Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **43** (2017), 65–78

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

EXTENDING THE CONVERGENCE DOMAIN OF NEWTON'S METHOD FOR GENERALIZED EQUATIONS

Ioannis K. Argyros, Santhosh George

Communicated by A. L. Dontchev

Abstract. We present semi-local convergence results for Newton's method to solve generalized equations. Using a combination of Lipschitz and center-Lipschitz conditions on the operators involved instead of just Lipschitz conditions we show that our Newton–Kantorovich criteria are weaker than earlier sufficient conditions for the convergence of Newton's method. In particular, we provide finer error bounds and a better information on the location of the solution. Our results apply to solve generalized equations involving single as well as multivalued operators, which include variational inequalities, nonlinear complementarity problems and non smooth convex minimization problems. Numerical examples validate the theoretical results by showing that equations that could not be solved before can be solved using our new approach.

2010 *Mathematics Subject Classification:* 65B05, 65G99, 65N35, 47H17, 49M15.

Key words: Hilbert space, generalized equation, Newton's method, Lipschitz conditions, Newton–Kantorovich hypothesis, local-semilocal convergence theorems, coercivity, multivalued maximal monotone operator, radius of convergence.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution x^* of the generalized equation

$$F(x) + G(x) \ni 0, \tag{1.1}$$

where, $F\colon D_0 \subseteq D \subseteq H \to H$ is a continuous operator which is Fréchet-differentiable at each point of the interior D_0 of a closed convex subset D of a Hilbert space H with values in H, and G is a multivalued maximal monotone operator from H into H (to be precised later) [3, 11, 14, 15].

The generalized Newton iteration

$$F'(x_n)(x_{n+1}) + G(x_{n+1}) \ni F'(x_n)(x_n) - F(x_n) \quad (n \geq 0) \tag{1.2}$$

has already been used to generate a sequence approximating x^*. In particular Uko [14,15] has provided local and semi-local convergence results for method (1.2) as well as a procedure for the computation of the inner-iterative procedures for the computation of the generalized iterates x_n $(n \geq 0)$. This way he extended the classical Newton–Kantorovich results to hold for non smooth generalized equations. His results extend earlier works on non smooth equations [4–13]. As in the classical cases Uko used Lipschitz differentiability conditions on F' and the maximality properties of G.

Here using a combination of center-Lipschitz and Lipschitz conditions we provide local and semilocal convergence results for method (1.2) with the following advantages over earlier works and in particular [15]:

(a) our results hold whenever the corresponding ones in [15] hold but not vice versa;

(b) our Newton–Kantorovich hypotheses sufficient for the convergence of (1.2) is weaker than the corresponding one in [15]; and

(c) our error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are finer and the information on the location of the solution x^* more precise.

Problems that are special cases of equation (1.1) have been in the literature for a long time. For example if $H = \mathbf{R}^j$ and $G(x_1, \dots, x_j) = G_1(x_1) \times \cdots \times G_j(x_j)$, where G_i, $i = 1, 2, \dots, j$ then (1.1) is called separable [11]. Moreover set

$$F(x_1, x_2, \dots, x_j) = \big(F_1(x_1, \dots, x_j), \dots, F_j(x_1, \dots, x_j)\big),$$

in which case (1.1) reduces to

$$F_i(x_1, \ldots, x_j) + G_i(x_i) \ni 0, \quad i = 1, \ldots, j.$$

Moreover as in [15] let

$$G_i = \{0\} \times (-\infty, 0] \cup (0, \infty) \times \{0\} \quad (i \geq 0)$$

to obtain the complementarity problem

$$F_i(x_1, \ldots, x_j) \geq 0,\ x_i \geq 0,\ i = 1, \ldots, j,\ \sum_{i=1}^{j} x_i F_i(x_1, \ldots, x_j) = 0.$$

These type of special cases of (1.1) have been studied extensively [9,12,14]. Furthermore if $\phi\colon H \to (-\infty, \infty)$ is a proper lower semicontinuous convex operator and

$$G(x) = \partial\varphi(x) = \{y \in H\colon \varphi(v) - \varphi(w) \leq \langle y, v - w\rangle, \quad \text{for all } w \in H\}$$

becomes the variational inequality

$$F(x) + \partial\varphi(x) \ni 0.$$

Other examples of special cases of (1.1) can be found in [1–3, 12, 14–18] and the references there.

