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THE GULF BETWEEN THE CLIQUE NUMBER AND ITS UPPER ESTIMATE PROVIDED BY FRACTIONAL COLORING OF THE NODES

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Abstract. It is well known that coloring the nodes can be used to establish upper bounds for the clique number of a graph which in turn can be used to speed up practical clique search algorithms. E. Balas and J. Xue suggested factional coloring while S. Szabó and B. Zaválnij suggested triangle free coloring of the nodes to get tighter bounds. The main result of this paper is that the gap between the clique number and the upper bound provided by the coloring scheme combining the fractional and triangle free colorings still can be arbitrarily large.

1. Introduction. Let $G = (V, E)$ be a finite simple graph, where V is the set of nodes of the graph and E is the set of edges of the graph. The finiteness means that the graph has only finitely many nodes and edges. The simplicity refers to the fact that the graph does not have any loop or any double edge.

Let k be a fixed positive integer. A subgraph Δ of G is called a k-clique in G if it has k vertices and if each two distinct nodes of Δ are adjacent in G.

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The number of edges in Δ is equal to $k(k-1)/2$. The number k is called the size of the clique Δ.

A k-clique Δ in G is defined to be a maximal clique if Δ is not a subgraph of any $(k+1)$-clique in G. A k-clique Δ in G is defined to be a maximum clique if the graph G does not contain any $(k+1)$-clique. Typically a graph has more than one maximum cliques. However, all maximum cliques in G have a well defined common size. This well defined number is referred as the clique number of G, and it is denoted by $\omega(G)$.

The expression clique search problem refers to a number of problems related to finding cliques in a given graph. One may look for maximal cliques or maximum cliques. One might be interested in listing all maximal cliques or listing all maximum cliques. We maybe content with locating only one maximum clique. Or we maybe satisfied with just learning the clique size of G without exhibiting any maximum clique. We describe some relevant clique search problems more formally.

Problem 1. *Given a finite simple graph G and given a positive integer k. Decide if G contains a k-clique.*

Problem 1 is a decision problem and it is well-known that it belongs to the NP-complete complexity class.

Problem 2. *Given a finite simple graph G and a positive integer k. List all k-cliques in G.*

Problem 2 is not a decision problem and so it cannot belong to the NP-complete complexity class. However, it is clear that Problem 2 cannot be computationally less demanding than Problem 1. In other words Problem 2 is an NP-hard problem.

Determining the clique number $\omega(G)$ of G is not a decision problem either. It is an optimization problem. But again it must be clear that finding $\omega(G)$ is computationally at least as challenging as Problem 1. Therefore finding the clique number is an NP-hard problem.

Clique search problems have many practical applications and there is a considerable amount of research devoted to them. For details see for example [2], [4], [5], [6]. Many practical clique search algorithms utilize the coloring the nodes of a graph. We color the nodes of a given finite simple graph G with k colors satisfying the following conditions.

(1) Each node receives exactly two distinct colors.

(2) Adjacent nodes never receive the same color.

Table 1. The adjacency matrix of the Chvatal graph

	1	2	3	4	5	6	7	8	9	10	11	12
1	×	•		•	•							•
2	•	×	•			•	•					
3		•	×	•				•	•			
4	•		•	×					•	•		
5	•				×	•		•	•			
6		•			•	×				•	•	
7		•					×	•		•	•	
8			•		•		•	×				•
9			•		•				×	•		•
10				•		•	•		•	×		
11				•		•	•				×	•
12	•							•	•		•	×

This coloring of the nodes of the graph G is referred as a 2-fold legal coloring of the nodes of G. In connection with each finite simple graph G there is a number of colors k such that the nodes of G have a 2-fold legal coloring with k colors and the nodes of G does not have any 2-fold legal coloring with $k-1$ colors. This well defined number k is called the 2-fold chromatic number of G and it is denoted by $\chi^{(\text{frac})}(G)$. In this paper we take a rather narrow view of coloring. We are interested in coloring only from one reason. Coloring provides upper estimates for $\omega(G)$. Namely, $\omega(G) \leq \chi^{(\text{frac})}(G)/2$.

