Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **43** (2017), 35–48

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DIFFERENTIAL GEOMETRY OF CONCIRCULAR SUBMANIFOLDS OF EUCLIDEAN SPACES

Bang-Yen Chen, Shihshu Walter Wei*

Communicated by O. Mushkarov

Abstract. A Euclidean submanifold is called a rectifying submanifold if its position vector field $\mathbf{x}$ always lies in its rectifying subspace [7]. It was proved in [7] that a Euclidean submanifold M is rectifying if and only if the tangential component $\mathbf{x}^T$ of its position vector field is a concurrent vector field.

Since concircular vector fields are natural extension of concurrent vector fields, it is natural and fundamental to study a Euclidean submanifold M such that the tangential component $\mathbf{x}^T$ of the position vector field $\mathbf{x}$ of M is a concircular vector field. We simply call such a submanifold a *concircular submanifold*. The main purpose of this paper is to study concircular submanifolds in a Euclidean space. Our main result completely classifies concircular submanifolds in an arbitrary Euclidean space.

1. Introduction. Let $\mathbb{E}^3$ denote the Euclidean 3-space with inner product $\langle\ ,\ \rangle$. Consider a unit speed space curve $x : I \to \mathbb{E}^3$, where $I = (\alpha, \beta)$ is an open interval. Let $\mathbf{x}$ denote the position vector field of x and its derivative $\mathbf{x}'$ be denoted by $\mathbf{t}$. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ the Frenet-Serret apparatus of x with curvature κ, torsion τ, unit tangent vector field $\mathbf{t}$, the principal normal vector

2010 *Mathematics Subject Classification:* 53A07, 53C40, 53C42.
Key words: Euclidean submanifold, position vector field, concurrent vector field, concircular vector field, rectifying submanifold.
*Research supported in part by NSF (DMS-1447008).

field $\mathbf{n}$ and the binormal vector field $\mathbf{b}$. Then the famous Frenet-Serret equations are given by

$$\begin{cases} \mathbf{t}' = \kappa\mathbf{n}, \\ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' = -\tau\mathbf{n}. \end{cases} \tag{1.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$, and $\{\mathbf{n}, \mathbf{b}\}$ are well-known as the *osculating plane*, the *rectifying plane*, and the *normal plane* of the curve, respectively.

The fundamental theorem of curves states that for two given smooth functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a curve $x : I \to \mathbb{E}^3$ such that s is the arc length, $\kappa(s)$ is the curvature function, and $\tau(s)$ is the torsion function of x; moreover, any other curve satisfying the same conditions differs from x by a rigid motion.

From elementary differential geometry, it is well known that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point, and it lies on a sphere if its position vector always lies in its normal plane. In view of these basic facts, the first author called a space curve a *rectifying curve* in [3] if its position vector field always lies in its rectifying plane.

The first author extended the notion of rectifying plane to the notion of *rectifying subspace* in [7]. Furthermore, he introduced the notion of *rectifying submanifolds*, by defining a Euclidean submanifold to be a rectifying submanifold if its position vector field always lies in its rectifying subspace. The first author also investigated and classified rectifying submanifolds in [7, 9]. In particular, he showed that a Euclidean submanifold is rectifying if and only if the tangential component $\mathbf{x}^T$ of its position vector field $\mathbf{x}$ is a concurrent vector field.

Since concircular vector fields are natural extension of concurrent vector fields, it is natural and fundamental to study a Euclidean submanifold M such that the tangential component $\mathbf{x}^T$ of the position vector field $\mathbf{x}$ of M is a concircular vector field. We simply call such a submanifold a *concircular submanifold.*

In this paper, we study some fundamental properties of concircular submanifolds. Our main result completely classifies concircular submanifolds of Euclidean spaces.

2. Preliminaries. Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into a Euclidean m-space $\mathbb{E}^m$. For each point $p \in M$, we denote by T_pM and $T_p^{\perp}M$ the tangent and the normal spaces of M at p, respectively.

Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and $\mathbb{E}^m$, respectively. Then the formulas of Gauss and Weingarten are then given respectively by (cf. [5, 10])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connection, and A the shape operator of M.

At a given point $p \in M$, the *first normal space* of M in $\mathbb{E}^m$, denoted by $\operatorname{Im} h_p$, is the subspace given by

$$\operatorname{Im} h_p = \operatorname{Span}\{h(X,Y) : X, Y \in T_pM\}. \tag{2.3}$$

For each normal vector ξ at p, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle\,, \tag{2.4}$$

where $\langle\ ,\ \rangle$ is the inner product on M as well as on the ambient Euclidean space.

