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HOMEOMORPHISMS OF FUNCTION SPACES AND TOPOLOGICAL DIMENSION OF DOMAINS

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Abstract. It is known since 1982, that $\dim X = \dim Y$ whenever the function spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. This statement was later extended to uniform homeomorphisms of the spaces $C_p(X)$ and $C_p(Y)$. We obtain, in the case of separable function spaces, a generalization of the first result to another direction.

We introduce, for each X, some subspace $E(X) \subset C_pC_p(X)$, which is significantly wider, than the space $L_p(X)$ of all linear continuous functionals on $C_p(X)$. Our generalization includes homeomorphisms $h : C_p(X) \to C_p(Y)$, such that the image of Y under the dual mapping h^* of h is contained in $E(X)$ and the image of X under $(h^{-1})^*$ is contained in $E(Y)$.

0. Introduction. The problem of coincidence of dimensions dim X and dim Y under homeomorphism of function spaces $C_p\,(X)$, $C_p\,(Y)$ does not leave the research agenda in C_p-theory for rather long time. V. G. Pestov [14], generalizing previous particular results (see [1, 13]), proved that for arbitrary

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Tykhonoff spaces X and Y we have $\dim X = \dim Y$, if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Later in [6] S. P. Gul'ko extended this result to the case of uniformly homeomorphic function spaces. However up till now the answer for an arbitrary homeomorphism $h : C_p(X) \to C_p(Y)$ is still unknown. For this case R. Cauty [3] proved only that metrisable compacts X and Y must have some finite powers X^k and Y^m, which are both strongly infinite dimensional.

In this paper we propose a new approach to the description of the properties of homeomorphisms of function spaces $C_p(X)$ and $C_p(Y)$. We formulate these properties as some requirements to the dual mappings. We show that a series of properties of such homeomorphisms as being an isomorphism of topological rings, linearity, uniformity may be described uniformly inside our approach.

Furthermore, we have found another such property P, which is more general than linearity and different from uniformity. We have proved that if the Tykhonoff spaces X and Y have countable i-weight, and a homeomorphism $h : C_p(X) \to C_p(Y)$ has the property P, then $\dim X = \dim Y$. Therefore, for spaces of countable i-weight, this result generalizes the theorem of Pestov, and it is different from the theorem of Gul'ko.

Both in [14] and [6] the main results are established first for second countable spaces X and Y. For this aim the approach used is basing on finite-valued mappings from X onto Y, which are generated by a linear or uniform homeomorphism between function spaces. Then the statements are extended over the Tykhonoff spaces of an arbitrary weight by application of technics of the inverse sequences of the second countable spaces. In this paper we basically follow this pattern as well.

1. Notation, terminology and preliminaries. We use standard topological notations and terms, which may be found, for example, in [4]. All topological spaces under consideration are assumed Tykhonoff, and named below simply "spaces". Given a space X we denote by $C_p(X)$ the set of all continuous functions $\varphi : X \to \mathbb{R}$, endowed with pointwise convergence topology. In C_p-theory we follow the terminology of [2] or [15]. Here we recall those symbols and facts, which are most important later on. The space $C_p(X)$ is a dense subspace in the space $\mathbb{R}^X$ of all functions $\varphi : X \to \mathbb{R}$. If $A \subset \mathbb{R}$, then A^X denotes the set of all functions $\varphi : X \to A$. If $A = \{0\}$, we write 0^X instead of $\{0\}^X$ to denote the zero-function on X.

If $A \subset \mathbb{R}^X$, then the *diagonal of* A is the mapping $\Delta A : X \to \mathbb{R}^A$, defined by the rule $\Delta A(x)(\varphi) = \varphi(x)$ for all $x \in X$, and $\varphi \in A$. The mapping ΔA

is continuous if and only if $A \subset C_p(X)$. A subset $A \subset C_p(X)$ is said to be *regular*, if for each disjoined pair $(\{x\}, F)$, where $x \in X$, $F = \bar{F} \subset X$, there exists a $\varphi \in A$, such that $\varphi(x) \notin \overline{\varphi(F)}$. In this case $\Delta A : X \to C_p(A)$ is a homeomorphism "into". By this reason we identify X with $\Delta A(X)$. Thus each point $x \in X$ is, at the same time, the (continuous) mapping $x : A \to \mathbb{R}$. Similarly, for each $\varphi \in A$ we may consider the continuous mapping $\hat{\varphi} : C_p(A) \to \mathbb{R}$ defined by the rule $\hat{\varphi}(f) = f(\varphi)$. If $A \subset B \subset \mathbb{R}^X$ then the natural projection $p_A^B : \Delta B(X) \to \Delta A(X)$ is well-defined by the formula $p_A^B(f) = f|_A$. If we have an increasing sequence $\gamma = \{A_n : n \in \mathbb{N}\}$ of subsets in $\mathbb{R}^X$ and if $A = \cup\gamma$, then $\Delta A(X)$ is the limit of the inverse sequence $\{\Delta A_n(X) : n \in \mathrm{N}\}$.