2. Semi-local analysis. Throughout this section, we suppose

$$\|F'(x) - F'(y)\| \leq q\|x - y\| \tag{2.1}$$

$$\|F'(x) - F'(x_0)\| \leq q_0\|x - x_0\| \tag{2.2}$$

for all $x, y \in D_0$ and some fixed $x_0 \in D_0$. G is a nonempty subset of $H \times H$ so that there exists $a \geq 0$ such that

$$[x, y] \in G \text{ and } [v, w] \in G \Rightarrow \langle w - y, v - x\rangle \geq a\|x - v\|^2, \tag{2.3}$$

and which is not contained in any larger subset of $H \times H$.

We will use Lemma 2.2. from [15, p. 256]:

Lemma 2.1. *Let G be a maximal monotone operator satisfying* (2.3)*, and let M be a bounded linear operator from H into H. If there exists $c \in \mathbf{R}$ such that $c > -a$, and*

$$\langle M(x), x\rangle \geq c\|x\|^2 \quad \text{for all } x \in H, \tag{2.4}$$

then there exists a unique $z \in H$ for any $b \in H$ such that

$$M(z) + G(z) \ni b. \tag{2.5}$$

We need the following auxiliary result on majorizing sequences:

Lemma 2.2 ([2]). *Let $L > 0$, $L_0 > 0$ and $n \geq 0$ be parameters. Define parameter γ by*

$$\gamma = \frac{2L}{L + \sqrt{L^2 + 8L_0L}}. \tag{2.6}$$

Suppose that

$$h_1 = L_1\eta \leq 1, \tag{2.7}$$

where

$$L_1 = \frac{1}{4}(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}). \tag{2.8}$$

Then, scalar sequence $\{t_n\}$ defined for each $n = 0, 1, 2, \ldots$ by

$$t_0 = 0, \quad t_1 = \eta, t_2 = \eta + \frac{L_0\eta^2}{2(1 - L_0\eta)}, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0t_{n+1})} \tag{2.9}$$

is well defined, non-decreasing, bounded above by

$$t^{**} = \eta + \frac{L\eta^2}{2(1 - \gamma)(1 - L_0\eta)} \tag{2.10}$$

and converges quadratically to its unique least upper bound t^*, which satisfies

$$\eta \leq t^* \leq t^{**}. \tag{2.11}$$

Next, we present the main semilocal convergence theorem for method (1.2) using Lipschitz (2.1) and center-Lipschitz conditions (2.2).

Theorem 2.3. *Let F and G satisfy* (2.1), (2.2), (2.3) *and* (2.4), *respectively, for $M = F'(x_0)$. Let $x_0 \in D_0$. Suppose: there exists $y_0 \in H$ such that $G(x_0) \ni y_0$ and $\|F(x_0) + y_0\| \leq b_0$ for $b_0 > 0$. Moreover suppose* (2.7) *holds for*

$$L_0 = \frac{q_0}{c_0 + a}, \quad c_0 = c, \quad L = \frac{q}{c_0 + a}, \tag{2.12}$$

and

$$\overline{B}(x_0, t^*) \subseteq D. \tag{2.13}$$

Then generalized Newton's iteration $\{x_n\}$ $(n \geq 0)$ generated by (1.2) *is well defined, remains in $\overline{B}(x_0, t^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{B}(x_0, t^*)$. Moreover the following error bounds hold for all $n \geq 0$*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{2.14}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.15}$$

where $\{t_n\}$ is given in (2.9).