The *equation of Gauss* of M in $\mathbb{E}^m$ is given by

$$R(X,Y;Z,W) = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \tag{2.5}$$

for X, Y, Z, W tangent to M, where R is the Riemann curvature tensor of M defined by

$$R(X,Y;Z,W) = \langle \nabla_X \nabla_Y Z, W\rangle - \langle \nabla_Y \nabla_X Z, W\rangle - \left\langle \nabla_{[X,Y]} Z, W\right\rangle.$$

The mean curvature vector H of a submanifold M is defined by

$$H = \left(\frac{1}{n}\right) \operatorname{trace} h, \quad n = \dim M. \tag{2.6}$$

A Riemannian manifold is called a *flat space* if its curvature tensor R vanishes identically. Further, a submanifold M is called *totally umbilical* (respectively, *totally geodesic*) if its second fundamental form h satisfies $h(X,Y) = \langle X, Y\rangle H$ identically (respectively, $h = 0$ identically).

Let B and Q be two Riemannian manifolds with metric tensors g_B and g_Q, respectively, and f be a positive smooth function on B. Then the *warped product* $B \times_f Q$ is the product manifold $B \times Q$ equipped with the metric tensor

$$g = g_B + f^2 g_Q,$$

where f is called the *warping function* (cf. [1, 10, 13]).

3. Basic results on $\mathbf{x}$, $\mathbf{x}^T$ and $\mathbf{x}^N$. It follows from the definition of a rectifying curve $x : I \to \mathbb{E}^3$ that the position vector field $\mathbf{x}$ of a rectifying curve satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \eta(s)\mathbf{b}(s) \tag{3.1}$$

for some functions λ and η.

For a curve $x : I \to \mathbb{E}^3$ with $\kappa(s_0) \neq 0$ at $s_0 \in I$, the first normal space at s_0 is the line spanned by the principal normal vector $\mathbf{n}(s_0)$. Thus the rectifying plane at s_0 is nothing but the plane orthogonal to the first normal space. For an arbitrary submanifold M of $\mathbb{E}^m$, we simply call the orthogonal complement subspace to the first normal space $\operatorname{Im} h_p$ at $p \in M$ the *rectifying space of M at p* (cf. [7]).

Analogous to rectifying curves in [3], the first author introduced the notion of rectifying submanifolds in [7] defined as follows.

Definition 3.1. *A submanifold M of a Euclidean m-space $\mathbb{E}^m$ is called a* rectifying submanifold *if its position vector field $\mathbf{x}$ always lies in its rectifying space. In other words, M is called a rectifying submanifold if and only if*

$$\langle \mathbf{x}(p), \operatorname{Im} h_p \rangle = 0 \tag{3.2}$$

holds at every point $p \in M$.

Definition 3.2. *A non-trivial vector field V on a Riemannian manifold M is called a concurrent vector field if it satisfies (cf. e.g.* [10, 16]*)*

$$\nabla_X V = X \tag{3.3}$$

for any $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of M and $\Gamma(TM)$ is the space of smooth cross sections in the tangent bundle TM of M.

Definition 3.3. *A non-trivial vector field Z on a Riemannian manifold M is called a concircular vector field if it satisfies (cf. e.g.* [6, 10, 15]*)*

$$\nabla_X Z = \varphi X, \quad X \in TM, \tag{3.4}$$

where φ is a smooth function on M, called the concircular function.

By a *cone* in $\mathbb{E}^m$ with vertex at the origin o we mean a ruled submanifold generated by a family of half lines through o. Obviously, a linear subspace of $\mathbb{E}^m$ containing the origin o is a special case of cone in this sense. A submanifold of $\mathbb{E}^m$ is called a *conic submanifold* with vertex at o if it is an open part of a cone with vertex at o.

For a Euclidean submanifold M, there exists a natural orthogonal decomposition of the position vector field $\mathbf{x}$ of M; namely,

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \tag{3.5}$$

where $\mathbf{x}^T$ and $\mathbf{x}^N$ are the tangential and normal components of $\mathbf{x}$, respectively. Let $|\mathbf{x}^T|$ and $|\mathbf{x}^N|$ denote the length of $\mathbf{x}^T$ and $\mathbf{x}^N$, respectively.

The following results can be found in [7].