If $A \subset \mathbb{R}^X$, $0^X \in A$, then we denote by $C_p^0(A)$ the subspace in $C_p(A)$ consisting of all functions, that are equal to zero at the point 0^X. The symbols $L_p(X)$ and $U_p(X)$ mean subspaces in $C_pC_p(X)$, consisting, respectively, of the linear and the uniformly continuous functions.

For the spaces A and B let $h : A \to B$ be a homeomorphism. We denote by h^* the dual mapping $h^* : C_p(B) \to C_p(A)$, where $h^*(g) = g \circ h$ for each $g \in C_p(B)$. It is clear, that without loss of generality we may assume $h(0^X) = 0^Y$, if $h : C_p(X) \to C_p(Y)$ is a homeomorphism.

2. Satisfactory homeomorphisms of function spaces. We start from the consideration of the following general construction of families of homeomorphisms between subspaces in $C_p(X)$ and $C_p(Y)$. Let us assume that for each pair (X, A), where X is a space, and A is a regular subfamily in $C_p(X)$, some subspace $E_A(X) \subset C_p(A)$ is fixed. In this notation we formulate the following definition.

Definition 2.1. *Let X and Y be spaces and let $A \subset C_p(X)$, $B \subset C_p(Y)$ be regular subfamilies. A homeomorphism $h : A \to B$ is said to be* $(E_A(X), E_B(Y))$-satisfactory, *if and only if $h^*(Y) \subset E_A(X)$ and $\left(h^{-1}\right)^*(X) \subset E_B(Y)$. The (may be empty) set of all $(E_A(X), E_B(Y))$-satisfactory homeomorphisms will be denoted by $(E_A(X), E_B(Y))$. In the particular case $A = C_p(X)$, $B = C_p(Y)$ we shall write $(E(X), E(Y))$.*

Of course, if $E_A^1(X) \subset E_A^2(X)$ and $E_B^1(Y) \subset E_B^2(Y)$, then

$$\left(E_A^1(X), E_B^1(Y)\right) \subset \left(E_A^2(X), E_B^2(Y)\right).$$

Now we shall join a number of evident or known facts in the following proposition.

Proposition 2.2. *Let $A = C_p(X)$ and $B = C_p(Y)$. The following statements are true:*
1) $(X, Y) \neq \emptyset$ if and only if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic;
2) $(L_p(X), L_p(Y)) \neq \emptyset$ if and only if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic;
3) $(U_p(X), U_p(Y)) \neq \emptyset$ if and only if $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic;
4) $\left(C_p^0 C_p(X), C_p^0 C_p(Y)\right) \neq \emptyset$ if and only if $C_p(X)$ and $C_p(Y)$ are homeomorphic.

Proof. The item 1) follows from Nagata theorem [11]. The item 2) is well-known [2], the item 4) is obvious. To prove 3) it suffices to show, that each $(U_p(X), U_p(Y))$-satisfactory homeomorphism $h : C_p(X) \to C_p(Y)$ is uniform. Fix any basic neighborhood $W = W\left(0^Y, K, \varepsilon\right)$ of the function $0^Y \in C_p(Y)$. By $(U_p(X), U_p(Y))$-satisfactoriness of h, and $K \subset Y$, we have $h^*(K) \subset U_p(X)$. Since K is finite, then, by uniform continuity of each $h^*(y)$, $y \in K$, we can find $\delta > 0$ and finite subsets $M_y \subset X$, such that if $\varphi, \psi \in C_p(X)$, $\varphi - \psi \in V_y = V\left(0^X, M_y, \delta\right)$, then $|h^*(y)(\varphi) - h^*(y)(\psi)| < \varepsilon$. Let $M = \cup\{M_y : y \in K\}$ and $V = V\left(0^X, M, \delta\right)$. Now it is easy to check that $h(\varphi) - h(\psi) \in W$ whenever $\varphi - \psi \in V$. It means that the mapping h (and, similarly, h^{-1}) is uniformly continuous. So, the homeomorphism h is uniform. □

3. Functionals with a finite support. In this section we define and investigate some functions that may be considered as a natural generalization of linear continuous functionals on $C_p(X)$.

Definition 3.1. *Let $A \subset C_p(X)$ with $0^X \in A$. A function $f \in C_p^0(A)$ is said to be* functional with a finite support on A*, or* finitely supported functional on A*, or* A-FSF*, if there exists a finite subset $K \subset X$, such that the following two statements hold:*

(i) For each $\varepsilon > 0$ and each $\varphi \in A$ there is some $\delta > 0$ such that $|f(\varphi) - f(\psi)| < \varepsilon$ whenever $\psi \in W(\varphi, K, \delta) \cap A$.

(ii) There exists $\varepsilon_0 > 0$ such that for each $x' \in K$ and its arbitrary neighborhood $U \subset X$ one can find a function $\varphi \in A$, which coincides with 0^X on $X \backslash U$, although $|f(\varphi)| > \varepsilon_0$.

