Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **43** (2017), 127–146

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

OSCILLATION CRITERIA OF SOLUTIONS OF THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH CONTINUOUSLY DISTRIBUTED DELAY

O. Moaaz, E. M. Elabbasy, E. Shaaban

Communicated by I. D. Iliev

Abstract. In this paper, a class of third-order neutral delay differential equations with continuously distributed delay is studied. Also, we establish new oscillation results for the third-order equation by using the integral averaging technique due to Philos. Our results essentially improve and complement some earlier publications. Examples are provided to illustrate new results.

1. Introduction. We are concerned with the oscillation and the asymptotic behavior of solutions of the third-order nonlinear neutral differential equations with delayed argument

$$(r(l)[z''(l)]^{\alpha})' + \int_c^d q(l,\xi)x^{\alpha}(g(l,\xi))d\xi = 0,\ l \geq l_0, \tag{1.1}$$

2010 *Mathematics Subject Classification:* 34K10, 34K11.
Key words: Oscillation, third-order, neutral delay, differential equations.

where $z(l) = x(l) + \int_a^b p(l,\eta)x(\tau(l,\eta))d\eta$. In the sequel we will assume that the following conditions are satisfied:

(C) $r \in C([l_0,\infty),(0,\infty))$, $p \in C([l_0,\infty) \times [a,b],\mathbb{R}^+)$, $\tau \in C([l_0,\infty) \times [a,b],\mathbb{R})$; $q,\, g \in C([l_0,\infty) \times [c,d],\mathbb{R})$, $0 \le \int_a^b p(l,\eta)d\eta \le p < 1$, $\tau(l,\eta) \le l$, $g(l,\xi) \le l$, $\lim_{l\to\infty} \tau(l,\eta) = \lim_{l\to\infty} g(l,\xi) = \infty$, $q(l,\xi) > 0$, $g(l,\xi)$ is nondecreasing with respect to l,ξ, $\int_{l_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds = \infty$ and α is a quotient of odd positive integers.

By a solution of Eq. (1.1), we mean a function $x(l) \in C([l_x,\infty))$, $l_x \ge l_0$, which has the property $r(l)\left(z''(l)\right)^{\alpha} \in C^1([l_x,\infty))$ and satisfies Eq. (1.1) on $[l_x,\infty)$. We consider only those solutions $x(l)$ of Eq. (1.1) which satisfy $\sup\{|x(l)| : l \ge l\} > 0$ for any $l \ge l_x$. A solution of Eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of second/third-order delay differential equations, see for example [1]-[17] and the references quoted therein. Special cases of equation (1.1) include the delay equation

$$\left(r(l)\left((x(l)+p(l)\,x(\tau(l)))''\right)^{\alpha}\right)' + q(l)\,x^{\alpha}(g(l)) = 0. \tag{1.2}$$

The oscillatory behavior of solutions of (1.2) have been discussed in number of studies and we refer the reader, for example, to the monographs by Baculikova [3], Dzurina [7] and Thandapani [19].

Actually, we have greatly less results for third-order differential equations than for the first or second order equations. So, the main objective of this paper is to shed light on the class of third-order equation, through study the asymptotic behavior of this equation and comparison of results. Our results in this paper not only generalize some the previous results, but also improve the earlier ones (as described in the examples and remarks). First, we establish some new oscillation criteria for the equation (1.1), which in the special case (equation (1.2)) generalize and improve the results established by Baculikova [3] and are different from the results of [3] in the sense that our results not require $r'(l) > 0$. Also, we set a new criteria for oscillation of solutions of the equation

$$\left(l\left[(x(l)+(1/3)\,x(l/2))''\right]^{3}\right)' + \left(\lambda/l^{6}\right)x^{3}(l/2) = 0,$$

studied by [3] and [7]. Finally, we will apply the Riccati technique to establish some new oscillation results of Kamenev-type and our results in this paper improve the results established by Qin [16] and Tian [20].

2. Main results. For the sake of brevity, we define:

$$R_u(l) = \int_u^l \frac{ds}{r^{\frac{1}{\alpha}}(s)}, \quad \bar{R}_u(l) = \int_u^l R_u(s)\, ds.$$

First, we state and prove some useful lemmas.

Lemma 2.1. *Let $x(l)$ be a positive solution of Eq.* (1.1). *Then $z(l)$ has only one of the following two properties eventually*

$$\begin{aligned} (\mathbf{i})\ z(l) &> 0,\ z'(l) > 0 \textit{ and } z''(l) > 0; \\ (\mathbf{ii})\ z(l) &> 0,\ z'(l) < 0 \textit{ and } z''(l) > 0. \end{aligned}$$

Proof. Let $x(l)$ be a positive solution of Eq. (1.1). From (C), there exists a $l_1 \geq l_0$ such that $x(l) > 0$, $x(\tau(l,\eta)) > 0$ and $x(g(l,\xi)) > 0$, then $z(l) > 0$, and Eq. (1.1) implies that

$$\left(r(l)\left[z''(l)\right]^{\alpha}\right)' = -\int_c^d q(l,\xi)\, x^{\alpha}(g(l,\xi))\, d\xi \leq 0.$$

Hence $\left(r(l)\left[z''(l)\right]^{\alpha}\right)$ is a non-increasing function and of one sign. We claim that $\left(r(l)\left[z''(l)\right]^{\alpha}\right) > 0$ for $l \geq l_1$. Suppose that $\left(r(l)\left[z''(l)\right]^{\alpha}\right) < 0$ for $l \geq l_2 \geq l_1$, then there exists a $l_3 \geq l_2$ and constant $K_1 > 0$ such that

$$\left(r(l)\left[z''(l)\right]^{\alpha}\right) \leq -K_1,\ for\ l \geq l_3.$$

By integrating the last inequality from l_3 to l, we get

$$z'(l) \leq z'(l_3) - K_1^{\frac{1}{\alpha}} \int_{l_3}^l r^{-\frac{1}{\alpha}}(s)\, ds.$$

Letting $l \to \infty$, we have $\lim\limits_{l\to\infty} z'(l) = -\infty$. Then there exists a $l_4 \geq l_3$ and constant $K_2 > 0$ such that

$$z'(l) < -K_2,\ for\ l \geq l_4.$$

By integrating this inequality from l_4 to ∞ and using (C), we get $\lim\limits_{l\to\infty} z(l) = -\infty$, which contradicts $z(l) > 0$. Now we have $\left(r(l)\left[z''(l)\right]^{\alpha}\right) > 0$ for $l \geq l_0$. Therefore $z'(l)$ is increasing function. Thus (**i**) or (**ii**) holds for $z(l)$ eventually. □