Lemma 3.1. *Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian n-manifold into a Euclidean m-space $\mathbb{E}^m$. Then $\mathbf{x} = \mathbf{x}^T$ holds identically if and only if M is a conic submanifold with the vertex at the origin.*

Lemma 3.2. *Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a Riemannian n-manifold into $\mathbb{E}^m$. Then $\mathbf{x} = \mathbf{x}^N$ holds identically if and only if M lies in a hypersphere centered at the origin.*

In view of Lemma 3.1 and Lemma 3.2, we make the following.

Definition 3.4. *A submanifold M of $\mathbb{E}^m$ is called* proper *if its position vector field $\mathbf{x}$ satisfies $\mathbf{x} \neq \mathbf{x}^T$ and $\mathbf{x} \neq \mathbf{x}^N$ everywhere on M except a measure zero subset.*

We have the following characterization of rectifying submanifolds from [7].

Theorem 3.1. *Let M be a proper submanifold of a Euclidean m-space $\mathbb{E}^m$. Then M is a rectifying submanifold if and only if $\mathbf{x}^T$ is a concurrent vector field on M.*

Further basic results on $\mathbf{x}^T$ and $\mathbf{x}^N$ can be found in [2, 4, 8, 9] among others.

Obviously, concircular vector fields are natural extension of concurrent vector fields. Hence, in view of Theorem 3.1, we ask the following basic question.

Question 3.1. Which submanifolds of a Euclidean m-space $\mathbb{E}^m$ have concircular vector field $\mathbf{x}^T$?

For simplicity, we make the following.

Definition 3.5. *A proper submanifold M of a Euclidean space with* $\dim M \geq 2$ *is called a* concircular submanifold *if the tangential component $\mathbf{x}^T$ of its position vector field $\mathbf{x}$ is a concircular vector field on M.*

The *concircular function* of a concircular submanifold M is defined to be the concircular function φ of the concircular vector field $\mathbf{x}^T$ on M given in (3.4).

4. Some lemmas. Now, we provide five lemmas for the proof of our main result.

Lemma 4.1. *Let M a submanifold of a Euclidean m-space $\mathbb{E}^m$. Then the Levi-Civita connection ∇ and the normal connection D of M satisfy*

$$\nabla_Z \mathbf{x}^T = Z + A_{\mathbf{x}^N} Z, \tag{4.1}$$

$$D_Z \mathbf{x}^N = -h(\mathbf{x}^T, Z), \tag{4.2}$$

for any $Z \in \Gamma(TM)$.

Proof. Let M be a submanifold of $\mathbb{E}^m$. Then, by using the fact that the position vector field is a concurrent vector, we find from Gauss' and Weingarten's formulas that

$$Z = \tilde{\nabla}_Z \mathbf{x} = \nabla_Z \mathbf{x}^T + h(\mathbf{x}^T, Z) - A_{\mathbf{x}^N} Z + D_Z \mathbf{x}^N$$

for any $Z \in \Gamma(TM)$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathbb{E}^{n+1}$. Hence, by comparing the tangential and normal components of the last equation, we obtain formulas (4.1) and (4.2). □

Lemma 4.2. *A proper hypersurface M of $\mathbb{E}^{n+1}$ $(n \geq 2)$ is a concircular hypersurface if and only if either*

(1) *M is an open portion of a hyperplane L^n of $\mathbb{E}^{n+1}$ such that $o \notin L^n$, where o is the origin of $\mathbb{E}^{n+1}$, or*

(2) *M is an open portion of a hypersphere S^n such that the origin o of $\mathbb{E}^{n+1}$ is not the center of S^n.*

Further, M has constant concircular function $\varphi = 1$ in case (1)*; and M has non-constant concircular function $\varphi = 1 + \langle H, \mathbf{x} \rangle$ in case* (2)*.*

Proof. Let M be a concircular hypersurface of $\mathbb{E}^{n+1}$. Then we have $\nabla_Z \mathbf{x}^T = \varphi Z$ with a concircular function φ. Combining this with (4.1) gives

$$A_{\mathbf{x}^N} Z = (\varphi - 1) Z, \quad Z \in \Gamma(TM), \tag{4.3}$$

which shows that M is totally umbilical in $\mathbb{E}^{n+1}$.

Consequently, M is either an open portion of a hyperplane L^n or an open portion of a hypersphere S^n depending on M is totally geodesic or not totally geodesic.