Proof. We use induction on $k = 0, 1, 2, \ldots$ to show:

$$x_k \in \overline{B}(x_0, t^*), \tag{2.16}$$

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \tag{2.17}$$

$$\overline{B}(x_{k+1}, t^* - t_{k+1}) \subseteq \overline{B}(x_k, t^* - t_k), \tag{2.18}$$

$$\exists y_k \in H \text{ such that } y_k \in G(x_k), \tag{2.19}$$

$$\exists b_k > 0 \text{ such that } \|F(x_k) + y_k\| \leq b_k, \tag{2.20}$$

$$\exists c_k > -a \text{ such that } \langle F'(x_k)(x), x\rangle \geq c_k\|x\|^2 \text{ for all } x \in H. \tag{2.21}$$

The induction is true if $k = 0$ for (2.16), (2.19)–(2.21) by the hypotheses of the theorem. It then follows from (2.21) and Lemma 2.1 that there exists a unique $x_1 \in H$ satisfying (1.2). By (2.3), (2.4) and (1.2) we obtain in turn

$$a\|x_1 - x_0\|^2 + \langle y_0 + F(x_0) - F'(x_0)(x_0 - x_1), x_1 - x_0\rangle \leq 0,$$

or

$$a\|x_1 - x_0\|^2 + \langle F'(x_0)(x_1 - x_0), x_1 - x_0\rangle \le \langle -F(x_0) - y_0, x_1 - x_0\rangle \tag{2.22}$$

or

$$\|x_1 - x_0\| \le a_0 = \frac{b_0}{c_0 + a} = t_1 - t_0. \tag{2.23}$$

For every $z \in \overline{B}(x_1, t^* - t_1)$,

$$\|z - x_0\| \le \|z - x_1\| + \|x_1 - x_0\| \le t^* - t_1 + t_1 = t^* - t_0, \tag{2.24}$$

implies $z \in \overline{B}(x_0, t^* - t_0)$. It follows from (2.23) and (2.24) that (2.17) and (2.18) hold for $k = 0$. Given they hold for $n = 0, \dots, k$ and again using (2.21) and Lemma 2.1 we conclude that there exists a unique $x_{k+1} \in H$ satisfying (1.2),

$$\begin{aligned} \|x_{k+1} - x_0\| &\le \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k+1} (t_i - t_{i-1}) \\ &= t_{k+1} - t_0 = t_{k+1} \le t^*, \qquad (2.25) \\ \|x_k + \theta(x_{k+1} - x_k) - x_0\| &\le t_k + \theta(t_{k+1} - t_k) < t^* \quad \theta \in [0,1]. \qquad (2.26) \end{aligned}$$

Hence (2.16) holds if k is replaced by $k+1$. As in (2.22) we obtain in turn

$$a\|x_{k+1} - x_k\|^2 + \langle y_k + F(x_k) - F'(x_k)(x_k - x_{k+1}), x_{k+1} - x_k\rangle \le 0$$

or

$$a\|x_{k+1} - x_k\|^2 + \langle F'(x_k)(x_{k+1} - x_k), x_{k+1} - x_k\rangle \le \langle -F'(x_k) - y_k, x_{k+1} - x_k\rangle \tag{2.27}$$

or

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k. \tag{2.28}$$

That is (2.17) and (2.18) hold for k replaced by $k+1$.

By (2.2) and (2.25) we get

$$\|F'(x_{k+1}) - F'(x_0)\| \le q_0\|x_{k+1} - x_0\| \le q_0 t_{k+1}. \tag{2.29}$$

Set

$$c_{k+1} = c_0 - q_0 t_k. \tag{2.30}$$

Then by hypothesis (2.7) we get

$$c_{k+1} > -a. \tag{2.31}$$

Therefore

$$\langle F'(x_0)(x) - F'(x_{k+1})(x), x\rangle \leq \|F'(x_0) - F'(x_{k+1})\| \, \|x\|^2 \leq q_0 t_k \|x\|^2, \tag{2.32}$$

for all $x \in H$. Hence (2.21) holds for k replaced by $k+1$.