2. Reducing 2-fold legal coloring to legal coloring. Let $G = (V, E)$ be a finite simple graph. We would like to construct a 2-fold legal coloring of the nodes of G. One can accomplish this by constructing a legal coloring of the nodes of an auxiliary graph. Let $V = \{v_1, \ldots, v_n\}$. We construct a new graph $\Gamma = (W, F)$. We set

$$W = \{(v_1, 1), \ldots, (v_n, 1), (v_1, 2), \ldots, (v_n, 2)\}.$$

The nodes $(v_i, 1)$ and $(v_i, 2)$ are always connected by an edge in the graph Γ for each i, $1 \leq i \leq n$. If the unordered pair $\{v_i, v_j\}$ is an edge of G, then we add the unordered pairs

$$\{(v_i, 1), (v_j, 1)\}, \quad \{(v_i, 1), (v_j, 2)\},$$

$$\{(v_i, 2), (v_j, 1)\}, \quad \{(v_i, 2), (v_j, 2)\},$$

as edges to Γ. Note that if the nodes $(v_i, 1)$ and $(v_j, 1)$ are adjacent in Γ and $i \neq j$, then the nodes v_i and v_j must be adjacent in G. Similarly, if the nodes

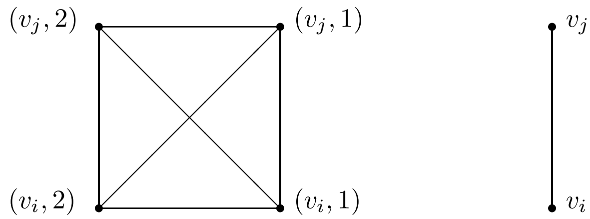

Fig. 1. The correspondence between the graphs G and Γ

$(v_i, 1)$ and $(v_j, 2)$ are adjacent in Γ and $i \neq j$, then the nodes v_i and v_j must be adjacent in G. Figure 1 depicts the subgraph of the auxiliary graph Γ that corresponds to two adjacent nodes in the graph G.

Observation 1. *If the nodes of G have a 2-fold legal coloring using k colors, then the nodes of the auxiliary graph Γ have a legal coloring with k colors.*

Proof. Let us suppose that the nodes of G have a 2-fold legal coloring using k colors. We define a coloring of the nodes of Γ. If the node v of G receives the distinct colors α, β, then the nodes $(v, 1)$, $(v, 2)$ of Γ receive colors α, β, respectively.

It is clear that each node of the graph Γ receives exactly one color. It remains to verify that if (u, i), (v, j) are distinct adjacent nodes of Γ, then they do not receive the same color. We distinguish two cases depending on $u = v$ or $u \neq v$.

Case 1: $u = v$. Now $i \neq j$ must hold since otherwise the nodes (u, i), (v, j) are identical. We may assume that $i = 1$ and $j = 2$ since this is only a matter of exchanging the nodes. As $\alpha \neq \beta$ then nodes $(u, 1)$ and $(u, 2)$ of Γ receive distinct colors.

Case 2: $u \neq v$. Now the unordered pair $\{u, v\}$ must be an edge of G. Let us suppose that the node u of G receives the colors α, β and the node v of G receives the colors γ, δ. The colors α, β, γ, δ are pair-wise distinct. Consequently the nodes (u, i) and (v, j) of Γ receive distinct colors. $\square$

Observation 2. *If the nodes of the auxiliary graph Γ have a legal coloring with k colors, then the nodes of the graph G have a 2-fold legal coloring with k colors.*

Proof. Let us assume that the nodes of the graph Γ have a legal coloring with k colors. We define a fractional coloring of the nodes of the graph

Table 2. The adjacency matrix of the auxiliary graph

	1 a	2 a	3 a	4 a	5 a	6 a	7 a	8 a	9 a	10 a	11 a	12 a	1 b	2 b	3 b	4 b	5 b	6 b	7 b	8 b	9 b	10 b	11 b	12 b
1,a	×	•		•	•							•	•	•		•	•							•
2,a	•	×	•			•	•						•	•	•			•	•					
3,a		•	×	•				•	•					•	•	•				•	•			
4,a	•		•	×					•	•			•		•	•					•	•		
5,a	•				×	•		•	•				•				•	•		•	•			
6,a		•			•	×				•	•			•			•	•				•	•	
7,a		•					×	•		•	•			•					•	•		•	•	
8,a			•		•		•	×				•			•		•		•	•				•
9,a			•		•				×	•		•			•		•				•	•		•
10,a				•		•	•		•	×						•		•	•		•	•		
11,a				•		•	•				×	•				•		•	•				•	•
12,a	•							•	•		•	×	•							•	•		•	•
1,b	•	•		•	•							•	×	•		•	•							•
2,b	•	•	•			•	•						•	×	•			•	•					
3,b		•	•	•				•	•					•	×	•				•	•			
4,b	•		•	•					•	•			•		•	×					•	•		
5,b	•				•	•		•	•				•				×	•		•	•			
6,b		•			•	•				•	•			•			•	×				•	•	
7,b		•					•	•		•	•			•					×	•		•	•	
8,b			•		•		•	•				•			•		•		•	×				•
9,b			•		•				•	•		•			•		•				×	•		•
10,b				•		•	•		•	•						•		•	•		•	×		
11,b				•		•	•				•	•				•		•	•				×	•
12,b	•							•	•		•	•	•							•	•		•	×