Lemma 2.2. *Assume that $x(l)$ be a positive solution of Eq.* (1.1) *and $z(l)$ has the property* (**ii**). *If*

$$\int_{l_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\xi)\,d\xi ds\right]^{\frac{1}{\alpha}}dudv=\infty, \tag{2.1}$$

then the solution $x(l)$ of Eq. (1.1) *is converges to zero as $l\to\infty$.*

Proof. Assume that $x(l)$ be a positive solution of Eq. (1.1). Since $z(l)$ satisfies (**ii**), we obtain

$$\lim_{l\to\infty} z(l)=\gamma\geq 0.$$

Next, we claim that $\gamma=0$. Let $\gamma>0$, then we get $\gamma<z(l)<\gamma+\varepsilon$ for all $\varepsilon>0$ and l enough large. Choosing $\varepsilon<\dfrac{1-p}{p}\gamma$, we obtain

$$\begin{aligned}
x(l) &= z(l)-\int_a^b p(l,\eta)\,x(\tau(l,\eta))\,d\eta\\
&> \gamma-\int_a^b p(l,\eta)\,z(\tau(l,\eta))\,d\eta\\
&> \gamma-z(\tau(l,a))\int_a^b p(l,\eta)\,d\eta\\
&> \gamma-p(\gamma+\varepsilon)>Lz(l),
\end{aligned}$$

where $L=\dfrac{\gamma-p(\gamma+\varepsilon)}{\gamma+\varepsilon}>0$. Hence from Eq. (1.1) and (C) we have

$$\begin{aligned}
\left(r(l)\left[z''(l)\right]^{\alpha}\right)' &= -\int_c^d q(l,\xi)\,x^{\alpha}(g(l,\xi))\,d\xi\\
&\leq -L^{\alpha}\int_c^d q(l,\xi)\,z^{\alpha}(g(l,\xi))\,d\xi\\
&\leq -L^{\alpha}\gamma^{\alpha}\int_c^d q(l,\xi)\,d\xi.
\end{aligned}$$

Integrating this inequality from l to ∞, we get

$$r(l)\left[z''(l)\right]^{\alpha}\geq L^{\alpha}\gamma^{\alpha}\int_l^{\infty}\int_c^d q(s,\xi)\,d\xi ds,$$

this inequality implies that

$$z''(l)\geq L\gamma\left[\frac{1}{r(l)}\int_l^{\infty}\int_c^d q(s,\xi)\,d\xi ds\right]^{\frac{1}{\alpha}}.$$

Integrating last inequality from l to ∞, we obtain

$$-z'(l) \geq L\gamma \int_l^\infty \left[\frac{1}{r(u)} \int_u^\infty \int_c^d q(s,\xi)\, d\xi ds\right]^{\frac{1}{\alpha}} du.$$

Integrating again from l_1 to ∞, we have

$$z(l_1) \geq L\gamma \int_{l_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty \int_c^d q(s,\xi)\, d\xi ds\right]^{\frac{1}{\alpha}} dudv.$$

This contradicts to the condition (2.1), hence $\lim\limits_{l\to\infty} z(l) = 0$, which implies that $\lim\limits_{l\to\infty} x(l) = 0$. $\square$

Lemma 2.3. *Let* $x(l)$ *be a positive solution of Eq.* (1.1) *and* $z(l)$ *has the property* (**i**). *Then*

$$\left(r(l)\left[z''(l)\right]^\alpha\right)' \leq -(1-p)^\alpha z^\alpha (g(l,c)) \int_c^d q(l,\xi)\, d\xi. \tag{2.2}$$

$$z'(g(l,\xi)) \geq \left(r(l)\left[z''(l)\right]^\alpha\right)^{\frac{1}{\alpha}} R_{l_0}(g(l,\xi)). \tag{2.3}$$

$$\bar{R}_{l_0}^\alpha (g(l,\xi)) \frac{(r(l)\left[z''(l)\right]^\alpha)}{z^\alpha (g(l,\xi))} \leq 1. \tag{2.4}$$

P r o o f. Let $x(l)$ be a positive solution of Eq. (1.1) from (C) there exists a $l_1 \geq l_0$ such that $x(l) > 0$, $x(\tau(l,\eta)) > 0$ and $x(g(l,\xi)) > 0$ for $l \geq l_1$. Since $z(l)$ satisfies the property (**i**) then we get

$$\begin{aligned} x(l) &= z(l) - \int_a^b p(l,\eta)\, x(\tau(l,\eta))\, d\eta \\ &\geq z(l) - z(l) \int_a^b p(l,\eta)\, d\eta \\ &\geq \left(1 - \int_a^b p(l,\eta)\, d\eta\right) z(l) \\ &\geq (1-p)\, z(l). \end{aligned}$$

Thus, by Eq. (1.1) and (C) we have

$$\left(r(l)\left[z''(l)\right]^\alpha\right)' = -\int_c^d q(l,\xi)\, x^\alpha (g(l,\xi))\, d\xi$$

$$\begin{aligned} &\leq -(1-p)^{\alpha} \int_{c}^{d} q(l,\xi) z^{\alpha}(g(l,\xi)) \, d\xi \\ &\leq -(1-p)^{\alpha} z^{\alpha}(g(l,c)) \int_{c}^{d} q(l,\xi) \, d\xi. \end{aligned}$$

Now, from property (**i**), there exists a $l \geq l_0$ such that

$$z'(l) = z'(l) + \int_{l}^{l} \frac{\left(r(s)\left[z''(s)\right]^{\alpha}\right)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)} ds.$$

Since $\left(r(l)\left[z''(l)\right]^{\alpha}\right)' < 0$, we obtain

$$z'(l) \geq \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} \int_{l}^{l} \frac{1}{r^{\frac{1}{\alpha}}(s)} ds.$$

This implies that

$$z'(l) \geq \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} R_l(l). \tag{2.5}$$

Since $g(l,\xi) \leq l$, we have

$$z'(g(l,\xi)) \geq \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} R_l(g(l,\xi)).$$

Next, integrating the inequality (2.5) from l to l and using $\left(r(l)\left[z''(l)\right]^{\alpha}\right)' < 0$, we get

$$\begin{aligned} z(l) &\geq z(l) + \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} \int_{l}^{l} R_l(s) \, ds \\ &\geq \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} \bar{R}_l(l). \end{aligned}$$

