in \Im$ *and* $\rho \in C(t_0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left(\Theta(s) - \frac{Q^2(t,s)}{\eta(s)} \right) ds = \infty, \tag{3.15}$$

where

$$\Theta(t) = k\rho(t) \sum_{i=1}^{n} q_i(t), \quad \eta(t) = \frac{R_{t_2}(g(t))\, g'(t)}{\rho(t)}$$

and

$$Q(t,s) = \frac{\rho'(t)}{\rho(t)} - p_*(t) R_{t_0}(g(t)) - \frac{h(t,s)}{\sqrt{H(t,s)}},$$

then every solution of Eq. (1.2) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.3, we see that (3.14) holds for $t \geq t_2$. Thus, we have

$$\omega'(t) \leq -\Theta(t) + \lambda(t)\omega(t) - \eta(t)\omega^2(t), \tag{3.16}$$

where

$$\lambda(t) = \frac{\rho'(t)}{\rho(t)} - p_*(t) R_{t_2}(g(t)).$$

Multiplying (3.16) by $H(t,s)$ and integrating from t_2 to t, we get

$$\begin{aligned}
\int_{t_2}^{t} H(t,s)\Theta(s)\,ds &\leq -\int_{t_2}^{t} H(t,s)\omega'(s)\,ds + \int_{t_2}^{t} H(t,s)\lambda(s)\omega(s)\,ds \\
&\quad - \int_{t_2}^{t} H(t,s)\eta(s)\omega^2(s)\,ds \\
&\leq H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H(t,s)\left(\lambda(s) - \frac{h(t,s)}{\sqrt{H(t,s)}}\right)\omega(s)\,ds \\
&\quad - \int_{t_2}^{t} H(t,s)\eta(s)\omega^2(s)\,ds,
\end{aligned}$$

and hence,

$$\int_{t_2}^{t} H(t,s)\Theta(s)\,ds \leq H(t,t_2)\omega(t_2) - \int_{t_2}^{t} H(t,s)\left(\eta(s)\omega^2(s) - Q(t,s)\omega(s)\right) ds.$$

It follows that

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\left(\Theta(s)-\frac{Q^2(t,s)}{4\eta(s)}\right) \le \omega(t_2)-\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\,\eta(s)\left(\omega(s)-\frac{Q(t,s)}{2\eta(s)}\right)^2 ds. \tag{3.17}$$

This implies

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\left(\Theta(s)-\frac{Q^2(t,s)}{\eta(s)}\right) ds \le \omega(t_2),$$

which contradicts (3.15). This completes the proof. □

The following oscillation criteria treat the cases when it is not possible to verify easily conditions (3.15).

Theorem 3.5. *Assume that* (C_3) *holds, the second-order differential equation* (3.1) *is nonoscillatory and let*

$$0 < \inf_{s\ge T}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)}\right] \le \infty \tag{3.18}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\,\frac{Q^2(t,s)}{\eta(s)}ds < \infty. \tag{3.19}$$

If there exists $\psi \in C([t_0,\infty),\mathbb{R})$ *such that for* $t \ge T$

$$\limsup_{t\to\infty}\int_{t_0}^{t} \eta(s)\,\psi_+^2(s)\,ds \tag{3.20}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t} H(t,s)\left(\Theta(s)-\frac{Q^2(t,s)}{4\eta(s)}\right) ds \ge \sup_{t\ge T}\psi(t), \tag{3.21}$$

where $\psi_+(t)=\max\{\psi(t),0\}$*, then every solution of Eq.* (1.2) *is either oscillatory or tends to zero as* $t\to\infty$*.*

Proof. As in the proof of Theorem 3.4, we get that (3.17) holds for $t \geq t_2$. Then, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} H(t,s) \left(\Theta(s) - \frac{Q^2(t,s)}{4\eta(s)} \right)$$
$$\leq \omega(t_2) - \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} H(t,s)\,\eta(s) \left(\omega(s) - \frac{Q(t,s)}{2\eta(s)} \right)^2 ds.$$

From (3.21), we obtain

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} H(t,s)\,\eta(s) \left(\omega(s) - \frac{Q(t,s)}{2\eta(s)} \right)^2 ds \\ &\leq \omega(t_2) - \psi(t_2) < \infty. \end{aligned} \tag{3.22}$$

Now, we define the functions

$$\begin{aligned} \Phi(t) &= \frac{1}{H(t,t_2)} \int_{t_2}^{t} H(t,s)\,\eta(s)\,\omega^2(s)\,ds, \\ \Psi(t) &= \frac{1}{H(t,t_2)} \int_{t_2}^{t} H(t,s)\,Q(t,s)\,\omega(s)\,ds. \end{aligned}$$

So that (3.22) implies that

$$\liminf_{t\to\infty} (\Phi(t) - \Psi(t)) < \infty.$$

The remainder of the proof is similar to the proof of Theorem 5.2 in [7] or [9] and hence is omitted. □

Remark 3.1. Consider Example 2.1, if $n = 1$. This implies that a sufficient condition for the oscillation of (2.12) is $2a + b > 1/2$. On the other hand, if we choose $k = 1$, $\rho \in C([t_0,\infty))$ and $\rho(t) = t$, then equation (3.1) becomes

$$v''(t) + \frac{a}{t^2} v(t) = 0. \tag{3.23}$$

Apply Theorem 3.2, it is clear that (C_3) is satisfied and the Euler equation (3.23) is nonoscillatory. Since $l < 1$, we see that condition (3.10) becomes $2a + b > 1/2$.

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