The subspace in $C_p^0(A)$ consisting of all functionals with a finite support will be denoted by $\hat{X}_A$. If $f \in C_p^0(A)$ and a finite subset $K \subset X$ satisfies (i) and *(ii), then K is said to be* support *of the functional f. If $A = C_p(X)$, we write $\hat{L}_p(X)$ instead of $\hat{X}_A$.*

The above definition immediately implies the following simple facts:

Proposition 3.2. *Let $f \in \hat{X}_A$ and let $K \subset X$ be a support of f. Then*
a) $f \equiv 0$ if and only if $K = \emptyset$.
b) If $\varphi, \psi \in A$ and φ coincides with ψ on K, then $f(\varphi) = f(\psi)$.

The important fact below follows from Proposition 3.2. It was proved in [9] for the case $A = C_p(X)$. The proof can be easily extended to the general case.

Proposition 3.3. *Every $f \in \hat{X}_A$ has a unique support $K = K(f)$.*

This fact means, that the finite-valued mapping $s : \hat{X}_A \to X$, $s(f) = K(f)$ is well-defined. It allows us "to stratify" any subspace $Z \subset \hat{X}_A$ by such a way. Fix $Z^{\geq n} = \{f \in Z\colon |s(f)| \geq n\}$, $Z^{\leq n} = Z \backslash Z^{\geq n}$ and $Z^{=n} = Z^{\leq n} \cap Z^{\geq n}$. Obviously, we have $Z = \cup \{Z^{=n}\colon n \in \mathbb{N}\} = \cup \left\{Z^{\leq n}\colon n \in \mathbb{N}\right\}$.

Proposition 3.4. *Let $f \in \hat{X}_A$, let $G \subset X$ be an open subset and $s(f) \cap G \neq \emptyset$. Then there exists a neighborhood V of the functional f in $\hat{X}_A$, such that the intersection $s(g) \cap G$ has cardinality greater than or equal to the cardinality of $s(f) \cap G$ for any $g \in V$.*

P r o o f. By Proposition 3.2 *a)* we have $f \not\equiv 0$. Pick any point $x \in s(f) \cap G$ and its neighborhood U_x such that $U_x \subset G$ and $U_x \cap U_y = \emptyset$, if $y \in s(f)$, $x \neq y$. It follows from *(ii)* that $|f(\varphi_x)| > \varepsilon_0 > 0$ for a suitable function $\varphi_x \in A$. The item *b)* of Proposition 3.2 implies that $\varphi_x(x) \neq 0$. Now it is easy to check that the neighborhood V given by the formula $V = \hat{X}_A \cap \left(\bigcap\limits_{x \in s(f)} (\hat{\varphi}_x)^{-1} (\mathbb{R} \backslash \{0\}) \right)$, satisfies the required property. □

One can easy deduce the next corollaries from Proposition 3.4.

Corollary 3.5. *The mapping $s : \hat{X}_A \to X$ is lower semicontinuous.*

Corollary 3.6. *In the notation above, all $Z^{\geq n}$ are open and, therefore, all $Z^{\leq n}$ are closed in Z.*

Corollary 3.7. *For each $n \in \mathbb{N}$ the mapping $s : Z^{=n} \to Fin_n(X)$ is continuous (with respect to Vietoris topology on $Fin_n(X)$).*

Let us now consider the case $A = C_p(X)$. It is well known (see [2]), that each linear continuous functional f on the space $C_p(X)$ can be represented (in a unique way) in the form $f = \alpha_1 x_1 + \cdots + \alpha_k x_k$, where $\{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{R}$, $K = \{x_1, \ldots, x_k\} \subset X$. It is easy to check that the set K satisfies the conditions (i) and (ii) in Definition 3.1. Therefore we may conclude that $L_p(X) \subset \hat{L}_p(X)$. Moreover, in [10] it was established the following statement:

Proposition 3.8. *$\hat{L}_p(X)$ is dense in $C_p^0 C_p(X)$.*

Let us show that the space $U_p(X)$ does not contain $\hat{L}_p(X)$.

Example 3.9. Pick any point $x \in X$ and define the mapping $x^2 : C_p(X) \to \mathbb{R}$ by the rule $x^2(\varphi) = (\varphi(x))^2$ for all $\varphi \in C_p(X)$. Obviously, the set $K = \{x\}$ satisfies items (i) and (ii) of Definition 3.1. In the same time one can easily check that the mapping x^2 is not uniformly continuous. Thus $\hat{L}_p(X) \setminus U_p(X) \neq \emptyset$.

Proposition 3.8 and Example 3.9 show that, generally speaking, for given spaces X and Y the class $\left(\hat{L}_p(X), \hat{L}_p(Y)\right)$ is wider than $(L_p(X), L_p(Y))$ and does not coincide with $(U_p(X), U_p(Y))$.