G. If v is a node of G, then $(v, 1)$ and $(v, 2)$ are nodes of the graph Γ. Suppose that the node $(v, 1)$ receives the colors α and the node $(v, 2)$ receives the color β. We assign the colors α, β to the node v of G. As the nodes $(v, 1)$ and $(v, 2)$ are adjacent in Γ, the colors α and β must be distinct. In other words each node of G receives exactly two distinct colors.

It remains to show that if the unordered pair $\{u, v\}$ is an edge of G, then the nodes u and v cannot have the same color. The nodes $(u, 1)$, $(u, 2)$, $(v, 1)$, $(v, 2)$ of Γ are the nodes of a 4-clique in Γ. Since the nodes of Γ are legally colored, these nodes receive pair-wise distinct colors. Therefore the nodes u and v of G cannot receive the same color. $\square$

We work out an example in details.

Table 3. The simple greedy sequential coloring procedure applied to the auxiliary graph

(1, a)	[1]	1	[1]	[1]	1	1	1	1	1	1	[1]	[1]
(2, a)	→	[2]	2	2	[2]	[2]	2	2	2	2	2	[2]
(3, a)		→	[1]	1	1	1	[1]	[1]	1	1	1	1
(4, a)			→	2	2	2	2	2	[2]	[2]	2	[2]
(5, a)				→	[2]	2	[2]	[2]	2	2	2	[2]
(6, a)					→	1	1	1	[1]	[1]	1	1
(7, a)						→	[1]	1	[1]	[1]	1	1
(8, a)							→	3	3	3	[3]	3
(9, a)								→	[3]	3	[3]	3
(10, a)									→	4	4	4
(11, a)										→	[3]	3
(12, a)											→	2
(1, b)												→
(2, b)												
(3, b)												
(4, b)												
(5, b)												
(6, b)												
(7, b)												
(8, b)												
(9, b)												
(10, b)												
(11, b)												
(12, b)												

Example 1. Let us consider the graph $G = (V, E)$ given by its adjacency matrix in Table 1. The graph is the so-called Chvatal graph. It has 12 nodes and 24 edges. The chromatic number of G is equal to 4 and the clique number of G is equal to 2.

The adjacency matrix of the auxiliary graph $\Gamma = (W, F)$ can be seen in Table 2. The set of vertices of G and Γ are the following

$$V = \{1, 2, \ldots, 12\} \quad \text{and} \quad W = \{(1, \mathrm{a}), \ldots, (12, \mathrm{a}), (1, \mathrm{b}), \ldots, (12, \mathrm{b})\}.$$

We used the simplest sequential greedy coloring procedure to construct a legal coloring of the nodes of the auxiliary graph Γ. The procedure is recorded in Tables 3 and 4. The 2-fold legal coloring of the nodes of G can be found in Table 5. Applying the simple sequential greedy coloring procedure for the auxiliary graph we get the upper estimate $7/2$ for the clique number of G. This estimate is better than one can get from the chromatic number of G.