Thus, we get

$$z(g(l,\xi)) \geq \left(r(l)\left[z''(l)\right]^{\alpha}\right)^{\frac{1}{\alpha}} \bar{R}_l(g(l,\xi)).$$

This inequality implies that

$$\bar{R}_l^{\alpha}(g(l,\xi)) \frac{\left(r(l)\left[z''(l)\right]^{\alpha}\right)}{z^{\alpha}(g(l,\xi))} \leq 1.$$

This completes the proof. □

Now, for simplicity, we introduce the following notation:

$$P=\liminf_{l\to\infty}\bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\int_{l}^{\infty}\theta\left(s\right)ds$$

and

$$Q=\limsup_{l\to\infty}\frac{1}{\bar{R}_{l_0}\left(g\left(l,c\right)\right)}\int_{l_0}^{l}\bar{R}_{l_0}^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds,$$

where

$$\theta\left(l\right)=\left(1-p\right)^{\alpha}\int_{c}^{d}q\left(l,\xi\right)d\xi.$$

Moreover for $z\left(l\right)$ satisfying property $(\mathbf{i})$, we define

$$\omega\left(l\right)=r\left(l\right)\left(\frac{z''\left(l\right)}{z\left(g\left(l,c\right)\right)}\right)^{\alpha} \tag{2.6}$$

and

$$\varsigma=\liminf_{l\to\infty}\bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\omega\left(l\right),\ U=\limsup_{l\to\infty}\bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\omega\left(l\right). \tag{2.7}$$

Lemma 2.4. *Let $x\left(l\right)$ be a positive solution of Eq.* (1.1).

(I) *Let $P<\infty$, $Q<\infty$ and $z\left(l\right)$ satisfies property* $(\mathbf{i})$. *If*

$$\lim_{l\to\infty}\bar{R}_{l_0}\left(l\right)=\infty, \tag{2.8}$$

then

$$P\leq\varsigma-\varsigma^{\frac{1+\alpha}{\alpha}}\ \text{and}\ P+Q\leq1. \tag{2.9}$$

(II) *If $P=\infty$ or $Q=\infty$, then $z\left(l\right)$ does not have property* $(\mathbf{i})$.

Proof. **Part (I).** Let $x\left(l\right)$ be a positive solution of Eq. (1.1) and $z\left(l\right)$ satisfies property $(\mathbf{i})$. By Lemma 2.3, we have (2.2), (2.3) and (2.4) hold. From definition of $\omega\left(l\right)$, we see that $\omega\left(l\right)$ is positive and satisfies

$$\omega'\left(l\right)=\frac{\left(r\left(l\right)\left[z''\left(l\right)\right]^{\alpha}\right)'}{z^{\alpha}\left(g\left(l,c\right)\right)}-\alpha\frac{\left(r\left(l\right)\left[z''\left(l\right)\right]^{\alpha}\right)}{z^{\alpha+1}\left(g\left(l,c\right)\right)}z'\left(g\left(l,c\right)\right)g'\left(l,c\right).$$

Thus from (2.2) and (2.3) there exists a $l\geq l_0$ such that

$$\omega'\left(l\right)\leq-\left(1-p\right)^{\alpha}\int_{c}^{d}q\left(l,\xi\right)d\xi-\alpha\frac{\left(r\left(l\right)\left[z''\left(l\right)\right]^{\alpha}\right)^{\frac{1+\alpha}{\alpha}}}{z^{1+\alpha}\left(g\left(l,c\right)\right)}R_{l}\left(g\left(l,c\right)\right)g'\left(l,c\right),$$

for $l \geq l$. This implies that

$$\omega'(l) \leq -\theta(l) - \alpha R_l(g(l,c))\, g'(l,c)\, \omega^{\frac{1+\alpha}{\alpha}}(l). \tag{2.10}$$

From (2.4), we get

$$\bar{R}_l^{\alpha}(g(l,c))\,\omega(l) \leq 1,$$

which with (2.8) gives

$$\lim_{l\to\infty} \omega(l) = 0. \tag{2.11}$$

On the other hand, from the definition of $\omega(l)$, ς and U, we see that

$$0 \leq \varsigma \leq U \leq 1. \tag{2.12}$$

Now, we prove that the first inequality in (2.9) holds. Let $\varepsilon > 0$, then from the definition of P and ς we can choose $l_2 \geq l$ sufficiently large that

$$\bar{R}_l^{\alpha}(g(l,c)) \int_l^{\infty} \theta(s)\, ds \geq P - \varepsilon$$

and

$$\bar{R}_l^{\alpha}(g(l,c))\,\omega(l) \geq \varsigma - \varepsilon,$$

for $l \geq l_2$. By integrating (2.10) from l to ∞ and using (2.11), we have

$$\omega(l) \geq \int_l^{\infty} \theta(s)\, ds + \alpha \int_l^{\infty} R_l(g(s,c))\, g'(s,c)\, \omega^{\frac{1+\alpha}{\alpha}}(s)\, ds. \tag{2.13}$$

Multiplying the above inequality by $\bar{R}_l^{\alpha}(g(l,c))$, we obtain

$$\begin{aligned}
\bar{R}_l^{\alpha}(g(l,c))\,\omega(l) \geq{}& \alpha \bar{R}_l^{\alpha}(g(l,c)) \int_l^{\infty} \frac{R_l(g(s,c))\, g'(s,c)}{\bar{R}_l^{\alpha+1}(g(s,c))} \left(\bar{R}_l^{\alpha}(g(s,c))\,\omega(s)\right)^{\frac{1+\alpha}{\alpha}} ds \\
&+ \bar{R}_l^{\alpha}(g(l,c)) \int_l^{\infty} \theta(s)\, ds \\
\geq{}& (P-\varepsilon) + (\varsigma - \varepsilon)^{\frac{1+\alpha}{\alpha}} \bar{R}_l^{\alpha}(g(l,c)) \int_l^{\infty} \frac{\alpha R_l(g(s,c))\, g'(s,c)}{\bar{R}_l^{\alpha+1}(g(s,c))} ds \\
\geq{}& (P-\varepsilon) + (\varsigma - \varepsilon)^{\frac{1+\alpha}{\alpha}}.
\end{aligned}$$