From (2.4) and (4.3) we have

$$\nabla_Z \mathbf{x}^T = (1 + \langle H, \mathbf{x}^N \rangle) Z, \quad Z \in \Gamma(TM), \tag{4.4}$$

where H is the mean curvature vector of M.

Suppose that M is an open portion of a hyperplane L^n. Then $o \notin L^n$ since we have $\mathbf{x} \neq x^T$ according to Definition 3.5. Also, in this case it follows from (4.4) that $\nabla_Z \mathbf{x}^T = Z$. Thus M has constant concircular function $\varphi = 1$.

If M is an open portion of a hypersphere S^n. Then we know that the center of S^{n-1} is not the origin of $\mathbb{E}^{n+1}$ due to $\mathbf{x} \neq \mathbf{x}^N$. Thus, in this case, it is easy to show that the concircular function $\varphi = 1 + \langle H, \mathbf{x}\rangle$ of M is non-constant.

Conversely, if M is a hypersurface given either by case (1) or case (2), then it follows easily from (4.4) that M is a concircular hypersurface. □

Lemma 4.3. *Let M be a concircular submanifold of a Euclidean m-space $\mathbb{E}^m$ with codimension ≥ 2. Then there exists a local coordinate system $\{s, u_2, \ldots, u_n\}$ of M such that*

(a) $e_1 = \dfrac{\partial}{\partial s}$ *and* $\left\langle \dfrac{\partial}{\partial s}, \dfrac{\partial}{\partial u_j} \right\rangle = 0$ *for* $j = 2, \ldots, n$;

(b) $\dfrac{\partial}{\partial u_j}\langle \mathbf{x}^N, \mathbf{x}^N\rangle = 0$ *for* $j = 2, \ldots, n$;

(c) $\mu = \mu(s)$ *and* $\dfrac{\partial}{\partial s}\langle \mathbf{x}^N, \mathbf{x}^N\rangle = 2\mu(s)(1 - \mu'(s))$;

(d) $A_{\mathbf{x}^N} = (\mu'(s) - 1)I$, *where I denotes the identity map.*

Proof. Assume that M is a concircular submanifold of $\mathbb{E}^m$ with codimension ≥ 2. Let us define the unit vector field e_1 and the function μ on M by

$$\mathbf{x}^T = \mu e_1, \quad \mu = |\mathbf{x}^T|. \tag{4.5}$$

We may extend e_1 to a local orthonormal frame $e_1, \ldots, e_n$ on M. Since $\mathbf{x}^T$ is a concircular vector field on M, we derive from (3.4) and (4.5) that

$$\varphi Z = \nabla_Z \mathbf{x}^T = (Z\mu)e_1 + \mu \nabla_Z e_1, \quad Z \in \Gamma(TM), \tag{4.6}$$

where φ is the concircular function of $\mathbf{x}^T$. From (4.6) we find

$$e_1\mu = \varphi, \quad \nabla_{e_1} e_1 = 0, \tag{4.7}$$

$$e_j\mu = 0, \quad \nabla_{e_j} e_1 = \frac{\varphi}{\mu} e_j, \quad j = 2, \ldots, n. \tag{4.8}$$

If we define the connection forms ω_k^i, $i, k = 1, \ldots, n$, by

$$\nabla_Z e_k = \sum_{i=1}^{n} \omega_k^i(Z) e_i, \quad k = 1, \ldots, n, \tag{4.9}$$

then (4.8) and (4.9) yield

$$\omega_1^i(e_j) = \frac{\varphi}{\mu}\delta_{jk}, \quad j,k = 2,\ldots,n, \tag{4.10}$$

where δ_{jk} denote the Kronecker deltas.

Let us put

$$\mathcal{D} = \text{Span}\{e_1\}, \quad \mathcal{D}^{\perp} = \text{Span}\{e_2,\ldots,e_n\}.$$

Then it follows from (4.10) that $\mathcal{D}^{\perp}$ is an integrable distribution. Moreover, we know from the second equation in (4.7) that $\mathcal{D}$ is an integrable distribution whose integral curves are geodesics of M and hence $\mathcal{D}$ is a totally geodesic distribution. Therefore there exists a local coordinate system $\{s, u_2, \ldots, u_n\}$ on M such that

$$e_1 = \frac{\partial}{\partial s} \text{ and } \mathcal{D}^{\perp} = \text{Span}\left\{\frac{\partial}{\partial u_2},\ldots,\frac{\partial}{\partial u_n}\right\}. \tag{4.11}$$

Hence we have statement (a) of the lemma.