Define

$$y_{k+1} = -F(x_k) - F'(x_k)(x_{k+1} - x_k). \tag{2.33}$$

Then (2.19) holds by (2.5) and

$$\begin{aligned} \|F(x_{k+1}) + y_{k+1}\| &\leq \|F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k)\| \\ &= \left\| \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)](x_{k+1} - x_k) dt \right\| \\ &\leq \frac{\bar{q}}{2} \|x_{k+1} - x_k\|^2 = b_{k+1}, \end{aligned} \tag{2.34}$$

where

$$\bar{q} = \begin{cases} q_0, & k = 0 \\ q, & k = 1, 2, \ldots \end{cases} \tag{2.35}$$

and $a_k = \dfrac{b_k}{c_k + a}$ $(k \geq 0)$. Thus for every $z \in \overline{B}(x_{k+1}, t^* - t_{k+1})$, we have

$$\|z - x_k\| \leq \|z - x_{k+1}\| + \|x_{k+1} - x_k\| \leq t^* - t_{k+1} + t_{k+1} - t_k = t^* - t_k. \tag{2.36}$$

That is

$$z \in \overline{B}(x_k, t^* - t_k). \tag{2.37}$$

The induction for (2.16)–(2.21) is now completed.

Lemma 2.2 implies that $\{t_n\}$ $(n \geq 0)$ is a complete sequence. By (2.9) and (2.28) it follows that it is a complete sequence too, and as such it converges to some $x^* \in \overline{B}(x_0, t^*)$ (since $\overline{B}(x_0, t^*)$ is a closed set). By letting $m \to \infty$ in

$$\|x_{k+m} - x_k\| \leq \sum_{i=k}^{k+m-1} \|x_{i+1} - x_i\| \leq \sum_{i=k}^{k+m-1} (t_{i+1} - t_i) = t_{k+m} - t_k \tag{2.38}$$

we obtain (2.15). Moreover, since $\lim_{k\to\infty} x_{k+1} = x^*$,

$$\lim_{k\to\infty}[F'(x_k)(x_k - x_{k+1}) - F(x_k)] = -F(x^*),$$

and

$$G(x_{k+1}) \ni F'(x_k)(x_{k+1} - x_k) - F(x_k)$$

it follows that $G(x^*) \ni -F(x^*)$. Hence x^* is a solution of equation $F(x) = 0$.

Finally, to show uniqueness in $\overline{B}(x_0, t^*)$, let us assume there exists a solution $y^* \in \overline{B}(x_0, t^*)$. Then, we obtain in turn

$$a\|x_{k+1} - y^*\|^2 + \langle F'(x_k)(x_{k+1} - y^*), x_{k+1} - y^*\rangle \\ \le \langle F(y^*) - F(x_k) - F'(x_k)(y^* - x_k), x_{k+1} - y^*\rangle$$

or (as in (2.27))

$$\|x_{k+1} - y^*\| \le \frac{q}{2(c_k + a)}\|x_k - y^*\|^2 < \|x_k - y^*\| \tag{2.39}$$

(since $\frac{q}{2(c_k + a)}\|x_k - y^*\| < 1$ by (2.7)). Hence we get $x^* = \lim_{k\to\infty} x_k = y^*$. □

Remark 2.4. Note that t^* can be replaced by t^{**} given in closed form by (2.10) in condition (2.13).