Taking the limit inferior on both sides as $l \to \infty$, we get

$$\varsigma \geq (P-\varepsilon) + (\varsigma - \varepsilon)^{\frac{1+\alpha}{\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result

$$P \leq \varsigma - \varsigma^{\frac{1+\alpha}{\alpha}}.$$

Next, we prove the second inequality in part (I). Multiplying (2.10) by $\bar{R}_l^{\alpha+1}(g(l,c))$ and integrating it from l_2 to l, we obtain

$$\int_{l_2}^{l} \bar{R}_l^{\alpha+1}\left(g\left(s,c\right)\right)\omega'\left(s\right)ds \leq -\alpha\int_{l_2}^{l} R_l\left(g\left(s,c\right)\right)g'\left(s,c\right)\left(\bar{R}_l^{\alpha}\left(g\left(s,c\right)\right)\omega\left(s\right)\right)^{\frac{1+\alpha}{\alpha}}ds - \int_{l_2}^{l} \bar{R}_l^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds.$$

Integrating by parts, we have

$$\begin{aligned}\bar{R}_l^{\alpha+1}\left(g\left(l,c\right)\right)\omega\left(l\right) \leq{}& \bar{R}_l^{\alpha+1}\left(g\left(l_2,c\right)\right)\omega\left(l_2\right) - \int_{l_2}^{l} \bar{R}_l^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds \\ &+ \int_{l_2}^{l} R_l\left(g\left(s,c\right)\right)g'\left(s,c\right)\left(\left(\alpha+1\right)X - \alpha X^{\frac{\alpha+1}{\alpha}}\right)ds,\end{aligned}$$

where $X = \bar{R}_l^{\alpha}\left(g\left(s,c\right)\right)\omega\left(s\right)$. Using the inequality

$$Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{\left(\alpha+1\right)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \tag{2.14}$$

for $B \geq 0$, $A \geq 0$ and $u \geq 0$, with $u = X$, $B = (\alpha+1)$ and $A = \alpha$. Thus, we get

$$\begin{aligned}\bar{R}_l^{\alpha+1}\left(g\left(l,c\right)\right)\omega\left(l\right) \leq{}& \bar{R}_l^{\alpha+1}\left(g\left(l_2,c\right)\right)\omega\left(l_2\right) - \int_{l_2}^{l} \bar{R}_l^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds \\ &+ \bar{R}_l\left(g\left(l,c\right)\right) - \bar{R}_l\left(g\left(l_2,c\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}\bar{R}_l^{\alpha}\left(g\left(l,c\right)\right)\omega\left(l\right) \leq{}& 1 + \frac{\bar{R}_l^{\alpha+1}\left(g\left(l_2,c\right)\right)\omega\left(l_2\right)}{\bar{R}_l\left(g\left(l,c\right)\right)} - \frac{\bar{R}_l\left(g\left(l_2,c\right)\right)}{\bar{R}_l\left(g\left(l,c\right)\right)} \\ &- \frac{1}{\bar{R}_l\left(g\left(l,c\right)\right)}\int_{l_2}^{l} \bar{R}_l^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds.\end{aligned}$$

Taking the limit superior on both sides as $l \to \infty$ and using (2.8) we get

$$U \leq 1 - Q.$$

Thus from (2.12), we have

$$P \leq \varsigma - \varsigma^{\frac{\alpha+1}{\alpha}} \leq \varsigma \leq U \leq 1 - Q, \tag{2.15}$$

which completes the proof of part (I).

Part (II). Let $x(l)$ is a positive solution of Eq. (1.1). We shall proof that $z(l)$ does not have property (**i**). On the other contrary, we assume that $P = \infty$. Then from (2.13) we get

$$\bar{R}_l^\alpha (g(l,c))\, \omega(l) \geq \bar{R}_l^\alpha (g(l,c)) \int_l^\infty \theta(s)\, ds.$$

Taking the lim inf of both sides as $l \to \infty$ we get in view of (2.12) that

$$1 \geq \varsigma \geq P \geq \infty.$$

This is a contradiction. Now we admit that $Q = \infty$. Then by (2.15), $U = -\infty$, which contradicts. The proof is complete □

Now we are ready to present the following oscillation criterion

Theorem 2.1. *Assume that* (2.1) *and* (2.8) *hold. If*

$$P = \liminf_{l\to\infty} \bar{R}_{l_0}^\alpha (g(l,c)) \int_l^\infty \theta(s)\, ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{2.16}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to zero as* $l \to \infty$

Proof. Let $x(l)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality we may assume that $x(l) > 0$. If $P = \infty$, then by Lemma 2.4 $z(l)$ does not have property (**i**). That is, $z(l)$ satisfies property (**ii**). Therefore, from Lemma 2.2, we have $\lim_{l\to\infty} x(l) = 0$. Now, let $P < \infty$. By Lemma 2.1, we have that $z(l)$ has the property (**i**) or the property (**ii**). If $z(l)$ has the property (**ii**), from Lemm 2.2, we obtain $\lim_{l\to\infty} x(l) = 0$. Next, we assume that for $z(l)$ property (**i**) holds. Let $\omega(l)$ and ς be defined by (2.6) and (2.7), respectively. Then from Lemma 2.4, we have $P \leq \varsigma - \varsigma^{\frac{1+\alpha}{\alpha}}$. Using inequality (2.14) with $u = \varsigma$ and $B = A = 1$,we get

$$P \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

which contradicts (2.16). The proof is complete. □

Corollary 2.1. *Assume that* (2.1) *and* (2.8) *hold. If*

$$\liminf_{l\to\infty} \bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\int_l^{\infty}\int_c^d q\left(s,\xi\right)d\xi ds \geq \frac{\alpha^{\alpha}}{\left(\alpha+1\right)^{\alpha+1}\left(1-p\right)^{\alpha}}. \tag{2.17}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to zero as* $l\to\infty$.