From (4.7) and (4.8) we find

$$\mu = \mu(s), \quad \varphi = \mu'(s), \quad \mu = \langle \mathbf{x}, e_1\rangle. \tag{4.12}$$

Thus, by applying (4.3) and (4.12), we get

$$A_{\mathbf{x}^N} Z = (\mu'(s) - 1)Z, \tag{4.13}$$

which gives statement (d).

After applying (2.4) and (4.13), we find

$$\left\langle h(Z,\mathbf{x}^T), \mathbf{x}^N\right\rangle = \left\langle A_{\mathbf{x}^N} Z, \mathbf{x}^T\right\rangle = (\mu'(s)-1)\left\langle Z,\mathbf{x}^T\right\rangle. \tag{4.14}$$

On the other hand, it follows from (4.2) and (4.14) that

$$\begin{aligned} Z\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle &= 2\left\langle D_Z\mathbf{x}^N, \mathbf{x}^N\right\rangle \\ &= -2\left\langle h(\mathbf{x}^T, Z), \mathbf{x}^N\right\rangle = 2(1-\mu'(s))\left\langle Z, \mathbf{x}^T\right\rangle, \end{aligned} \tag{4.15}$$

which implies statement (b).

Finally, we see from (4.15) that $\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle$ is a function depending only on s. This if we choose $Z = \dfrac{\partial}{\partial s}$, then we obtain statement (c) from (4.12) and (4.15). □

Lemma 4.4. *If M is a concircular submanifold of a Euclidean m-space with codimension ≥ 2, then M is locally a warped product $I \times_{\mu(s)} Q$ with warping function μ, where Q is a Riemannian manifold, $\mu = |\mathbf{x}^T|$ and $\mathbf{x}^T = \mu\dfrac{\partial}{\partial s}$.*

P r o o f. If M is a concircular submanifold of a Euclidean space, then it follows from [6, Theorem 3.1] that M is locally a warped product $I \times_{f(s)} N$ with warping function $f(s)$ for some Riemannian manifold N such that $\frac{\partial}{\partial s}$ is parallel to $\mathbf{x}^T$.

Since the metric tensor of $I \times_{f(s)} N$ is

$$g = ds^2 + f^2(s) g_N, \tag{4.16}$$

the Levi-Civita connection ∇ of M satisfies

$$\nabla_V \frac{\partial}{\partial s} = \frac{d(\ln f)}{ds} V \tag{4.17}$$

for any tangent vector V of N (see, e.g., [10, 13]).

On the other hand, (4.6) and (4.12) imply that the Levi-Civita connection of M also satisfies

$$\nabla_V \frac{\partial}{\partial s} = \frac{d(\ln \mu)}{ds} V. \tag{4.18}$$

Hence, after comparing (4.17) and (4.18), we obtain $(\ln f)' = (\ln \mu)'$, which implies $f(s) = \lambda \mu(s)$ for some nonzero constant λ. Consequently, M is locally a warped product $I \times_{\mu(s)} Q$ such that the metric tensor of Q is given by $g_Q = \lambda^2 g_N$. □

The next lemma is an easy consequence of Nash's embedding theorem [12].

Lemma 4.5. *For sufficiently large integer m, every Riemannian manifold M can be isometrically immersed in the unit hypersphere $S_o^{m-1}(1)$ of $\mathbb{E}^m$ centered at the origin $o \in \mathbb{E}^m$.*

P r o o f. Nash's embedding theorem states that every Riemannian manifold can be isometrically embedding in a Euclidean k-space $\mathbb{E}^k$ for some large k. Clearly, $\mathbb{E}^k$ can be isometrically mapped into a flat k-torus T^k in $S_o^{2k-1}(1) \subset \mathbb{E}^{2k}$. Therefore, for sufficiently large m, every Riemannian manifold can be isometrically immersed into the unit hypersphere $S_o^{m-1}(1)$ of $\mathbb{E}^m$ centered at the origin. □

5. Main results.

The following main result completely classifies concircular submanifolds.