Remark 2.5. In order for us to compare our Theorem 2.3 with earlier ones, and in particular to Theorem 2.11 in [15] we define the scalar function p by

$$p(s) = \frac{L}{2}s^2 - s + a_0, \tag{2.40}$$

where L is given by (2.12). Uko's Newton–Kantorovich hypothesis (see [15]) becomes

$$h = 2La_0 \le 1. \tag{2.41}$$

But

$$q_0 \le q, \text{ so } L_0 \le L_1 \tag{2.42}$$

holds in general and $\frac{q}{q_0}$ can be arbitrarily large. Hence (2.41) always implies (2.7) but not vice versa. If strict inequality holds in (2.42) then (2.7) may hold but not (2.41). Moreover,

$$\frac{h_1}{h} \longrightarrow 0 \text{ as } \frac{L_0}{L} \longrightarrow 0. \tag{2.43}$$

Hence, the applicability of Newton's method can be extended infinitely many times over old approach.

In example that follows we show that $\frac{L}{L_0}$ may be arbitrarily large. Moreover define sequence $\{u_n\}$ by

$$u_{n+1} = u_n + \frac{\frac{L}{2}s_n^2 - u_n + a_0}{1 - Lu_n}, \quad s_0 = 0 \quad (n \geq 0), \tag{2.44}$$

and

$$u^* = \lim_{n \to \infty} u_n. \tag{2.45}$$

Then it is known [3, 10] that

$$u^* = \frac{1 - \sqrt{1 - 2L_0 a_0}}{L}, \tag{2.46}$$

$$u_{n+1} - u_n = -\frac{p(u_n)}{p'(u_n)} = \frac{\frac{L}{2}(u_n - u_{n-1})^2}{1 - Lu_n} \quad (n \geq 1), \tag{2.47}$$

and

$$u^* - u_{n+1} = \frac{\frac{L}{2}(u^* - u_n)^2}{1 - Lu_n} \leq \frac{1}{L2^{n+1}} h^{2^{n+1}} \quad (n \geq 0). \tag{2.48}$$

Uko used the error bounds (2.14) and (2.15) with sequence $\{u_n\}$, and point u^* replacing $\{t_n\}$, and point t^* respectively. That is for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq u_{n+1} - u_n \tag*{(2.14)$'$}$$

and

$$\|x_n - x^*\| \leq u^* - u_n. \tag*{(2.15)$'$}$$

We show that our error bounds are finer and the location of the solution x^* more precise:

Proposition 2.6. *Under hypotheses of Theorem* 2.3 *(for* $L_0 < L$*) and* (2.41) *the following error bounds hold:*

$$t_{n+1} < s_{n+1} \quad (n \geq 1), \tag{2.49}$$

$$t_{n+1} - t_n < u_{n+1} - u_n \quad (n \geq 1), \tag{2.50}$$

$$t^* - t_n \leq u^* - u_n \quad (n \geq 0), \tag{2.51}$$

$$t^* \leq u^*, \tag{2.52}$$

$$\begin{aligned} 0 &\leq t_{n+1} - t_n \leq \alpha^{2^{n-1}}(u_{n+1} - u_n) \ (n \geq 1), \\ \alpha &= \frac{1 - L\eta}{1 - L_0\eta} \in [0, 1) \end{aligned} \tag{2.53}$$

and

$$0 \leq t^* - t_n \leq \alpha^{2^{n-1}}(u^* - u_n) \quad (n \geq 1). \tag{2.54}$$

Moreover we have: $t_n = u_n$ $(n \geq 0)$ *if* $L = L_0$*.*

Proof. We use induction on the integer k to show (2.49) and (2.50) first. For $n = 0$ in (2.9) we obtain

$$t_2 - \eta = \frac{L\eta^2}{2(1 - L_0\eta)} \leq \frac{L\eta^2}{2(1 - L\eta)} = u_2 - u_1$$

and

$$t_2 < u_2.$$

Assume:

$$t_{k+1} < u_{k+1}, \quad t_{k+1} - t_k < u_{k+1} - u_k \quad (k \leq n + 1).$$

Using (2.9), and (2.44) we get

$$t_{k+2} - t_{k+1} = \frac{\frac{L}{2}(t_{k+1} - t_k)^2}{1 - L_0 t_{k+1}} < \frac{\frac{L}{2}(u_{k+1} - u_k)^2}{1 - L u_{k+1}} = u_{k+2} - u_{k+1}$$