Given a $\left(\hat{X}_A, \hat{Y}_B\right)$-satisfactory homeomorphism $h : C_p(X) \to C_p(Y)$ we can deduce, keeping the notation from Definition 3.1, the following corollary from Definition 2.1 and Proposition 3.2 b):

Corollary 3.10. *The restriction s_Y of the mapping $s : \hat{X}_A \to X$ from Corollary 3.5 on the subspace $h^*(Y) \subset \hat{X}_A$ is surjective. More precisely,*

$$x \in \cup \left\{ K\left(h^*(y)\right) : y \in K\left(\left(h^{-1}\right)^*(x)\right)\right\}.$$

Of course, an analogous statement holds with respect to the support-mapping $s' : \hat{Y}_B \to Y$ as well.

4. Preservation of domain's dimension. First we shall establish our result for spaces X, Y with countable base.

Definition 4.1. *The subset $A \subset C_p(X)$ is said to be* 0-sufficient *or, briefly,* ZSS *if and only if it contains the zero-function 0^X and for any real interval $(a;b)$, any point $x \in X$ and an arbitrary neighborhood $U \subset X$ of x there exists $\varphi \in A$, such that $\varphi(x) \in (a;b)$ and $\varphi(x') = 0$ for all $x' \in X \setminus U$.*

Obviously, if X is a second countable space, then $C_p(X)$ has a countable ZSS. Also it is clear that $A = C_p(X)$ is ZSS. In what follows the 0-sufficient subsets play the role similar with the role of QS-algebras in [6].

Theorem 4.2. *Let X and Y be second countable spaces, and let A and B be ZSS in $C_p(X)$ and $C_p(Y)$, respectively. If there exists a $\left(\hat{X}_A, \hat{Y}_B\right)$-satisfactory homeomorphism $h : A \to B$, then* $\dim X = \dim Y$.

P r o o f. We shall identify the spaces X and Y with their images in $\hat{Y}_B$ and $\hat{X}_A$ under the homeomorphisms $\left(h^{-1}\right)^*$ and h^*, respectively. So, using the notation introduced after Proposition 3.3, we can write $Y = \cup\{Y^{=n} : n \in \mathbb{N}\}$. Since Y is a second countable space, then each $Y^{=n}$ is an F_σ-subset in Y, being intersection of open $Y^{\geq n}$ and closed $Y^{\leq n}$ subsets. So, in its turn, we can write $Y^{=n} = \cup\{F(n,k) : k \in \mathbb{N}\}$, where $\{F(n,k) : k \in \mathbb{N}\}$ is the family of closures in Y of some elements of a countable base in $Y^{=n}$. In the same way we obtain the representation $X^{=n} = \cup\{D(n,k) : k \in \mathbb{N}\}$, where all $D(n,k)$ are closures of elements of some countable base in $X^{=n}$.

Take any point $y \in Y$. Applying support mappings, we can find $S(y) = \{x_1, \ldots, x_n\} \subset X$, $S'(x_i) = \left\{y_1^i, \ldots, y_{n(i)}^i\right\} \subset Y$, where $i \in \{1, \ldots, n\} \subset \mathbb{N}$. Fix the disjoined family $\{F(n(i,j), k(i,j))\colon 1 \leq i \leq n,\ 1 \leq j \leq n(i)\}$ of closed (in Y) neighborhoods of the points of the set $\cup\left\{S'(x_i)\colon i = 1, \ldots, n\right\}$. By Corollary 3.7 the mappings $S' : X^{=n(i)} \to \mathrm{Fin}_{n(i)}(Y)$ are continuous and one can find (closed in X) disjoined neighborhoods $D(n(i), k(i))$ of the points x_i, $i \in \{1, \ldots, n\}$, such that the intersection $S'(\xi) \cap F(n(i,j), k(i,j))$ is a single point for all $\xi \in D(n(i), k(i))$ and $j \in \{1, \ldots, n(i)\}$. Therefore we can correctly define the continuous single-valued mappings $S'_{ij}\colon D(n(i), k(i)) \to F(n(i,j), k(i,j))$ by the rule $S'_{ij}(\xi) = S'(\xi) \cap F(n(i,j), k(i,j))$. By the same reasons there exist continuous single-valued mappings $S_i\colon F(n,k) \to D(n(i), k(i))$, $i \in \{1, \ldots, n\}$, for a suitable neighborhood $F(n,k)$ of the point $y \in Y^{=n}$. Fix such $F(n,k)$.

Corollary 3.10 implies that for each $\eta \in F(n,k)$ there exist some indices i, j such that $S'_{ij}(S_i(\eta)) = \eta$. It follows from this fact that $F(n,k) = \cup\{\Phi_{ij}\colon i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n(i)\}\}$, where each Φ_{ij} is the (closed in Y) set of fixed points of the (continuous) mapping $S'_{ij} \circ S_i$. It is clear that each subspace $S_i(\Phi_{ij}) \subset X$ is homeomorphic to Φ_{ij}.