P r o o f. We shall show that (2.17) implies (2.16). First note that

$$\theta\left(l\right)=\left(1-p\right)^{\alpha}\int_c^d q\left(l,\xi\right)d\xi,$$

this inequality implies that

$$\liminf_{l\to\infty} \bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\int_l^{\infty}\frac{\theta\left(s\right)}{\left(1-p\right)^{\alpha}}ds=\liminf_{l\to\infty} \bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\int_l^{\infty}\int_c^d q\left(s,\xi\right)d\xi ds. \tag{2.18}$$

On the other hand, (2.17) implies that

$$\liminf_{l\to\infty} \bar{R}_{l_0}^{\alpha}\left(g\left(l,c\right)\right)\int_l^{\infty}\int_c^d q\left(s,\xi\right)d\xi ds \geq \frac{\alpha^{\alpha}}{\left(\alpha+1\right)^{\alpha+1}\left(1-p\right)^{\alpha}}. \tag{2.19}$$

Combining (2.18) with (2.19), we get (2.16). □

Theorem 2.2. *Assume that* (2.1) *and* (2.8) *hold. If*

$$P+Q>1, \tag{2.20}$$

then all solution $x\left(l\right)$ *of Eq.* (1.1) *is oscillatory or* $\lim_{l\to\infty} x\left(l\right)=0$.

P r o o f. Let Eq. (1.1) has one nonoscillatory solution $x\left(l\right)$. Then we may assume, without loss of generality that $x\left(l\right)>0$. If $P=\infty$ or $Q=\infty$, then by Lemma 2.4 $z\left(l\right)$ does not have property (**i**). That is mean, $z\left(l\right)$ must has property (**ii**). Then from Lemma 2.2, we get that $\lim_{l\to\infty} x\left(l\right)=0$. Next, let $P<\infty$ and $Q<\infty$. From Lemma 2.1, we have that $z\left(l\right)$ either has case (**i**) or case (**ii**). If $z\left(l\right)$ has case (**ii**), then exactly as above we are led by Lemma 2.2 to $\lim_{l\to\infty} x\left(l\right)=0$. Now, we assume that $z\left(l\right)$ has the case (**i**). Then from Lemma 2.4, we obtain $P+Q\leq 1$ is contradicts (2.20). The Theorem is hold. □

Corollary 2.2. *Assume that* (2.1) *and* (2.8) *hold.If*

$$Q=\limsup_{l\to\infty}\frac{1}{\bar{R}_{l_0}\left(g\left(l,c\right)\right)}\int_{l_0}^{l}\bar{R}_{l_0}^{\alpha+1}\left(g\left(s,c\right)\right)\theta\left(s\right)ds>1, \tag{2.21}$$

then every solution of Eq. (1.1) *is either oscillatory or tends to zero as* $l\to\infty$.

Example 2.1. Consider the equation

$$\left(l\left[z''(l)\right]^3\right)' + \frac{\lambda}{l^6}x^3\left(\frac{l}{2}\right) = 0,\ \lambda > 0, \tag{2.22}$$

where $z(l) = x(l) + \frac{1}{3}x\left(\frac{l}{2}\right)$, $p = \frac{1}{3}$ and $l \geq 1$. According to Corollary 1 in [3], every nonoscillatory solution of Eq. (2.22) converges to zero provided that

$$\lambda > \frac{9^3}{2} = 364.5.$$

Also, by Theorem 2.4 in [5], every nonoscillatory solution of Eq. (2.22) converges to zero provided that

$$\lambda > \left(\frac{10}{3}\right)^3 \frac{4}{e\ln 2} \simeq 78.628.$$

If we choose $a = 0$, $b = 1$, $c = 1$ and $d = 2$.Then (2.1) and (2.8) are satisfied and (2.16) hold for

$$\lambda > \frac{5^4}{8} = 78.125.$$

Hence, by Theorem 2.1 every solution of Eq. (2.22) is either oscillatory or tends to zero if $\lambda > 78.125$. Then, our results supplement and improve the results obtained in [3] and [5].

Example 2.2. Consider the equation

$$\left(\frac{1}{l}\left[z''(l)\right]\right)' + \frac{\upsilon}{l^4}x\left(\frac{l}{2}\right) = 0,\ \upsilon > 0, \tag{2.23}$$

where $z(l) = x(l) + \frac{1}{3}x\left(\frac{l}{2}\right)$, $p = \frac{1}{2}$ and $l \geq 1$. Furthermore, we choose $a = 1$, $b = 2$, $c = 2$ and $d = 3$. We note that $\alpha = 1$,

$$\bar{R}_{l_0}(l) = \frac{1}{2}\left(\frac{1}{3}l^3 - l + \frac{2}{3}l_0\right) \text{ and } \theta(l) = \frac{\upsilon}{2}\frac{1}{l^4}.$$

Hence, it easy to see that (2.1) and (2.8) hold and

$$\lim_{l\to\infty} \inf \bar{R}_{l_0}^{\alpha}(g(l,c)) \int_l^{\infty} \theta(s)\,ds = \frac{\upsilon}{288}.$$

Thus, by Theorem 2.1, if $\upsilon > \frac{3^5}{8}$, we have that every solution of Eq. (2.23) is either oscillatory or tends to zero.

Remark 2.1. In Example 2.2, we note that, $r(l) = 1/l$ and hence $r'(l) < 0$. So, the results of [3] cannot be applied in Eq. (2.23).

Example 2.3. Consider the equation

$$\left(\frac{1}{l}\left[z''(l)\right]^3\right)' + \int_0^1 54ml^{-8}\xi x^3(l-\xi)\,d\xi = 0,\ m > 0, \tag{2.24}$$

where $z(l) = x(l) + \int_{-1}^{0} \frac{1}{2l} e^{\frac{1}{2l}\eta} x(l\eta)\,d\eta$, $p = \frac{2}{3}$ and $l \geq 1$. We note that $\alpha = 3$, $r(l) = \frac{1}{l}$, $p(l,\eta) = \frac{1}{2l}e^{\frac{1}{2l}\eta}$, $\tau(l,\eta) = l\eta$, $q(l,\xi) = 54ml^{-8}\xi$, $g(l,\xi) = l-\xi$, $a = -1$, $b = 0$, $c = 0$, $d = 1$ and

$$\bar{R}_{l_0}(l) = \frac{3}{4}\left(\frac{3}{7}l^{\frac{7}{3}} - l + \frac{4}{7}l_0\right) \text{ and } \theta(l) = ml^{-8}.$$

Hence, it easy to see that (2.1), (2.8) hold and

$$\liminf_{l\to\infty} \bar{R}^{\alpha}_{l_0}(g(l,c)) \int_l^\infty \theta(s)\,ds = \frac{729m}{153664}.$$

Thus, by Theorem 2.1, if $m > \frac{2401}{108}$, we have that every solution of Eq. (2.24) is either oscillatory or tends to zero.