and

$$t_{k+2} - t_{k+1} < u_{k+2} - u_{k+1}.$$

Let $m \geq 0$, we can obtain

$$\begin{aligned} t_{k+m} - t_k &< (t_{k+m} - t_{k+m-1}) + (t_{k+m-1} - t_{k+m-2}) + \cdots + (t_{k+1} - t_k) \\ &< (u_{k+m} - u_{k+m-1}) + (u_{k+m-1} - u_{k+m-2}) + \cdots + (u_{k+1} - u_k) \\ &< u_{k+m} - u_k. \end{aligned} \tag{2.55}$$

By letting $m \to \infty$ in (2.55) we obtain (2.51). For $n = 1$ in (2.51) we get (2.52).

Finally, (2.53) and (2.54) follow easily from (2.9) and (2.44). Note also that (2.53) holds as a strict inequality if $n \geq 2$. □

3. Numerical examples. We complete this study with numerical examples when, $G = 0$ on D. In the first one we show that hypothesis (2.41) fails whereas (2.7) holds. In the second example we compare estimates (2.14), (2.15) and (2.14)′, (2.15)′, respectively.

Example 3.1. Let $H = \mathbf{R}$, $D = [\sqrt{2} - 1, \sqrt{2} + 1]$, $x_0 = \sqrt{2}$ and define function f on D by

$$f(x) = \frac{1}{6}x^3 - \left(\frac{2^{3/2}}{6} + .23\right). \tag{3.1}$$

Using (2.1), (2.2), (2.3) and (2.4), we obtain

$$a = 0, \quad c = 2, \quad a_0 = .23, \quad L = 2.4142136 \tag{3.2}$$

$$L_0 = 1.914213562, L = 3.9080, h = 2La_0 = 1.1105383 > 1, \tag{3.3}$$

and by (2.7)

$$L_1 a_0 = 0.8988 < 1. \tag{3.4}$$

That is, there is no guarantee that Newton's method $\{x_n\}$ $(n \geq 0)$ starting at x_0 converges to a solution x^* of equation $F(x) = 0$, since (2.41) is not satisfied. However since (3.5) holds, Theorem 2.3 guarantees the convergence of Newton's method to $x^* = 1.614507018$.

Example 3.2. Let $H = \mathbf{R}$, $x_0 = 1.3$, $D = [x_0 - 2\eta, x_0 + 2\eta]$ and define function f on D by

$$f(x) = \frac{1}{3}(x^3 - 1). \tag{3.5}$$

As in Example 3.1, we obtain

$$
\begin{aligned}
a_0 &= .236094674, \eta = 0.2463784, \quad L = 2.097265501 \\
L_0 &= 1.817863519, \ L_1 = 3.6810, \ h = 2L\eta = .990306428 < 1 \\
h_1 &= L_1\eta = 0.9069188 < 1, \quad (\text{for } \delta = 1) \\
t^* &= .369677842 \quad \text{and} \quad u^* = .429866445.
\end{aligned}
$$

That is, we provide a better information on the location of the solution x^* since

$$\overline{U}(x_0, t^*) \subset \overline{U}(x_0, u^*). \tag{3.6}$$

Moreover using (2.14), (2.15) and (2.14)$'$ and (2.15)$'$ we can tabulate the following, which shows the superiority of our results:

Comparison table

x_n	Estimates (2.14)	Estimates (2.15)	Estimates (2.14)$'$	Estimates (2.15)$'$
$x_1 = 1.0639053254$	0.0999320677420	0.13084952171864	.236094674	.193771771
$x_2 = 1.0037617275$	0.0245022382979	0.00264927072600	.115780708	.0779910691
$x_3 = 1.0000140800$	0.0016743296484	0.000023014227062	.053649732	.024342893
$x_4 = 1.0000000002$	0.0000078919862	0.00000000176714	.020186667	.004156226
$n = 5$	0.00000000176714	0.0000000001753	.0000000000000039	.00016902
$n = 6$	0	0	.000000000000003	.000002259

Example 3.3. Let $H = \mathbf{R}$, $x_0 = 0$ and define function f on $\mathbb{R}$ by

$$f(x) = c_0 x + c_1 + c_2 \sin e^{b_3 x},$$

where c_i, $i = 0, 1, 2, 3$ are given parameters. It can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{q_0}{q}$ may be arbitrarily small. That is (2.7) may be satisfied but not (2.41).