Table 4. The simple greedy sequential coloring procedure applied to the auxiliary graph

(1, a)	[1]	1	[1]	[1]	1	1	1	1	1	1	1	1
(2, a)	[2]	[2]	2	2	[2]	[2]	2	2	2	2	2	2
(3, a)	[1]	[1]	[1]	1	1	1	[1]	[1]	1	1	1	1
(4, a)	2	[2]	[2]	2	2	2	2	2	[2]	[2]	2	2
(5, a)	2	2	2	[2]	[2]	2	[2]	[2]	2	2	2	2
(6, a)	[1]	1	1	[1]	[1]	1	1	1	[1]	[1]	1	1
(7, a)	[1]	1	1	1	1	[1]	1	1	[1]	[1]	1	1
(8, a)	3	[3]	3	[3]	3	[3]	[3]	3	3	3	[3]	3
(9, a)	3	[3]	3	[3]	3	3	[3]	[3]	[3]	3	[3]	3
(10, a)	4	4	[4]	4	[4]	[4]	4	[4]	[4]	4	4	4
(11, a)	3	3	[3]	3	[3]	[3]	3	3	3	[3]	[3]	3
(12, a)	2	2	2	2	2	2	[2]	[2]	2	[2]	[2]	2
(1, b)	[3]	3	[3]	[3]	3	3	3	3	3	3	3	3
(2, b)	$\rightarrow$	[4]	4	4	[4]	[4]	4	4	4	4	4	4
(3, b)		$\rightarrow$	[5]	5	5	5	[5]	[5]	5	5	5	5
(4, b)			$\rightarrow$	6	6	6	6	6	[6]	[6]	6	6
(5, b)				$\rightarrow$	[4]	4	[4]	[4]	4	4	4	4
(6, b)					$\rightarrow$	5	5	5	[5]	[5]	5	5
(7, b)						$\rightarrow$	[5]	5	[5]	[5]	5	5
(8, b)							$\rightarrow$	6	6	6	[6]	6
(9, b)								$\rightarrow$	[6]	6	[6]	6
(10, b)									$\rightarrow$	7	7	7
(11, b)										$\rightarrow$	[4]	4
(12, b)											$\rightarrow$	1

3. A probabilistic argument. In 1955 J. Mycielski [10] has proved the next result.

Theorem 1. *For each positive integer k there is a graph G such that $\chi(G) = k$ and G does not contain any 3-clique.*

This section contains the construction of a family of graphs for which the gap between the clique number and the 2-fold chromatic number can be arbitrarily large. The result can be proved using a probabilistic argument of P. Erdős [8]. We do not claim any originality of this argument. We simply included

Table 5. The vertices of G and their colors

vertex	1	2	3	4	5	6	7	8	9	10	11	12
color	1	2	1	2	2	1	1	3	3	4	3	2
color	3	4	5	6	4	5	5	6	6	7	4	1

it for the convenience of the reader. One result later critically depends on the existence of the family of such graphs.

We establish a connection between the independence and chromatic numbers.

Let $G = (V, E)$ be a finite simple graph whose independence number is $\alpha(G)$ and whose 2-fold chromatic number is $\chi^{(\mathrm{frac})}(G)$. The next observation shows that one can find a lower estimate for $\chi^{(\mathrm{frac})}(G)$ by using an upper bound for $\alpha(G)$.

Observation 3. *The inequality $\alpha(G)\chi^{(\mathrm{frac})}(G) \geq 2|V|$ holds.*

Proof. Let us consider a 2-regular fractional coloring of the nodes of G, where $C_1, \ldots, C_k$ are the color classes. We assume that $|C_1| \geq \cdots \geq |C_k|$ and $\chi^{(\mathrm{frac})}(G) = k$. It is clear that

$$\begin{array}{rcl} 2|V| & = & |C_1| + \cdots + |C_k| \\ & \leq & |C_1| + \cdots + |C_1| \\ & = & |C_1|k \\ & \leq & \alpha(G)k. \end{array}$$

□

We will use the fact that deleting a node from the graph cannot increase the independence number. Let G' be the graph we get from G by deleting the node v.

Observation 4. *The inequality $\alpha(G') \leq \alpha(G)$ holds.*

Proof. Let I be a maximum independent set in G. If $v \notin I$, then I is an independent set in G' and so $\alpha(G') = \alpha(G)$ holds. We are left with the case when $v \in I$. After deleting the node v from G it may happen that some neighbors $u_1, \ldots, u_s$ of v can be added to $I \setminus \{v\}$ to get a larger independent set $J = (I \setminus \{v\}) \cup \{u_1, \ldots, u_s\}$ in G'. In this situation $s \geq 2$ must hold to get $|I| < |J|$. Further u_i cannot be adjacent to any node of $I \setminus \{v\}$ for each i, $1 \leq i \leq s$. Now J is an independent set in G contrary to the fact that I is a maximum independent set of G. □

Next we focus our attention to the number of the triangles.