In the next Theorems, we establish new oscillation results for Eq. (1.1) by using the integral averaging technique due to [15]. Following [15], let us introduce now the class of functions X which will be used in these Theorems. Let

$$D = \left\{(l,s) \in \mathbb{R}^2 : l \geq s \geq l_0\right\} \text{ and } D_0 = \left\{(l,s) \in \mathbb{R}^2 : l > s \geq l_0\right\}.$$

The function $H(l,s) \in C(D,\mathbb{R})$ said to belong to the class X (denoted by $H \in X$) if it satisfies

1. $H(l,l) = 0$, $l \geq l_0$, $H(l,s) > 0$, $(l,s) \in D_0$;

2. $\partial H(l,s)/\partial s \leq 0$, there exist $\rho, \delta \in C^1([l_0,\infty),[0,\infty))$, $\rho(l) \neq 0$ and $h(l,s) \in C(D_0,\mathbb{R})$ satisfying

$$-\frac{\partial H(l,s)}{\partial s} = H(l,s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta^{\frac{1}{\alpha}}(s)\right] + h(l,s).$$

Theorem 2.3. *Assume that conditions* (C) *and* (2.1) *are satisfied. If there exists a function* $H \in X$ *such that*

(2.25)
$$\limsup_{l\to\infty} \frac{1}{H(l,l_0)} \int_{l_0}^{l} \left[H(l,s)\,\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)\, r(s)\, |h(l,s)|^{\alpha+1}}{H^{\alpha}(l,s)} \right] ds = \infty$$

and

(2.26)
$$\psi(l) = \rho(l)\,\theta(l) + \rho(l)\, R_{l_0}(g(l,c))\, g'(l,c)\, (r(l)\,\delta(l))^{1+\frac{1}{\alpha}} - \rho(l)\,(r(l)\,\delta(l))',$$

then every solution $x(l)$ *of Eq.* (1.1) *is either oscillatory or satisfies* $\lim_{l\to\infty} x(l) = 0$.

P r o o f. Assume that Eq. (1.1) has a nonoscillatory solution $x(l)$. Without loss of generality, we may assume that $x(l)$ is an eventually positive solution of Eq. (1.1). By Lemma 2.1, we observe that $z(l)$ satisfies either (**i**) or (**ii**) for $l \geq l_1$. We consider each of two cases separately. Suppose first that $z(l)$ has the property (**i**). From Lemma 2.3, we see that (2.2) and (2.3) hold. Now, we define a generalized Riccati transformation $\omega(l)$ by

(2.27)
$$\omega(l) = \rho(l) \left[\frac{r(l)\,[z''(l)]^{\alpha}}{z^{\alpha}(g(l,c))} + r(l)\,\delta(l) \right], \; l \geq l_1.$$

Then we have $\omega(l) > 0$ and

$$\omega'(l) = \rho'(l) \left[\frac{r(l)\,[z''(l)]^{\alpha}}{z^{\alpha}(g(l,c))} + r(l)\,\delta(l) \right] + \rho(l) \left[\frac{r(l)\,[z''(l)]^{\alpha}}{z^{\alpha}(g(l,c))} + r(l)\,\delta(l) \right]',$$

this implies that

$$\omega'(l) = \frac{\rho'(l)}{\rho(l)}\omega(l) + \rho(l)\,(r(l)\,\delta(l))' + \rho(l)\, \frac{(r(l)\,[z''(l)])'}{z^{\alpha}(g(l,c))} - \alpha\rho(l)\, r(l)\, \frac{[z''(l)]^{\alpha}}{z^{\alpha+1}(g(l,c))} z'(g(l,c))\, g'(l,c).$$

By using (2.2) and (2.3) from Lemma 2.3, we get

$$\omega'(l) \leq \frac{\rho'(l)}{\rho(l)}\omega(l) + \rho(l)\,(r(l)\,\delta(l))' - \rho(l)\,(1-p)^{\alpha} \int_c^d q(l,\xi)\, d\xi$$

$$-\alpha\rho(l)\,R_{l_1}(g(l,c))\,g'(l,c)\,r^{\frac{1+\alpha}{\alpha}}(l)\left(\frac{z''(l)}{z(g(l,c))}\right)^{\alpha+1}.$$

By virtue of (2.27), we conclude that

$$\frac{z''(l)}{z(g(l,c))}=\frac{1}{r^{\frac{1}{\alpha}}(l)}\left(\frac{\omega(l)}{\rho(l)}-r(l)\,\delta(l)\right)^{\frac{1}{\alpha}}. \tag{2.28}$$

Which implies that

$$\begin{aligned}\omega'(l) \leq\ & -\alpha\rho(l)\,R_{l_1}(g(l,c))\,g'(l,c)\left(\frac{\omega(l)}{\rho(l)}-r(l)\,\delta(l)\right)^{1+\frac{1}{\alpha}}\\ & +\frac{\rho'(l)}{\rho(l)}\omega(l)+\rho(l)\,(r(l)\,\delta(l))'-\rho(l)\,\theta(l).\end{aligned} \tag{2.29}$$

Define

$$A^*=\frac{\omega(l)}{\rho(l)} \text{ and } B^*=r(l)\,\delta(l).$$

Using the inequality (see [15])

$$(A^*)^{1+\frac{1}{\alpha}}-(A^*-B^*)^{1+\frac{1}{\alpha}}\leq(B^*)^{\frac{1}{\alpha}}\left[\left(1+\frac{1}{\alpha}\right)A^*-\frac{1}{\alpha}B^*\right],$$

$A^*B^*\geq 0$, $\alpha=\dfrac{odd}{odd}\geq 1$, we have

$$\begin{aligned}\left(\frac{\omega(l)}{\rho(l)}-r(l)\,\delta(l)\right)^{1+\frac{1}{\alpha}} \geq\ & \frac{\omega^{1+\frac{1}{\alpha}}(l)}{\rho^{1+\frac{1}{\alpha}}(l)}+\frac{1}{\alpha}(r(l)\,\delta(l))^{1+\frac{1}{\alpha}}\\ & -\left(1+\frac{1}{\alpha}\right)\frac{(r(l)\,\delta(l))^{\frac{1}{\alpha}}}{\rho(l)}\omega(l).\end{aligned} \tag{2.30}$$