So, we express the space Y as a countable union of its closed subsets, which are homeomorphic to some subspaces of X. Now our statement follows from the theorem of monotonicity and the theorem for the sum for the dimensions dim (see [5], Theorem 3.1.4 and Proposition 3.1.7, respectively). □

Now we need some additional facts to extend Theorem 4.2 over spaces of countable i-weight.

Lemma 4.3 ([14]). $\dim X \le m$ *if and only if for each continuous surjection* $\varphi : X \to X_0$, *with second countable* X_0, *there exist continuous surjections* $\theta : X \to X_1$, $p : X_1 \to X_0$, *with second countable* X_1, *such that* $\varphi = p \circ \theta$ *and* $\dim X_1 \le m$.

Definition 4.4. *The subset* $A \subset C_p(X)$ *is said to be* projectively 0-sufficient *or, briefly,* PZS *if the family* $P_A = \{p_a \colon \Delta A(X) \to \mathbb{R},\ p_a\ \Delta A(x) = a(x) \colon a \in A\}$ *is a* 0*-sufficient subset in* $C_p(\Delta A(X))$.

Lemma 4.5. *For any countable subset* $A \subset C_p(X)$ *there exists a countable PZS subset* $B \subset C_p(X)$ *with* $A \subset B$.

Proof. Since the space $\Delta A(X)$ is second countable, we can choose some countable 0-sufficient subset $A' \subset C_p(\Delta A(X))$. Put $B' = \{b = a \circ \Delta A \colon a \in A'\}$ and $B = A \cup B'$. It is clear that $A \subset B \subset C_p(X)$ and B is countable. Let us show that B is PZS.

Take any real interval $(s;t)$, $\varepsilon > 0$, and a neighborhood W of an arbitrary point $y = \Delta B(x) \in \Delta B(X)$. We have $\Delta B : X \to \Delta B(X) \subset \mathbb{R}^B$, therefore we may assume that W in the form $W = W(y, a_1, \ldots, a_l, b_{l+1}, \ldots, b_m, \varepsilon) \cap \Delta B(X)$, where $\{a_1, \ldots, a_l\} \subset A$, $\{b_{l+1}, \ldots, b_m\} \subset B' \backslash A$. Notice that each b_k is of the form $b_k = a'_k \circ \Delta A$ for some $a'_k \in A'$. Choose the standard neighborhood $V = V(\Delta A(x), u_1, \ldots, u_n, \delta)$ of the point $\Delta A(x)$ in $\Delta A(X)$ such as $|a'_j(v) - a'_j(\Delta A(x))| < \varepsilon$ for all $j > l$, whenever $v \in V$. Of course, we may assume, that $\{a_1, \ldots, a_l\} \subset \{u_1, \ldots, u_n\}$and $\delta < \varepsilon$.

Since the set A' is 0-sufficient in $C_p(\Delta A(X))$, there is a function $a' \in A'$ such that as $a'(\Delta A(x)) \in (s;t)$ and $a'(v') = 0$ for all $v' \in \Delta A(X) \backslash V$. Now take $y' = \Delta B(x') \notin W$. Two cases are possible.

Case 1. $|\Delta B(x')(a_j) - \Delta B(x)(a_j)| \ge \varepsilon$ for some j, $1 \le j \le l$. Then $|\Delta A(x')(a_j) - \Delta A(x)(a_j)| \ge \varepsilon > \delta$, because $\Delta B(x)(a) = \Delta A(x)(a) = a(x)$ for each $a \in A \subset B$ and each $x \in X$. The inclusion $a_j \in \{u_1, \ldots, u_n\}$ implies $\Delta A(x') \notin V$. Therefore $a'(\Delta A(x')) = 0$.

Case 2. For some j, $l+1 \le j \le m$, it holds $|\Delta B(x')(b_j) - \Delta B(x)(b_j)| \ge \varepsilon$. We can rewrite the latter inequality in the form

$$|\Delta B(x')(a'_j \circ \Delta A) - \Delta B(x)(a'_j \circ \Delta A)| \ge \varepsilon, \quad \text{or} \quad |a'_j(\Delta A(x')) - a'_j(\Delta A(x))| \ge \varepsilon.$$

By the definition of V we conclude that $\Delta A(x') \notin V$. We obtain $a'\left(\Delta A(x')\right) = 0$ again.

It remains to observe that the function $b' = a' \circ \Delta A$ belongs to $B' \subset B$, and $p'_b\left(\Delta B(z)\right) = b'(z) = a'\left(\Delta A(z)\right)$ for all $z \in X$. □

Lemma 4.6 ([6]). *If* $\dim Y \leq m$ *and the subset* $B \subset C_p(Y)$ *is countable, then there exists a countable subset* $B' \subset C_p(Y)$ *with* $\dim\left(\Delta\left(B \cup B'\right)(Y)\right) \leq m$.

Lemma 4.7 ([12]). *If the space* Y *is a limit of the inverse sequence* $(Y_n)_{n\in\mathbb{N}}$, *where* $\dim Y_n \leq m$ *for all* $n \in \mathbb{N}$, *then* $\dim Y \leq m$.