Theorem 5.1. *Let M be a proper submanifold of a Euclidean m-space $\mathbb{E}^m$ with origin o. If $n = \dim M \geq 2$, then M is a concircular submanifold if and only if one of the following three cases occurs:*

(i) *M is an open portion of a linear n-subspace L^n of $\mathbb{E}^m$ such that $o \notin L$.*

(ii) *M is an open portion of a hypersphere S^n of a linear $(n+1)$-subspace L^{n+1} of $\mathbb{E}^m$ such that the origin of $\mathbb{E}^m$ is not the center of S^n.*

(iii) *$m \geq n+2$. Moreover, with respect to some suitable local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M the immersion x of M in $\mathbb{E}^m$ takes the following form:*

$$x(s, u_2, \ldots, u_n) = \sqrt{2\rho}\, Y(s, u_2, \ldots, u_n), \quad \langle Y, Y\rangle = 1, \tag{5.1}$$

where $Y : M \to S_o^{m-1}(1) \subset \mathbb{E}^m$ is an immersion of M into the unit hypersphere $S_o^{m-1}(1)$ such that the induced metric g_Y via Y is given by

$$g_Y = \frac{2\rho - \rho'^2}{4\rho^2} ds^2 + \frac{\rho'^2}{2\rho} \sum_{i,j=2}^{n} g_{ij}(u_2, \ldots, u_n) du_i du_j. \tag{5.2}$$

where $\rho = \rho(s)$ satisfies $2\rho > \rho'^2 > 0$ on an open interval I.

Proof. Assume that M is a concircular submanifold of $\mathbb{E}^m$ with $n = \dim M \geq 2$. If M lies in a totally geodesic $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$, then we obtain (i) or (ii) according to Lemma 4.2. Hence from now on we may assume that $m \geq n+2$.

Since M is a concircular submanifold, Lemma 4.4 implies that M is locally a warped product $I \times_{\mu(s)} Q$ such that $\dfrac{\partial}{\partial s}$ is parallel to $\mathbf{x}^T$, where $\mu = |\mathbf{x}^T|$ and Q is a Riemannian $(n-1)$-manifold. Thus the metric tensor of M is

$$g = ds^2 + \mu^2(s) g_Q, \tag{5.3}$$

where

$$g_Q = \sum_{i,j=2}^{n} g_{ij}(u_2, \ldots, u_n) du_i du_j \tag{5.4}$$

is the metric tensor of Q. Moreover, we also know that

$$\mathbf{x}^T = \mu(s) \frac{\partial}{\partial s}. \tag{5.5}$$

It follows from (4.5) or (5.5) and Lemma 4.3(3) that

$$|\mathbf{x}|^2 = |\mathbf{x}^T|^2 + |\mathbf{x}^N|^2 = \mu^2 + 2\int_{s_0}^{s} \mu(t)(1 - \mu'(t)) dt.$$

Hence we have

$$|\mathbf{x}|^2 = 2\rho \geq 0, \tag{5.6}$$

where $\rho(s)$ is an anti-derivative of $\mu(s)$, i.e., $\mu(s) = \rho'(s)$. If we put

$$F(s) = \sqrt{2\rho}, \tag{5.7}$$

then, according to (5.6), the position vector field of M takes the form:

$$\mathbf{x}(s,u_2,\ldots,u_n)=F(s)Y(s,u_2,\ldots,u_n), \tag{5.8}$$

where $Y : M \to S_o^{m-1}(1) \subset \mathbb{E}^m$ is a map of M into the unit hypersphere $S_o^{m-1}(1)$ centered at the origin o. Clearly, from (5.7) and (5.8) we have

$$\frac{\partial \mathbf{x}}{\partial s}=\frac{\rho'}{\sqrt{2\rho}}Y+\sqrt{2\rho}Y_s,\quad \frac{\partial \mathbf{x}}{\partial u_j}=\sqrt{2\rho}Y_{u_j},\quad j=2,\ldots,n. \tag{5.9}$$

Also, we find from (5.3), (5.9), $\langle Y,Y\rangle=1$ and $\rho'=\mu$ that

$$\begin{aligned}&\langle Y_s,Y_s\rangle=\frac{2\rho-\rho'^2}{4\rho^2},\quad \left\langle Y_s,Y_{u_j}\right\rangle=0,\\ &\left\langle Y_{u_i},Y_{u_j}\right\rangle=\frac{1}{2\rho}\left\langle \mathbf{x}_{u_i},\mathbf{x}_{u_j}\right\rangle,\quad i,j=2,\ldots,n.\end{aligned} \tag{5.10}$$

So, we conclude from (5.3) and (5.10) that the induced metric tensor g_Y of the spherical submanifold defined by Y is given by

$$g_Y=\frac{2\rho-\rho'^2}{4\rho^2}ds^2+\frac{\rho'^2}{2\rho}\sum_{i,j=2}^{n} g_{ij}(u_2,\ldots,u_n)du_idu_j. \tag{5.11}$$

Clearly, in order that g_Y to be well-defined, it requires that $2\rho>\rho'^2>0$.