Using inequality (2.29) and (2.30) for $l\geq l$, we have

$$\begin{aligned}\omega'(l) \leq\ & \rho(l)\,(r(l)\,\delta(l))'-\rho(l)\,\theta(l)-\rho(l)\,R_{l_1}(g(l,c))\,g'(l,c)\,(r(l)\,\delta(l))^{1+\frac{1}{\alpha}}\\ & +\left[\frac{\rho'(l)}{\rho(l)}+(\alpha+1)\,R_{l1}(g(l,c))\,g'(l,c)\,(r(l)\,\delta(l))^{\frac{1}{\alpha}}\right]\omega(l)\\ & -\frac{\alpha R_{l_1}(g(l,c))\,g'(l,c)}{\rho^{\frac{1}{\alpha}}(l)}\omega^{1+\frac{1}{\alpha}}(l),\end{aligned}$$

this implies that

$$\omega'(l) \leq -\psi(l) + A(l)\omega(l) - G(l)\omega^{1+\frac{1}{\alpha}}(l), \tag{2.31}$$

where $\psi(l)$ is defined as in (2.26),

$$A(l) = \left(\frac{\rho'(l)}{\rho(l)}\right) + (\alpha+1) R_{l_1}(g(l,c)) g'(l,c) (r(l)\delta(l))^{\frac{1}{\alpha}}$$

and

$$G(l) = \frac{\alpha}{\rho^{\frac{1}{\alpha}}(l)} R_{l_1}(g(l,c)) g'(l,c).$$

Multiplying inequality (2.31) by $H(l,s)$ and integrating the resulting inequality from l to l, we have

$$\begin{aligned}
\int_l^l H(l,s)\psi(s)\,ds &\leq \int_l^l H(l,s)\left(-\omega'(s) + A(s)\omega(s) - G(s)\omega^{1+\frac{1}{\alpha}}(s)\right) ds \\
&= H(l,l)\omega(l) + \int_l^l \left(\frac{\partial H(l,s)}{\partial s} + H(l,s)A(s)\right)\omega(s)\,ds \\
&\quad - \int_l^l H(l,s)G(s)\omega^{1+\frac{1}{\alpha}}(s)\,ds \\
&= H(l,l)\omega(l) - \int_l^l h(l,s)\omega(s)\,ds \\
&\quad - \int_l^l H(l,s)G(s)\omega^{1+\frac{1}{\alpha}}(s)\,ds,
\end{aligned}$$

this implies that

$$\begin{aligned}
\int_l^l H(l,s)\psi(s)\,ds &\leq H(l,l)\omega(l) + \int_l^l \left[|h(l,s)|\omega(s)\right. \\
&\quad \left. -H(l,s)G(s)\omega^{1+\frac{1}{\alpha}}(s)\right] ds.
\end{aligned} \tag{2.32}$$

Letting $B = |h(l,s)|, A = H(l,s)G(s), u = \omega(s)$ and using the inequality (2.14)

$$Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha},$$

we obtain

$$\int_l^l H(l,s)\psi(s)\,ds \leq H(l,l)\omega(l) + \int_l^l \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(l,s)|^{\alpha+1}}{H^\alpha(l,s)} ds.$$

Hence

$$\frac{1}{H(l,l)}\int_{l}^{l}\left[H(l,s)\psi(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\rho(s)r(s)|h(l,s)|^{\alpha+1}}{H^{\alpha}(l,s)}\right]ds\le\omega(l), \tag{2.33}$$

for all sufficiently large l, which contradicts (2.25). Now assume that $z(l)$ has the property (**ii**). By Lemma 2.2 we have $\lim_{l\to\infty} x(l)=0$. The proof is complete. □

It may happen that assume (2.25) in Theorem 2.3 fails to hold. Consequently, Theorem 2.3 can not be applied. The following Theorem provides a new oscillation criterion for Eq. (1.1).

Theorem 2.4. *Let conditions* (C) *and* (2.1) *be satisfied. Assume that there exists a function* $H\in X$ *such that*

$$0<\inf_{s\ge l_0}\left\{\lim_{l\to\infty}\inf\frac{H(l,s)}{H(l,l_0)}\right\}\le\infty \tag{2.34}$$

and

$$\lim_{l\to\infty}\sup\frac{1}{H(l,l_0)}\int_{l_0}^{l}\frac{\rho(s)r(s)|h(l,s)|^{\alpha+1}}{H^{\alpha}(l,s)}ds<\infty \tag{2.35}$$

hold. If there exists a function $\Gamma(l)\in C([l_0,\infty),\mathbb{R})$ *such that, for all* $l\ge l_0$,

$$\lim_{l\to\infty}\sup\int_{l_0}^{l}\rho^{-\frac{1}{\alpha}}(s)r^{-\frac{1}{\alpha}}(s)[\Gamma_{+}(s)]^{\frac{\alpha+1}{\alpha}}ds=\infty \tag{2.36}$$

and

$$\lim_{l\to\infty}\sup\frac{1}{H(l,l)}\int_{l}^{l}\left[H(l,s)\psi(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\rho(s)r(s)|h(l,s)|^{\alpha+1}}{H^{\alpha}(l,s)}\right]ds\ge\Gamma(l), \tag{2.37}$$

where $\psi(l)$ *is defined by* (2.26) *and* $\Gamma_{+}(l)=\max\{\Gamma(l),0\}$, *then the conclusion of Theorem* 2.3 *remain intact.*

Proof. The proof of this Theorem is similar to that of Theorem 3.2 in [20]. So it can be omitted. □

Example 2.4. For $l\ge1$, consider the equation

$$\left(\frac{1}{l}\left[z''(l)\right]\right)'+\int_{0}^{1}\frac{4q_0\xi}{l^3}x\left(\frac{l+\xi}{2}\right)d\xi=0,\ q_0>0, \tag{2.38}$$

where $z(l) = x(l) + \int_{\frac{1}{2}}^{1} \frac{4\eta}{3l^2} x\left(\frac{l+\eta}{3}\right) d\eta$. We note that: $\alpha = 1$, $r(l) = \frac{1}{l}$, $p(l,\eta) = \frac{4\eta}{3l^2}$, $\tau(l,\eta) = \left(\frac{l+\eta}{3}\right)$, $q(l,\xi) = \frac{4q_0\xi}{l^3}$, $g(l,\xi) = \left(\frac{l+\xi}{2}\right)$, $a = \frac{1}{2}$, $b = 1$, $c = 0$, $d = 1$.Then

$$\int_a^b p(l,\eta)\, d\eta = \int_{\frac{1}{2}}^{1} \frac{4\eta}{3l^2} d\eta = \frac{1}{2l^2} \leq \frac{1}{2} \text{ and } g(l,c) = g(l,0) = \frac{l}{2}.$$