The previous three lemmas allow us to establish the next statement:

Lemma 4.8. *Let* $\dim Y \leq m$. *Then for each countable subset* $B_0 \subset C_p(Y)$ *there exists countable PZS subset* $B \subset C_p(Y)$ *with* $B_0 \subset B$ *and* $\dim\left(\Delta B(Y)\right) \leq m$.

Proof. Starting from B_0, we can apply Lemma 4.5 at odd steps, and Lemma 4.6 at even steps each time to the previous subset in $C_p(Y)$. In such a manner we can construct some increasing sequence of countable PZS subsets $B_{2i-1} \subset C_p(Y)$ and some increasing sequence of countable subsets $B_{2i} \subset C_p(Y)$, such that $\dim\left(\Delta B_{2i}(Y)\right) \leq m$ for all $i \in \mathbb{N}$. In addition we have $B_{2i-1} \subset B_{2i} \subset B_{2i+1}$ for all $i \in \mathbb{N}$.

Let $B = \cup\{B_k : k \in \mathbb{N}\}$. It is clear that B is countable. Moreover, $\dim\left(\Delta B(Y)\right) \leq m$ because the space $\Delta B(Y)$ is the limit of the inverse sequence of the spaces $\Delta B_{2i}(Y)$ (by Lemma 4.7).

It remains to show that B is PZS. In other words, we have to show that the family $Q_B = \{q_b : \Delta B(Y) \to \mathbb{R},\ q_b\left(\Delta B(y)\right) = b(y) :\ b \in B\}$ is 0-sufficient. To this end fix a real interval $(s;t)$, $\varepsilon > 0$, an arbitrary finite family $\{d_1, \ldots, d_n\} \subset B$ and a neighborhood $W = W\left(z, d_1, \ldots, d_n, \varepsilon\right) \cap \Delta B(Y)$ of an arbitrary point $z = \Delta B(y) \in \Delta B(Y)$. Of course, $\{d_1, \ldots, d_n\} \subset B_{2i-1}$ for suitable $i \in \mathbb{N}$. Let $Y_i = \Delta B_{2i-1}(Y)$, $z_i = \Delta B_{2i-1}(y) \in Y_i$, $V = W\left(z_i, d_1, \ldots, d_n, \varepsilon\right) \cap Y_i$. Since B_{2i-1} is PZS, one can find $d_0 \in B_{2i-1} \subset B$, such that $q_{d_0}(z_i) = d_0(y) \in (s;t)$ and $q_{d_0}\left(z'\right) = d_0(y') = 0$ whenever $z' = \Delta B_{2i-1}(y') \in Y_i \backslash V$. So we already have $q_{d_0}(z) = d_0(y) \in (s;t)$. Now take any $z' = \Delta B(y') \in \Delta B(Y) \backslash W$. It means, by the definition of W, that $\left|\Delta B(y')(d_j) - \Delta B(y)(d_j)\right| = \left|d_j(y') - d_j(y)\right| \geq \varepsilon$ for some j, $1 \leq j \leq n$. This implies, that $\Delta B_{2i-1}(y') \in Y_i \backslash V$, because $d_j \in B_{2i-1}$. Now we may conclude, that $q_{d_0}\left(z'\right) = d_0(y') = 0$. So, the projection q_{d_0} satisfies $q_{d_0} \in Q_B$, as required and the lemma is proved. □

Theorem 4.9. *Let X and Y be spaces and $iw(X) = iw(Y) = \aleph_0$. Let also a homeomorphism $h : C_p(X) \to C_p(Y)$ be $\left(\hat{L}_p(X), \hat{L}_p(Y)\right)$-satisfactory. Then* $\dim X = \dim Y$.

P r o o f. Standard arguments show that it is sufficient to deduce the inequality $\dim X \le m$ from the condition $\dim Y \le m$.

So, let $\dim Y \le m$ and let $\xi : X \to X_0$ be an arbitrary continuous mapping onto the second countable space X_0. Take any countable 0-sufficient family $F \subset C_p(X_0)$ and define $A_0 = \{f \circ \xi : f \in F\}$. Note that, by the regularity of F, the spaces $X_0 = \xi(X)$ and $\Delta A_0(X) = \Delta F(\xi(X))$ are homeomorphic. By Lemma 4.5, we can choose a countable PZS subset $A_1 \subset C_p(X)$ with $A_0 \subset A_1$. Using Lemma 4.8, we can enlarge the set $h(A_1)$ to some PZS subset $B_1 \subset C_p(Y)$, such that $h(A_1) \subset B_1$ and $\dim(\Delta B_1(Y)) \le m$. By Lemma 4.5 again, we can enlarge the set $h^{-1}(B_1)$ to some countable PZS $A_2 \subset C_p(X)$.