We work with the random graph $G(n, p)$. Here the parameter n is then number of the nodes of the random graph. The probability of the event that two distinct randomly chosen nodes are adjacent is equal to the parameter p. In our argument we will set p to be $1/(n^{2/3})$.

We set $r = \binom{n}{3}$. Let $A_1, \ldots, A_r$ be all the 3 element subsets of V. We define a random variable X_i for each i, $1 \le i \le r$.

$$X_i = \begin{cases} 1, & \text{if } A_i \text{ induces a triangle in } G, \\ 0, & \text{otherwise.} \end{cases}$$

The quantity $X = X_1 + \cdots + X_r$ is the number of the 3-cliques in the graph G. We compute the expected value of X_i.

$$\mathrm{M}[X_i] = (0)(1 - p^3) + (1)(p^3) = p^3.$$

We are looking for an upper estimate for the expected value of X.

$$\begin{aligned} \mathrm{M}[X] &= \mathrm{M}[X_1 + \cdots + X_r] \\ &= \mathrm{M}[X_1] + \cdots + \mathrm{M}[X_r] \\ &= r\mathrm{M}[X_1] \\ &= rp^3 \\ &= \binom{n}{3} p^3 \\ &= \Big(\frac{n(n-1)(n-2)}{6}\Big)\Big(\frac{1}{n^2}\Big) \\ &< \frac{n}{6}. \end{aligned}$$

We use Markov's inequality $\Pr[X \ge \alpha] \le \mathrm{M}[X]/\alpha$ with $\alpha = n/2$ to get

$$\Pr\Big[X \ge \frac{n}{2}\Big] \le \Big(\frac{n}{6}\Big)/\Big(\frac{n}{2}\Big) = \Big(\frac{n}{6}\Big)\Big(\frac{2}{n}\Big) = \frac{1}{3}.$$

It follows that

$$\Pr\Big[X < \frac{n}{2}\Big] \ge \frac{2}{3}.$$

The random graph G contains less than $n/2$ triangles with probability at least 2/3.

We have a closer look at the number of independent sets.

Set $t = \lceil \frac{n}{k} \rceil$ and $s = \binom{n}{t}$. We are interested in independent sets of size t and s is the number of t element subsets of an n element set. Let $I_1, \ldots, I_s$ be all t element subsets of V. We introduce a random variable Y_i for each i, $1 \le i \le s$.

$$Y_i = \begin{cases} 1, & \text{if } I_i \text{ is an independent set in } G, \\ 0, & \text{otherwise.} \end{cases}$$

The quantity $Y = Y_1 + \cdots + Y_s$ is the number of the independent sets of size t in the graph G. We compute the expected value of Y_i.

$$\mathrm{M}[Y_i] = (0)\left[1 - (1-p)^{\binom{t}{2}}\right] + (1)\left[(1-p)^{\binom{t}{2}}\right] = (1-p)^{\binom{t}{2}}.$$

We compute the expected value of Y.

$$\begin{aligned}
\mathrm{M}[Y] &= \mathrm{M}[Y_1 + \cdots + Y_s] \\
&= \mathrm{M}[Y_1] + \cdots + \mathrm{M}[Y_s] \\
&= s\mathrm{M}[Y_1] \\
&= s(1-p)^{\binom{t}{2}} \\
&= \binom{n}{t}(1-p)^{\binom{t}{2}}.
\end{aligned}$$

We are looking for an upper estimate for the expected value of Y. Clearly, $\binom{n}{t}$ can be over estimated by $2^n = e^{(\ln 2)n}$. We use the inequality $1 - x \leq e^{-x}$ to replace $(1-p)^{\binom{t}{2}}$ by

$$(e^{-p})^{\binom{t}{2}} = e^{-p\frac{t(t-1)}{2}}.$$

We get the upper estimate

$$\mathrm{M}[Y] \leq e^{(\ln 2)n - p\frac{t(t-1)}{2}}.$$

We denote the exponent on the right hand side by Q and we focus our attention to this quantity.