It is not difficult to verify

$$\int_1^\infty s^{\frac{1}{3}} ds = \infty \textit{ and } \int_1^\infty \int_v^\infty \left[u \int_u^\infty \int_0^1 \frac{4q_0\xi}{s^3} d\xi ds\right] du dv = \infty.$$

Therefore, the condition (C) and (2.1) are satisfied. Furthermore, we choose $p = \frac{1}{2}$, $\rho(l) = l^2$, $\delta(l) = 0$ and $H(l,s) = (l-s)^2$. Then $h(l,s) = 2(l-s)\left(2 - ls^{-1}\right)$, $\psi(l) = \left(1 - \frac{1}{2}\right)^1 \left(l^2\right) \int_0^1 \frac{4q_0}{l^3} \xi d\xi = \frac{q_0}{l}$ and

$$\begin{aligned}
&\limsup_{l\to\infty} \frac{1}{H(l,l_0)} \int_{l_0}^{l} \left[H(l,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s) r(s) |h(l,s)|^{\alpha+1}}{H^\alpha(l,s)}\right] ds \\
= &\limsup_{l\to\infty} \frac{1}{(l-1)^2} \int_1^l \left[q_0 (l-s)^2 \frac{1}{s} - \frac{1}{2^2} \frac{s^2 \frac{1}{s} 2^2 (l-s)^2 \left(2 - ls^{-1}\right)^2}{(l-s)^2}\right] ds \\
= &\limsup_{l\to\infty} \frac{1}{(l-1)^2} \int_1^l \left[(q_0 - 1) l^2 s^{-1} + (4 - 2q_0) l + (q_0 - 4) s\right] ds = \infty,
\end{aligned}$$

if $q_0 > 1$. Hence, by Theorem 2.3 every solution $x(l)$ of Eq. (2.38) is either oscillatory or converges to zero as $l \to \infty$.

Remark 2.2. In Example (2.4), we note that, $r(l) = 1/l$ and hence $r'(l) < 0$. So, the results in [16] and [20] cannot be applied in Eq. (2.38).

REFERENCES

[1] R. P. Agarwal, M. F. Aktas, A. Tiryaki. On oscillation criteria for third-order nonlinear delay differential equations. *Arch. Math. (Brno)* **45**, *1* (2009), 1–18.

[2] R. P. Agarwal, S. R. Grace, D. O'Regan. Oscillation criteria for certain nth order differential equations with devating arguments. *J. Math. Anal. Appl.* **262**, *2* (2001), 601–622.

[3] B. Baculíková, J. Džurina. Oscillation of third-order neutral differential equations. *Math. Comput. Modelling* **52**, *1–2* (2010), 215–226.

[4] B. Baculíková, J. Džurina. Oscillation of third-order nonlinear differential equations. *Appl. Math. Lett.* **24**, *4* (2011), 466–470.

[5] B. Baculíková, J. Džurina. On the asymptotic behavior of class of third-order nonlinear neutral differential equations. *Cent. Eur. J. Math.* **8**, *6* (2010), 1091–1103.

[6] B. Baculíková E. M. Elabbasy, S. H. Saker. Oscillation criteria for third-order nonlinear differential equations. *Math. Slovaca* **58**, *2* (2008), 1–20.

[7] J. Dzurina, R. Kotorova. Comparison theorems for the third-order trinomial differential equations with argument. *Czechoslovak Math. J.* **59**, *2* (2009), 353–370.

[8] E. M. Elabbasy, M. Y. Barsoum, O. Moaaz. Boundedness and oscillation of third-order neutral differential equations with deviating arguments. *J. Appl. Math. Physics* **3**, *11* (2015), 1367–1375.

[9] E. M. Elabbasy, T. S. Hassan, O. Moaaz. Oscillation behavior of second-order nonlinear neutral differential equations with deviating arguments. *Opuscula Math.* **32**, *4* (2012), 719–730.

[10] E. M. Elabbasy and O. Moaaz. On the asymptotic behavior of third-order nonlinear functional differential equations. *Serdica Math. J.* **42**, *2* (2016), 157–174.

[11] E. M. Elabbasy, O. Moaaz. On the oscillation of third-order neutral differential equations. *Asian J. Math. Appl.* **2016** (2016), article ID ama0274, 11 pages.

[12] L. H. Erbe, Q. Kong, B. G. Zhang. Oscillation theory for functional-differential equations. Monographs and Textbooks in Pure and Applied Mathematics, vol. **190**, New York, Marcel Dekker Inc., 1995.

[13] J. R. Graef, S. H. Saker. Oscillation theory of third-order nonlinear functional differential equations. *Hiroshima Math. J.* **43**, *1* (2013), 49–72.

[14] O. Moaaz. Oscillation theorems for cartain second-order differential equations. Saarbrücken, Germany, LAP Lambert Academic Publishing, 2014.

[15] C. G. Philos. Oscillation theorems for linear differential equations of second-order. *Arch. Math. (Basel)* **53**, *5* (1989), 482–492.

[16] G. Qin, Ch. Huang, Y. Xie, F. Wen. Asymptotic behavior for third-order quasi-linear differential equations. *Adv. Difference Equ.* **2013** (2013), arcicle 305, 8 pp.

[17] S. H. Saker. Oscillation criteria of third-order nonlinear delay differential equations. *Math. Slovaca* **56**, *4* (2006), 433–450.

[18] M. Su, Z. Xu. Oscillaion criteria of cetain third-order neutral differential equations. *Differ. Equ. Appl.* **4**, *2* (2012), 221–232.

[19] E. Thandapani, S. Tamilvanan, E. S. Jambulingam. Oscillation of third-order half linear neutral delay differential equations. *Int. J. Pure Appl. Math.* **75**, *4* (2012), 511–520.

[20] Y. Tian, Y. Cai, Y. Fu, T. Li. Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *Adv. Difference Equ.* **2015** (2015), article 267, 14 pp.

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, 35516, Egypt
e-mail: o_moaaz@mans.edu.eg *(O. Moaaz)*
e-mail: emelabbasy@mans.edu.eg *(E. M. Elabbasy)*

E. Shaaban
Department of Mathematics
Faculty of Education
Al Asmarya University
Zliten, Libya
e-mail: ebtesamshaaban@asmarya.edu.ly

Received November 8, 2016