Let us suppose, that the countable PZS subsets $A_n \subset C_p(X)$ and $B_n \subset C_p(Y)$ with $\dim(\Delta B_n(Y)) \le m$ are already chosen. Then by Lemma 4.5 again, we can enlarge the set $h^{-1}(B_n)$ to some countable PZS $A_{n+1} \subset C_p(X)$. After this we apply Lemma 4.8 to $h(A_{n+1})$, in order to obtain a PZS subset $B_{n+1} \subset C_p(Y)$ with $h(A_{n+1}) \subset B_{n+1}$ and $\dim(\Delta B_{n+1}(Y)) \le m$.

So, we constructed by induction the increasing sequence $\{A_n : n \in \mathbb{N}\}$ of countable PZS subsets in $C_p(X)$, and the increasing sequence $\{B_n : n \in \mathbb{N}\}$ of countable PZS subsets in $C_p(Y)$, such that $\dim(\Delta B_n(Y)) \le m$ for all $n \in \mathbb{N}$.

Now let $A = \cup\{A_n : n \in \mathbb{N}\}$ and $B = \cup\{B_n : n \in \mathbb{N}\}$. It is clear that $h(A) = B$, so A and B are countable homeomorphic subsets in $C_p(X)$ and $C_p(Y)$, respectively.

Repeating our arguments from the proof of Lemma 4.8, we can prove that A and B are PZS. Let us consider the corresponding families $P_A \subset C_p(\Delta A(X))$ and $P_B \subset C_p(\Delta B(X))$ from Definition 4.4. Let us define the mapping $\tilde{h} : P_A \to P_B$ by the rule $\left(\tilde{h}(p_a)\right)(\Delta B(y)) = (h(a))(y) = p_{h(a)}(\Delta B(y))$ for all $a \in A$, $y \in Y$. We can define the inverse mapping $\left(\tilde{h}\right)^{-1} : P_B \to P_A$ similarly, by using the mapping h^{-1}. The continuity of $\tilde{h}$ and $\left(\tilde{h}\right)^{-1}$ can be easily deduced from the equalities $a(x) = p_a(\Delta A(x))$ and $b(y) = p_b(\Delta B(y))$ and from the fact, that h is a homeomorphism.

In order to apply Theorem 4.2 to the 0-sufficient subsets P_A and P_B it remains to establish two inclusions: $\tilde{h}^*(\Delta B(y)) \in \widehat{\Delta A(X)}_{P_A}$ for all $y \in Y$ and

$\widetilde{h^{-1}}^{*}(\Delta A(x)) \in \widehat{\Delta B(Y)}_{P_B}$ for all $x \in X$. It is easy to see by the symmetry of the situation, that we may establish just first of them. For this aim fix any $y \in Y$, positive ε and $p_u \in P_A$. By the condition of this theorem $h^*(y) \in \hat{L}_p(X)$. Therefore there exist a finite set $K \subset X$ and a positive δ, such that the inequalities $|u(x) - v(x)| < \delta$ for all $x \in K$ imply that $|h^*(y)(u) - h^*(y)(v)| < \varepsilon$.

Consider the set $K' = \Delta A(K) \subset \Delta A(X)$ and take any $p_v \in P_A$, such that $\left|p_v(x') - p_u(x')\right| < \delta$ for each $x' \in K'$. Then for each $x \in K$ we have $|p_v(\Delta A(x)) - p_u(\Delta A(x))| = |v(x) - u(x)| < \delta$. It follows that

$$\begin{aligned}
&\left|\tilde{h}^*(\Delta B(y))(p_u) - \tilde{h}^*(\Delta B(y))(p_v)\right| \\
&\quad = \left|\tilde{h}(p_u)(\Delta B(y)) - \tilde{h}(p_v)(\Delta B(y))\right| = \left|p_{h(u)}(\Delta B(y)) - p_{h(v)}(\Delta B(y))\right| \\
&\quad = |h(u)(y) - h(v)(y)| = |h^*(y)(u) - h^*(y)(v)| < \varepsilon.
\end{aligned}$$

So, the condition (i) of Definition 3.1 is verified.

Let us verify its condition (ii). To this end we shall use the condition $iw(X) = iw(Y) = \aleph_0$. It permits us to assume, that the mappings ΔA and ΔB are one-to-one. Pick any point $x_0' = \Delta A(x_0) \in K'$ and choose a neighborhood V of the point $x_0 \in X$, such that $V \cap (K \setminus \{x_0\}) = \emptyset$. Again by the fact $h^*(y) \in \hat{L}_p(X)$, one may find a positive ε_0 and a function $\varphi' \in C_p(X)$ such as $\varphi'(x') = 0$ for all $x' \in X \setminus V$, but $\left|h^*(y)(\varphi')\right| = \left|h(\varphi')(y)\right| \geq 2\varepsilon_0$.