$$\begin{aligned}
Q &= (\ln 2)n - p\frac{t(t-1)}{2} \\
&= (\ln 2)n - \left(\frac{1}{2}\right)\left(\frac{1}{n^{2/3}}\right)\left\lceil\frac{n}{k}\right\rceil\left(\left\lceil\frac{n}{k}\right\rceil - 1\right) \\
&\leq (\ln 2)n - \left(\frac{1}{2}\right)\left(\frac{1}{n^{2/3}}\right)\left(\frac{n}{k}\right)\left(\left(\frac{n}{k}\right) - 1\right) \\
&= (\ln 2)n - \left(\frac{1}{2}\right)\left(\frac{1}{n^{2/3}}\right)\left(\frac{n}{k}\right)\left(\frac{n-k}{k}\right) \\
&= (\ln 2)n - \left(\frac{1}{2}\right)\left(\frac{1}{k^2}\right)n^{1/3}(n-k)
\end{aligned}$$

Note that Q tends to $-\infty$ as n tends to ∞. Therefore $\mathrm{M}[Y]$ tends to 0 as n tends to ∞. Consequently we may choose n such that $\mathrm{M}[Y] < 1/10$ holds.

We apply Markov's inequality $\Pr[Y \geq \alpha] \leq \mathrm{M}[Y]/\alpha$ with $\alpha = 1$ to get $\Pr[Y \geq 1] \leq (1/10)/1 = 1/10$ It follows that $\Pr[Y < 1] \geq 9/10$. The random graph G does not contain any independent set of size $\lceil \frac{n}{k} \rceil$ with probability at least $9/10$.

As a next step we delete nodes from our graph.

Let A be the event that the graph G contains less than $n/2$ triangles. Let B be the event that G does not contain any independent set of size $\lceil \frac{n}{k} \rceil$. We know that $\Pr[A] \geq 2/3$ and $\Pr[B] \geq 9/10$. Further $\Pr[A \cup B] \leq 1$ implies that $-\Pr[A \cup B] \geq -1$.

From $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$ we get that

$$\begin{aligned} \Pr[A \cap B] &= \Pr[A] + \Pr[B] - \Pr[A \cup B] \\ &\geq \frac{2}{3} + \frac{9}{10} - 1 \\ &= \frac{20}{30} + \frac{27}{30} - \frac{30}{30} \\ &= \frac{17}{30}. \end{aligned}$$

In plain English there is a finite simple graph G with n nodes such that G contains less than $n/2$ triangles and G does not contain any independent set of size $\lceil \frac{n}{k} \rceil$.

We delete at most one node from each triangle in G. In this way we end up with a graph G' which does not contain any triangle. The number of the nodes of G' is at least $n/2$.

When we delete a node from G in the way of getting the graph G' the size of the maximum independent set cannot increase. Thus the inequality $\alpha(G') \leq \alpha(G)$ holds.

Let k be a fixed positive integer. There is a finite simple graph $H^{(k)} = (V, E)$ with the following properties.

(1) The graph $H^{(k)}$ has at least $n/2$ nodes.

(2) The graph $H^{(k)}$ does not contain any triangle.

(3) The inequality $\alpha(H^{(k)}) \leq \lceil \frac{n}{k} \rceil$ holds.

Using the upper bounds n/k and $n/2$ for $\alpha(H^{(k)})$ and $|V|$, respectively from the inequality

$$\alpha(H^{(k)})\chi^{(\mathrm{frac})}(H^{(k)}) \geq 2|V|$$

we can get a lower bound for $\chi^{(\mathrm{frac})}(H^{(k)})$. Namely,

$$\begin{aligned} \chi^{(\mathrm{frac})}(H^{(k)}) &\geq (2)\left(\frac{n}{2}\right)/\left(\frac{n}{k}\right) \\ &= (2)\left(\frac{n}{2}\right)\left(\frac{k}{n}\right) \\ &= k. \end{aligned}$$

We may summarize the result of our considerations in the following theorem.

Theorem 2. *For each positive integer k there is a graph $H^{(k)}$ for which $\omega(H^{(k)}) \leq 2$ and $\chi^{(\mathrm{frac})}(H^{(k)}) \geq k$.*

4. 2-fold triangle free coloring of the nodes. Let $G = (V, E)$ be a finite simple graph. We color the nodes of G satisfying the following conditions.

(1) Each node of G receives exactly two distinct colors.

(2) If u, v, w are the nodes of 3-clique in G and $\{\alpha, \beta\}$, $\{\gamma, \delta\}$, $\{\epsilon, \mu\}$ are the colors assigned to these nodes, then at least two of the equations $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$, $\{\alpha, \beta\} \cap \{\epsilon, \mu\} = \emptyset$, $\{\gamma, \delta\} \cap \{\epsilon, \mu\} = \emptyset$ must hold.