Conversely, we know from Lemma 4.2 that submanifolds given by (i) and (ii) are concircular submanifolds.

Next, we would like to prove that a submanifold defined by (5.1) and (5.2) in (iii) gives rise to a concircular submanifold. In order to do so, let us assume that $\rho=\rho(s)$ is a function satisfying $2\rho>\rho'^2>0$ on an open interval I. We also assume that Q is a Riemannian $(n-1)$-manifold with metric tensor g_Q.

Let us consider the warped product $P=I\times Q$ with the warped product metric:

$$g_P=\frac{2\rho-\rho'^2}{4\rho^2}ds^2+\frac{\rho'^2}{2\rho}g_Q. \tag{5.12}$$

According to Lemma 4.5, for a sufficient large integer m, the warped product (P,g_P) admits an isometric immersion:

$$Y:(P,g_P)\to S_o^{m-1}(1)\subset \mathbb{E}^m \tag{5.13}$$

into the unit hypersphere $S_o^{m-1}(1)$ of $\mathbb{E}^m$ centered at the origin o.

Let us define the map $x : I\times Q\to \mathbb{E}^m$ by

$$x(s,u_2,\ldots,u_n)=\sqrt{2\rho(s)}\,Y(s,u_2,\ldots,u_n), \tag{5.14}$$

where $\{u_2, \ldots, u_n\}$ is a local coordinate system of Q. It is easy to verify from (5.12) and (5.14) that the induced metric tensor on $I \times Q$ via x is given by

$$g = ds^2 + \rho'(s)^2 g_Q. \tag{5.15}$$

A direct computation shows that the Levi-Civita connection of $(I \times Q, g)$ satisfies

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \;\; \nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial s} = \frac{\rho''(s)}{\rho'(s)} \frac{\partial}{\partial u_j}, \;\; j = 2, \ldots, n. \tag{5.16}$$

Also, it follows from (5.14) that the position vector field $\mathbf{x}$ of x satisfies

$$\frac{\partial \mathbf{x}}{\partial s} = \frac{\rho'}{\sqrt{2\rho}} Y + \sqrt{2\rho} Y_s, \;\; \frac{\partial \mathbf{x}}{\partial u_j} = \sqrt{2\rho} Y_{u_j}, \;\; j = 2, \ldots, n. \tag{5.17}$$

Therefore we obtain

$$\mathbf{x}^T = \rho'(s) \frac{\partial}{\partial s}. \tag{5.18}$$

By using (5.16), (5.17) and (5.18), it is easy to verify that $\nabla_Z \mathbf{x}^T = \rho''(s) Z$ holds for every $Z \in \Gamma(TM)$. Hence the immersion of $I \times Q$ into $\mathbb{E}^m$ via (5.14) is a concircular immersion whose concircular function is given by $\varphi = \rho''(s)$. Consequently, (5.1) together with (5.2) gives rise to a concircular submanifold in $\mathbb{E}^m$. □

6. An explicit example of concircular surfaces in $\mathbb{E}^4$. Theorem 5.1 shows that there exist ample examples of concircular submanifolds in Euclidean spaces.

The following provides one explicit example of concircular surface in $\mathbb{E}^4$.

Example 6.1. If we choose $n = 2$ and $\rho(s) = \frac{3}{8} s^2$, then the function defined by (5.7) becomes $F = \frac{\sqrt{3}}{2} s$. Thus (5.11) reduces to

$$g_Y = \frac{1}{3s^2} ds^2 + \frac{3}{4} du^2. \tag{6.1}$$

Let us define $Y : I_1 \times I_2 \to S_o^3(1) \subset \mathbb{E}^4$ to be the map of $I_1 \times I_2$ into $S_o^3(1)$ given by

$$Y(s, u) = \frac{1}{\sqrt{2}} \left(\cos\left(\tfrac{\sqrt{2}}{\sqrt{3}} \ln s\right), \sin\left(\tfrac{\sqrt{2}}{\sqrt{3}} \ln s\right), \cos\left(\tfrac{\sqrt{6}}{2} u\right), \sin\left(\tfrac{\sqrt{6}}{2} u\right) \right). \tag{6.2}$$

Then the induced metric tensor of $I_1 \times I_2$ via the map Y is given by (6.1). Therefore $P^2 = (I_1 \times I_2, g_Y)$ with the induced metric tensor g_Y is a flat surface.

