Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **42** (2016), 175–186

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

COEFFICIENT BOUNDS FOR TWO NEW SUBCLASSES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Communicated by O. Mushkarov

Abstract. In this paper, we consider two new subclasses $N^{\mu}_{\Sigma_m}(\alpha, \lambda)$ and $N^{\mu}_{\Sigma_m}(\beta, \lambda)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U. Furthermore, we establish bounds for the coefficients for these subclasses and several related classes are also considered and connections to earlier known results are made.

1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A consisting of the form (1) which are also univalent in U.

2010 *Mathematics Subject Classification:* 30C45, 30C50.
Key words: analytic functions, m-fold symmetric bi-univalent functions, Coefficient bounds.

The Koebe one-quarter theorem [6] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U.

For a brief history and interesting examples in the class Σ, see [15]. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familier Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [15]).

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in U, \quad m \in \mathbb{N}) \tag{3}$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [11], [14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in U, \quad m \in \mathbb{N}). \tag{4}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (4). In fact, the functions in the class S are *one*-fold symmetric.

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [17], is given as follows:

$$
\begin{aligned}
(5) \qquad g(w) = {} & w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1})\right] w^{2m+1} \\
& - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 \right. \\
& \left. \qquad -(3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots .
\end{aligned}
$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. For $m = 1$, the formula (5) coincides with the formula (2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$
\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right].
$$

Lewin [10] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [13] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^\star(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$ respectively (see [13]). The classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^\star(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [15] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [2], [7], [12], [15], [16], [18]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([5], [8], [9]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\};\ \ \mathbb{N} = \{1,2,3,\dots\})$ is still an open problem.

The aim of the this paper is to introduce two new subclasses of the function class Σ_m and derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. We have to remember here the following lemma here so as to derive our basic results:

Lemma 1 ([14]). *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ *is an analytic function in* U *with positive real part, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

2. Coefficient bounds for the function class $N_{\Sigma_m}^{\mu}(\alpha, \lambda)$.

Definition 2. *A function* $f \in \Sigma_m$ *is said to be in the class* $N_{\Sigma_m}^{\mu}(\alpha, \lambda)$ *if the following conditions are satisfied:*

$$\left| \arg \left((1-\lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \geq 0,\ z \in U)$$

and

$$\left| \arg \left((1-\lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1, \lambda \geq 1,\ \mu \geq 0,\ w \in U)$$

where the function $g = f^{-1}$.

Theorem 3. *Let* f *given by* (4) *be in the class* $N_{\Sigma_m}^{\mu}(\alpha, \lambda)$, $0 < \alpha \leq 1$. *Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)]+(1-\alpha)(\mu+\lambda m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha}{\mu + 2\lambda m} + \frac{2\left[\mu(m+1)(1-\lambda) + \lambda(2m+1)(m+\mu) + (1-\mu)\lambda m\right]\alpha^2}{(\mu+\lambda m)^2(\mu + 2\lambda m)}.$$

P r o o f. Let $f \in N^{\mu}_{\Sigma_m}(\alpha,\lambda)$. Then

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = [p(z)]^{\alpha} \tag{6}$$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = [q(w)]^{\alpha} \tag{7}$$

where $g = f^{-1}$, p, q in P and have the forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots .$$

Now, equating the coefficients in (6) and (7), we get

$$(\mu + \lambda m)a_{m+1} = \alpha p_m, \tag{8}$$

$$(\mu + 2\lambda m)a_{2m+1} + (\mu + 2\lambda m)\frac{(\mu-1)}{2}a^2_{m+1} = \alpha p_{2m} + \tfrac{\alpha(\alpha-1)}{2}p^2_m, \tag{9}$$

and

$$-(\mu + \lambda m)a_{m+1} = \alpha q_m, \tag{10}$$

$$\begin{aligned}\left[\mu(m+1)(1-\lambda) + \lambda(2m+1)(m+\mu) + \frac{\mu(\mu-1)}{2}\right] a^2_{m+1} \\ -(\mu + 2\lambda m)a_{2m+1} = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q^2_m.\end{aligned} \tag{11}$$

From (8) and (10) we obtain

$$p_m = -q_m. \tag{12}$$

and

$$2(\mu + \lambda m)^2 a^2_{m+1} = \alpha^2(p^2_m + q^2_m). \tag{13}$$

Also form (9), (11) and (13) we have

$$[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)]\,a_{m+1}^2$$

$$=\alpha\left(p_{2m}+q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

$$=\alpha\left(p_{2m}+q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\frac{2(\mu+\lambda m)^2}{\alpha^2}a_{m+1}^2.$$

Therefore, we have

$$(14)\quad a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{\alpha[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)]+(1-\alpha)(\mu+\lambda m)^2}.$$

Appying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha\left[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)\right]+(1-\alpha)(\mu+\lambda m)^2}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (11) from (9), we obtain

$$2(\mu+2\lambda m)a_{2m+1}+[(\mu-1)\lambda m-\mu(m+1)(1-\lambda)-\lambda(2m+1)(m+\mu)]\,a_{m+1}^2$$

$$=\alpha\left(p_{2m}-q_{2m}\right)+\tfrac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2).$$

Then, in view of (12) and (13) , and appying Lemma 1 for the coefficients p_{2m}, p_m and q_{2m}, q_m ,we have

$$|a_{2m+1}| \leq \frac{2\alpha}{\mu+2\lambda m}+\frac{2\left[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(1-\mu)\lambda m\right]\alpha^2}{(\mu+\lambda m)^2(\mu+2\lambda m)}.$$

This completes the proof of Theorem 3. □

3. Coefficient bounds for the function class $N^{\mu}_{\Sigma_m}(\beta, \lambda)$

Definition 4. *A function* $f \in \Sigma_m$ *given by* (4) *is said to be in the class* $N^{\mu}_{\Sigma_m}(\beta, \lambda)$ *if the following conditions are satisfied:*

$$\mathrm{Re}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \beta \qquad (0 \le \beta < 1,\ \lambda \ge 1,\ \mu \ge 0,\ z \in U) \tag{15}$$

and

$$\mathrm{Re}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \beta \qquad (0 \le \beta < 1,\ \lambda \ge 1,\ \mu \ge 0,\ w \in U) \tag{16}$$

where the function $g = f^{-1}$.