We say that two distinct adjacent nodes of the graph G is not colored correctly if to the two nodes the same color is assigned at least once. In a 2-fold triangle free coloring of the nodes of the graph G in each triangle there can be at most one not correctly colored pair of nodes.

For each finite simple graph $G = (V, E)$ there is a number of colors k such that the nodes of G have a 2-fold triangle free coloring with k colors, but the nodes of G do not admit any 2-fold triangle free coloring with $k - 1$ colors. This well defined number k is called the 2-fold triangle free chromatic number of G and it is denoted by $\chi^{(\mathrm{frac})}_{(3)}(G)$.

Our interest in the 2-fold triangle free chromatic number is motivated by the fact that it can be used to upper estimate the clique number.

Observation 5. *Let G be a finite simple graph. Then the inequality $\omega(G) \leq \chi^{(\mathrm{frac})}_{(3)}(G)$ holds.*

P r o o f. Let us suppose that $\omega(G) = l$ and let Δ be an l-clique in G. Let us suppose further that $\chi^{(\mathrm{frac})}_{(3)}(G) = k$. We consider a 2-fold triangle free coloring of the nodes of G. Let $C_1, \dots, C_k$ be the colors classes of this coloring of the nodes. Each node of G is contained in exactly two color classes. Each color class contains at most two nodes of Δ. It follows that $2l \leq 2k$.

The details are the following. We construct an incidence matrix M. The rows of M are labeled by the nodes of G. The columns of M are labeled by the colors classes $C_1, \dots, C_k$. If node v receives color c, then we put a 1 to the cell at the intersection of row v and column c. Otherwise we put 0 into the cell.

By rearranging the rows of M we may assume that the first l rows correspond to the nodes of the clique Δ. By rearranging the columns of M we may assume that each of the first p columns contains exactly two 1's in the first l rows. Similarly, we may assume that each of the next q columns contains exactly one 1 in the first l rows. Finally, we may assume that none of the last r columns contains any 1 in the first l rows.

The number of the 1's in the first l rows in M is equal to $2l$. We may count the 1's by going through the columns of M. The number of 1's in the first l rows of M is equal to $2p+q$. Thus $2l \leq 2p+q$. Using $2p+q \leq 2p+2q+2r = 2k$ we get $2l \leq 2k$, as required. □

The probabilistic method gives that for each given positive integer k there is a finite simple graph $H^{(k)}$ with the property that $\omega(H^{(k)}) \leq 2$ and $\chi^{(\mathrm{frac})}(H^{(k)}) \geq k$.

The main result of this section is the following theorem.

Theorem 3. *For each positive integer k there is a finite simple graph $L^{(k)}$ such that $\omega(L^{(k)}) \leq 4$ and $\chi^{(\mathrm{frac})}_{(3)}(L^{(k)}) \geq k$.*

P r o o f. Let us assume that the graph $H^{(k)}$ has n nodes. Let $u_1, \dots, u_n$ be the nodes of $H^{(k)}$. The graph $L^{(k)}$ we are constructing will have n^2 nodes

$$v_{1,1}, \dots, v_{1,n}, \dots, v_{n,1}, \dots, v_{n,n}.$$

The set of nodes $V_i = \{v_{i,1}, \dots, v_{i,n}\}$ induces a graph $H^{(k)}_i$ in $L^{(k)}$ such that $H^{(k)}_i$ is isomorphic to $H^{(k)}$. For the sake of a simple notation we assume that the map $u_j \to v_{i,j}$ is an isomorphism between $H^{(k)}$ and $H^{(k)}_i$.

If the unordered pair $\{u_i, u_j\}$ is an edge of the graph $H^{(k)}$, then each node of $H^{(k)}_i$ and each node of $H^{(k)}_j$ will be connected by an edge in the graph $L^{(k)}$.

We claim that the inequality $\omega(L^{(k)}) \leq 4$ holds.

In order to prove the claim we assume on the contrary that $\omega(L^{(k)}) \geq 5$. We choose a 5-clique Δ in the graph $L^{(k)}$. Since the graph $H_i^{(k)}$ does not contain any 3-clique the 5-clique Δ may have at most 2 nodes in the graph $H_i^{(k)}$ for each i, $1 \leq i \leq n$. There are at least 3 distinct values of i for which the graph $H_i^{(k)}$ contains a node of the 5-clique Δ. This provides that the graph $H_i^{(k)}$ contains a 3-clique. But we know that $\omega(H^{(k)}) \leq 2$. This contradiction completes the proof of the claim.