Consider $x(s,u) : I_1 \times I_2 \to \mathbb{E}^4$ given by $x(s,u) = F(s)Y(s,u)$, i.e.,

$$x(s,u) = \frac{\sqrt{3}s}{2\sqrt{2}}\left(\cos\left(\tfrac{\sqrt{2}}{\sqrt{3}}\ln s\right), \sin\left(\tfrac{\sqrt{2}}{\sqrt{3}}\ln s\right), \cos\left(\tfrac{\sqrt{6}}{2}u\right), \sin\left(\tfrac{\sqrt{6}}{2}u\right)\right). \tag{6.3}$$

Then it is easy to verify that the induced metric via x is

$$g = ds^2 + \frac{9}{16}s^2 du^2. \tag{6.4}$$

Hence the Levi-Civita connection of $M = (I_1 \times I_2, g)$ satisfies

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \ \ \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial s} = \frac{1}{s}\frac{\partial}{\partial u}. \tag{6.5}$$

Using (6.3) and (6.4), it is easy to verify that the tangential component $\mathbf{x}^T = \frac{3}{4}s\frac{\partial}{\partial s}$ of the position vector field $\mathbf{x}$ is a concircular vector field satisfying $\nabla_Z \mathbf{x}^T = \frac{3}{4}Z$ for $Z \in TM$. Consequently, M is a concircular surface in $\mathbb{E}^4$.

REFERENCES

[1] R. L. Bishop, B. O'Neill. Manifolds of negative curvature. *Trans. Amer. Math. Soc.* **145** (1969), 1–49.

[2] B.-Y. Chen. Geometry of position functions of Riemannian submanifolds in pseudo-Euclidean space. *J. Geom.* **74**, *1–2* (2002), 61–77.

[3] B.-Y. Chen. When does the position vector of a space curve always lie in its rectifying plane?. *Amer. Math. Monthly* **110**, *2* (2003), 147–152.

[4] B.-Y. Chen. Constant-ratio space-like submanifolds in pseudo-Euclidean space. *Houston J. Math.* **29**, *2* (2003), 281–294.

[5] B.-Y. Chen. Pseudo-Riemannian geometry, δ-invariants and applications. Hackensack, NJ, World Scientific Publishing Co. Pte. Ltd., 2011.

[6] B.-Y. Chen. Some results on concircular vector fields and their applications to Ricci solitons. *Bull. Korean Math. Soc.* **52**, *5* (2015), 1535–1547.

[7] B.-Y. Chen. Differential geometry of rectifying submanifolds. *Int. Electron. J. Geom.* **9**, *2* (2016), 1–8.

[8] B.-Y. Chen. Topics in differential geometry associated with position vector fields on Euclidean submanifolds. *Arab J. Math. Sci.* **23**, *1* (2017), 1–17 (in: Special Issue on Geometry and Global Analysis).

[9] B.-Y. Chen. Addendum to: Differential geometry of rectifying submanifolds. *Int. Electron. J. Geom.* **10**, *1* (2017), 81–82.

[10] B.-Y. Chen. Differential geometry of warped product manifolds and submanifolds. Hackensack, NJ, World Scientific, 2017.

[11] S. Hiepko. Eine innere Kennzeichnung der verzerrten Produkte. *Math. Ann.* **241**, *3* (1979), 209–215.

[12] J. F. Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math.(2)* **63** (1956), 20–63.

[13] B. O'Neill. Semi-Riemannian geometry with applications to relativity. New York, Academic Press, 1983.

[14] J. A. Schouten. Ricci-calculus. An introduction to tensor analysis and its geometrical applications. 2nd ed. Berlin-Göttingen-Heidelberg, Springer-Verlag, 1954.

[15] K. Yano. On the torse forming direction in a Riemannian space. *Proc. Imp. Acad. Tokyo* **20** (1944), 340–345.

[16] K. Yano and B.-Y. Chen, On the concurrent vector fields of immersed manifolds. *Kōdai Math. Sem. Rep.* **23** (1971), 343–350.

Bang-Yen Chen
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824–1027, U.S.A.
e-mail: bychen@math.msu.edu

Shihshu Walter Wei
Department of Mathematics
University of Oklahoma
Norman, Oklahoma 73019-0315, U.S.A.
e-mail: wwei@ou.edu

Received February 8, 2017