Theorem 5. *Let* f *given by* (4) *be in the class* $N^{\mu}_{\Sigma_m}(\beta, \lambda)$, $0 \le \beta < 1$. *Then*

$$|a_{m+1}| \le \sqrt{\frac{4\,(1-\beta)}{\mu(m+1)(1-\lambda) + \lambda(2m+1)(m+\mu) + (\mu-1)(\mu+\lambda m)}}$$

and

$$|a_{2m+1}| \le \frac{2\left[\mu(m+1)(1-\lambda) + \lambda(2m+1)(m+\mu) + (1-\mu)\lambda m\right](1-\beta)^2}{(\mu+\lambda m)^2(\mu+2\lambda m)} + \frac{2\,(1-\beta)}{\mu+2\lambda m}.$$

Proof. Let $f \in N^{\mu}_{\Sigma_m}(\beta, \lambda)$. Then

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \beta + (1-\beta)p(z) \tag{17}$$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = \beta + (1-\beta)q(w) \tag{18}$$

where $p, q \in P$ and $g = f^{-1}$.

It follows from (17) and (18) that

$$(\mu+\lambda m)a_{m+1}=(1-\beta)p_m, \tag{19}$$

$$(\mu+2\lambda m)a_{2m+1}+(\mu+2\lambda m)\frac{(\mu-1)}{2}a_{m+1}^2=(1-\beta)p_{2m}, \tag{20}$$

and

$$-(\mu+\lambda m)a_{m+1}=(1-\beta)q_m, \tag{21}$$

$$\left[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+\frac{\mu(\mu-1)}{2}\right]a_{m+1}^2 - (\mu+2\lambda m)a_{2m+1}=(1-\beta)q_{2m}. \tag{22}$$

From (19) and (21) we obtain

$$p_m=-q_m. \tag{23}$$

and

$$2(\mu+\lambda m)^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{24}$$

Adding (20) and (22), we have

$$\left[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)\right]a_{m+1}^2 = (1-\beta)\left(p_{2m}+q_{2m}\right).$$

Therefore, we obtain

$$a_{m+1}^2=\frac{(1-\beta)\left(p_{2m}+q_{2m}\right)}{\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)}.$$

Appying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\sqrt{\frac{4\left(1-\beta\right)}{\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(\mu-1)(\mu+\lambda m)}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (22) from (20), we obtain

$$2(\mu+2\lambda m)a_{2m+1}+\left[(\mu-1)\lambda m-\mu(m+1)(1-\lambda)-\lambda(2m+1)(m+\mu)\right]a_{m+1}^2 = (1-\beta)\left(p_{2m}-q_{2m}\right).$$

Then, in view of (23) and (24), appying Lemma 1 for the coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$|a_{2m+1}| \leq \frac{2\left[\mu(m+1)(1-\lambda)+\lambda(2m+1)(m+\mu)+(1-\mu)\lambda m\right](1-\beta)^2}{(\mu+\lambda m)^2(\mu+2\lambda m)} + \frac{2(1-\beta)}{\mu+2\lambda m}.$$

This completes the proof of Theorem 5. □

If we set $\mu=0$ and $\lambda=1$ in Theorems 3 and 5, then the classes $N_{\Sigma_m}^{\mu}(\alpha,\lambda)$ and $N_{\Sigma_m}^{\mu}(\beta,\lambda)$ reduce to the classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ and thus, we obtain following corollaries:

Corollary 6. *Let f given by (4) be in the class $S_{\Sigma_m}^{\alpha}$ $(0<\alpha\leq 1)$. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

Corollary 7. *Let f given by (4) be in the class $S_{\Sigma_m}^{\beta}$ $(0\leq\beta<1)$. Then*

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}.$$

The classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ are respectively defined as follows:

Definition 8. *A function $f\in\Sigma_m$ given by (4) is said to be in the class $S_{\Sigma_m}^{\alpha}$ if the following conditions are satisfied:*

$$f\in\Sigma, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0<\alpha\leq 1,\ w\in U)$$

where the function $g=f^{-1}$.

Definition 9. *A function $f \in \Sigma_m$ given by* (4) *is said to be in the class $S^{\beta}_{\Sigma_m}$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (0 \le \beta < 1, \quad z \in U)$$

and

$$\mathrm{Re}\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (0 \le \beta < 1, \quad w \in U)$$

where the function $g = f^{-1}$.

For *one*-fold symmetric bi-univalent functions and $\mu = 0$, $\lambda = 1$, Theorem 3 and Theorem 5 reduce to Corollary 10 and Corollary 11, respectively, which were proven earlier by Murugunsundaramoorthy et al. [12].

Corollary 10. *Let f given by* (4) *be in the class $S^{*}_{\Sigma}(\alpha)$ $(0 < \alpha \le 1)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$|a_3| \le 4\alpha^2 + \alpha.$$

Corollary 11. *Let f given by (4) be in the class $S^{*}_{\Sigma}(\beta)$ $(0 \le \beta < 1)$. Then*

$$|a_2| \le \sqrt{2\,(1-\beta)}$$

and

$$|a_3| \le 4(1-\beta)^2 + (1-\beta).$$

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Received August 19, 2015