Let $\varepsilon' = \left|h(\varphi')(y)\right| - \varepsilon_0$. Since $h^*(y) \in \hat{L}_p(X)$, there exists a positive σ, which has the property, that if $\left|\psi(z) - \varphi'(z)\right| < \sigma$ for each $z \in K$, then $\left|h^*(y)(\psi) - h^*(y)(\varphi')\right| < \varepsilon'$. By the 0-sufficiency of P_A, one can choose some $u \in A$, such that $p_u(z') = 0$ for all $z' \in K' \setminus \{x_0'\}$ and $p_u(x_0') \in \left(\varphi'(x_0) - \sigma; \varphi'(x_0) + \sigma\right)$. It follows that $\left|(p_u \circ \Delta A)(z) - \varphi'(z)\right| = \left|u(z) - \varphi'(z)\right| < \sigma$ for each $z \in K$. Therefore, $\left|h^*(y)(u) - h^*(y)(\varphi')\right| < \varepsilon' = \left|h^*(y)(\varphi')\right| - \varepsilon_0$. It implies $|h^*(y)(u)| = |h(u)(y)| = \left|p_{h(u)}(\Delta B(y))\right| = \left|\tilde{h}^*(\Delta B(y))(p_u)\right| > \varepsilon_0$. So, the condition (ii) is verified.

Now we may apply Theorem 4.2 to the homeomorphic 0-sufficient subsets P_A and P_B in the spaces $C_p(\Delta A(X))$ and $C_p(\Delta B(Y))$, respectively. We have $\dim(\Delta B(Y)) \leq m$, because the space $\Delta B(Y)$ is the limit of the inverse sequence $\{\Delta B_n(Y) : n \in \mathbb{N}\}$, where $\dim(\Delta B_n(Y)) \leq m$ for all $n \in \mathbb{N}$. We may now conclude, that $\dim(\Delta A(X)) \leq m$. Furthermore, we have $\xi = \left((\Delta F)^{-1} \circ \pi_F\right) \circ \Delta A$, where π_F is the restriction on $\Delta A(X)$ of the natural projection of $\mathbb{R}^A$ onto $\mathbb{R}^{A_0}$. Applying Lemma 4.3 to $\varphi = \xi$, $\theta = \Delta A$ and $p = (\Delta F)^{-1} \circ \pi_F$, we obtain $\dim X \leq m$. $\square$

Remark. Observe that it is possible to omit the restriction $iw(X) = iw(Y) = \aleph_0$ in the formulation of Theorem 4.9, if we modify Definition 3.1 by a convenient way. Namely, replace item (ii) by the following condition:

(ii') There exists $\varepsilon_0 > 0$ such that for each finite $M \subset K$ and an arbitrary open $U \subset X$ with $M \subset U$ one can find a function $\varphi \in A$, which is constant on M, coincides with 0^X on $X \backslash U$ and $|f(\varphi)| > \varepsilon_0$.

The requirements (i) and (ii') form another space of FSF, say $E(X)$, and we may formulate the counterpart for Theorem 4.9. But this case is not so interesting, because this $E(X)$ does not contain the space $L_p(X)$. Indeed, let us consider, for example, the functional $x_1 + x_2 - x_3 = f \in L_p(X)$. It is clear that $K = \{x_1, x_2, x_3\}$ is a support of f in the sense of Definition 3.1. Consider $M = \{x_2, x_3\}$. Evidently, for each function $\varphi \in A$, which is constant on M, we have $f(\varphi) = 0$ in a contradiction with (ii').

5. Open questions. Let us fix for each space X two subspaces $E_1(X)$ and $E_2(X)$ in $C_pC_p(X)$. Let S_1 and S_2 be the collections of all $(E_1(X), E_1(Y))$-satisfactory and $(E_2(X), E_2(Y))$-satisfactory homeomorphisms, respectively, with arbitrary spaces X and Y.

Definition 5.1. *We say, that S_2 is* irreducible to *S_1, if there exist two spaces X and Y such that the family $(E_2(X), E_2(Y))$ is nonempty, while $(E_1(X), E_1(Y))$ is empty. Otherwise S_2 is said to be reducible to S_1.*

Question 5.2. *What conditions on $E_1(X)$ and $E_2(X)$ must hold for the mutual irreducibility (or, conversely, reducibility) of S_2 and S_1?*

Question 5.3. *Is S_2 irreducible with respect to S_1 provided that $E_1(X) = L_p(X)$, $E_2(X) = \hat{L}_p(X)$ for each space X? What happens if $E_1(X) = U_p(X)$ and $E_2(X) = \hat{L}_p(X)$? In other words, is it true that the spaces $C_p(X)$, $C_p(Y)$ are linearly (or uniformly) homeomorphic whenever there exists an $\left(\hat{L}_p(X), \hat{L}_p(Y)\right)$-satisfactory homeomorphism $h : C_p(X) \to C_p(Y)$?*

Remark. If $E_1(X) = L_p(X)$, $E_2(X) = U_p(X)$ for each space X, it is known [7] that S_2 is irreducible with respect to S_1. If $E_1(X) = U_p(X)$, $E_2(X) = C_p^0C_p(X)$ for each space X, then [8] S_2 is also irreducible with respect to S_1.

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