Next we claim that the inequality $\chi_{(3)}^{(\text{frac})}(L^{(k)}) \geq k$ holds.

In order to prove the claim let us assume on the contrary that $\chi_{(3)}^{(\text{frac})}(L^{(k)}) \leq k-1$. There is an integer l such that $1 \leq l \leq k-1$ and $\chi_{(3)}^{(\text{frac})}(L^{(k)}) = l$. For the sake of definiteness we assume that $l = k-1$.

Set $V = V_1 \cup \cdots \cup V_n$ and let B be the family of all the 2-element subsets of the set $\{1, \ldots, k-1\}$. We consider a 2-fold triangle free coloring of the nodes of the graph $L^{(k)}$ with $k-1$ colors. We assume that this coloring is described by a map $f : V \to B$. The restriction of the map f to the set V_i defines a 2-fold triangle free coloring of the graph $H_i^{(k)}$. As the 2-fold triangle free chromatic number of the graph $H_i^{(k)}$ is at least k, there must be two distinct adjacent nodes $v_{i,i(1)}$, $v_{i,i(2)}$ of the graph $H_i^{(k)}$ such that the colors assigned to these nodes are $\{\alpha, \beta\}$, $\{\beta, \gamma\}$. (The point is that there is a color that is assigned to both of the nodes.) Figure 2 may serve as an aid to follow this argument.

We construct a coloring of the nodes of the graph $H^{(k)}$. To the node u_i of the graph $H^{(k)}$ we assign the colors $\{\alpha, \beta\}$. Assigning the colors $\{\beta, \gamma\}$ to the node u_i would work as well. For the sake of definiteness we assigned the colors of the node $v_{i,i(1)}$ of the graph $H_i^{(k)}$ to the node u_i of the graph $H^{(k)}$.

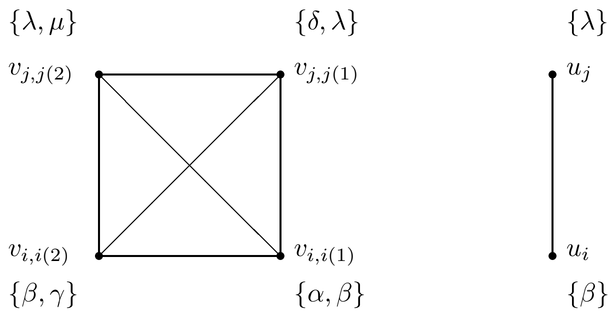

Fig. 2. The 4-clique Δ in the graph $L^{(k)}$

It is clear that each node of the graph $H^{(k)}$ receives exactly two distinct colors. It remains to show that if the unordered pair $\{u_i, u_j\}$ is an edge of the graph $H^{(k)}$, then the nodes u_i and u_j do not receive the same color.

The 2-fold triangle free chromatic number of the graph $H_j^{(k)}$ is at least k. There are two distinct adjacent nodes $v_{j,j(1)}$, $v_{j,j(2)}$ of the graph $H_i^{(k)}$ such that the colors assigned to these nodes are $\{\delta, \lambda\}$, $\{\lambda, \mu\}$. The colors assigned to the node u_j of $H^{(k)}$ are $\{\delta, \lambda\}$.

The nodes $v_{i,i(1)}$, $v_{i,i(2)}$, $v_{j,j(1)}$, $v_{j,j(2)}$ are the nodes of a 4-clique Δ in the graph $L^{(k)}$.

The nodes $v_{i,i(1)}$, $v_{i,i(2)}$ are not colored correctly because of the color β. Similarly, the nodes $v_{j,j(1)}$, $v_{j,j(2)}$ are not colored correctly because of the color λ. The end points of the two edges

$$\{v_{i,i(1)}, v_{i,i(2)}\}, \quad \{v_{j,j(1)}, v_{j,j(2)}\}$$

in the 4-clique Δ are not colored correctly. It follows that the end points of the remaining 4 edges of the 4-clique Δ must be correctly colored. In particular the colors α, γ, δ, μ are pair-wise distinct. $\square$

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