REFERENCES

[1] I. K. Argyros, F. Szidarovszky. The Theory and Applications of Iteration Methods. Systems Engineering Series. Boca Raton, FL, CRC Press, 1993.

[2] I. K. Argyros, S. Hilout. Weaker conditions for the convergence of Newton's method. *J. Complexity* **28**, *3* (2012), 364–387.

[3] I. K. Argyros, Á. A. Magreñán. Iterative methods and their dynamics with applications. Boca Raton, FL, CRC Press, 2017.

[4] A. L. Dontchev, W. W. Hager. An inverse mapping theorem for setvalued maps. *Proc. Amer. Math. Soc.* **121**, *2* (1994), 481–489.

[5] A. L. Dontchev. Local convergence of the Newton method for generalized equation. *C. R. Acad. Sci. Paris, Sér. I Math.* **322**, *4* (1996), 327–331.

[6] A. L. Dontchev. Uniform convergence of the Newton method for Aubin continuous maps. *Serdica Math. J.* **22**, *3* (1996), 385–398.

[7] A. L. Dontchev, M. Quincampoix, N. Zlateva. Aubin criterion for metric regularity. *J. Convex Anal.* **13**, *2* (2006), 281–297.

[8] A. L. Dontchev, R. T. Rockafellar. Implicit functions and solution mappings. A view from variational analysis. Springer Monographs in Mathematics. Dordrecht, Springer, 2009.

[9] N. H. Josephy. Newton's method for generalized equations. Technical Report No. 1965, University of Wisconsin, Mathematics Research Center, Madison, Wisconsin, 1979.

[10] L. V. Kantorovich, G. P. Akilov. Functional Analysis. Translated from the Russian by Howard L. Silcock, 2nd edition. Oxford-Elmsford, N.Y., Pergamon Press, 1982.

[11] J. S. Pang, D. Chan. Iterative methods for variational and complementarity problems. *Mathematical Programming* **24**, *3* (1982), 284–313.

[12] S. M. Robinson. Generalized equations. In: Mathematical Programming. The State of the Art (Eds A. Bachem, M. Grötschel, B. Korte). Berlin, Springer, 1982, 346–367.

[13] Stampacchia, G., Formes bilinéares coercitives sur les ensembles convexes. *C. R. Acad. Sci. Paris* **258** (1964), 4413–4416 (in French).

[14] L. U. Uko. Remarks on the generalized Newton method. *Math. Programming* **59**, *3*, Ser. A (1993), 405–412.

[15] L. U. Uko. Generalized equations and the generalized Newton method. *Math. Programming* **73**, *3*, Ser. A (1996), 251–268.

[16] R. U. Verma. Nonlinear variational and constrained hemivariational inequalities involving relaxed operators. *Z. Angew. Math. Mech.* **77**, *5* (1997), 387–391.

[17] R. U. Verma. A class of projection-contraction methods applied to monotone variational inequalities. *Appl. Math. Lett.* **13**, *8* (2000), 55–62.

[18] R. U. Verma. Generalized multivalued implicit variational inequalities involving the Verma class of mappings. *Math. Sci. Res. Hot-Line* **5**, *2* (2001), 57–64.

Ioannis K. Argyros
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
e- mail: iargyros@cameron.edu

Santhosh George
Department of Mathematical
and Computational Sciences
NIT Karnataka, India-575 025
e-mail: sgeorge@nitk.ac.in

Received May 5, 2017