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JORDAN ELEMENTARY MAPS ON ALTERNATIVE DIVISION RINGS

Bruno Leonardo Macedo Ferreira

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Abstract. In this work we prove that if $\mathcal{R}$ and $\mathcal{R}'$ are arbitrary alternative division rings, then under a mild condition every Jordan semi-isomorphism (M, M^*) of $\mathcal{R} \times \mathcal{R}'$ is a isomorphism or an anti-isomorphism.

1. Nonassociative rings and Jordan elementary maps. Let $\mathcal{R}$ be a nonassociative ring. For $x, y, z \in \mathcal{R}$ we denote the *associator* by

$$(x, y, z) = (xy)z - x(yz).$$

A nonassociative ring $\mathcal{R}$ is said to be *alternative* if

$$(x, x, y) = 0 = (y, x, x),$$

for all $x, y \in \mathcal{R}$. Evidently, these identities are satisfied by any associative ring.

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A nonassociative ring $\mathcal{R}$ is said to be *flexible* if

$$(x, y, x) = 0,$$

for all $x, y \in \mathcal{R}$. This identity is satisfied by any alternative ring and so by any associative ring. An alternative ring satisfies the Moufang identities

$$(xax)y = x[a(xy)], \tag{1}$$

$$y(xax) = [(yx)a]x, \tag{2}$$

$$(xy)(ax) = x(ya)x \tag{3}$$

for all $x, y, a \in \mathbb{R}$.

Let $\mathcal{R}$ and $\mathcal{R}'$ be two alternative rings and

$$M : \mathcal{R} \longrightarrow \mathcal{R}' \text{ and } M^* : \mathcal{R}' \longrightarrow \mathcal{R}$$

two maps. Call the ordered pair (M, M^*) a *Jordan elementary map of* $\mathcal{R} \times \mathcal{R}'$ if

$$\begin{cases} M(xM^*(y)x) = M(x)yM(x) \\ M^*(yM(x)y) = M^*(y)xM^*(y), \end{cases}$$

for all $x \in \mathcal{R}$ and $y \in \mathcal{R}'$. We say that the Jordan elementary map (M, M^*) of $\mathcal{R} \times \mathcal{R}'$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective). A map $\phi : \mathcal{R} \longrightarrow \mathcal{R}'$ is called a *Jordan triple map from* $\mathcal{R}$ *onto* $\mathcal{R}'$, if satisfies the condition

$$\phi(xyx) = \phi(x)\phi(y)\phi(x),$$

for all $x, y \in \mathcal{R}$. Of course, for every bijective Jordan triple map ϕ from $\mathcal{R}$ onto $\mathcal{R}'$, the ordered pair (ϕ, ϕ^{-1}) is a Jordan elementary map of $\mathcal{R} \times \mathcal{R}'$.

We say that a Jordan elementary additive map (M, M^*) of $\mathcal{R} \times \mathcal{R}'$ is a *Jordan semi-homomorphism* if it satisfies $M(1) = 1'$ and $M^*(1') = 1$. We say that the Jordan semi-homomorphism (M, M^*) of $\mathcal{R} \times \mathcal{R}'$ is a *homomorphism* or an *anti-homomorphism* (resp., *monomorphism, isomorphism or anti-monomorphism, anti-isomorphism*) if both maps M and M^* are homomorphisms or an anti-homomorphism (resp., monomorphism, isomorphism or anti-monomorphism, anti-isomorphism). A map $\phi : \mathcal{R} \longrightarrow \mathcal{R}'$ is called a *Jordan triple semi-homomorphism from* $\mathcal{R}$ *into* $\mathcal{R}'$, if ϕ is an additive map such that

$$\phi(xyx) = \phi(x)\phi(y)\phi(x),$$

for all $x, y \in \mathcal{R}$, and satisfies the condition $\phi(1) = 1'$. Of course, for every Jordan triple semi-homomorphism ϕ from $\mathcal{R}$ onto $\mathcal{R}'$ the ordered pair (ϕ, ϕ^{-1}) is a Jordan semi-homomorphism of $\mathcal{R} \times \mathcal{R}'$.

2. Jordan additive maps on alternative division rings. According to [1] "Since Hamilton's first example of non-commutative division algebra, the quaternion algebra and division algebra have received a great deal of attention. By comparison, infinite dimensional division algebras and sfields were neglected. Hua came onto the scene around 1950 and proved several theorems in this area by direct and elementary methods. The well known examples of semi-automorphisms are automorphisms, which satisfy $\sigma(ab) = \sigma(a)\sigma(b)$, and anti-automorphisms which satisfy $\sigma(ab) = \sigma(b)\sigma(a)$. An outstanding problem was whether there exists a semi-automorphism which is neither an automorphism nor an anti-automorphism. Hua [2] settled this problem in 1949 by proving that every semi-automorphism is either an automorphism or an anti-automorphism. The fundamental theorem of projective geometry on a line over a sfield of characteristics of 2, namely, any one-to-one mapping carrying the projective line over a sfield of characteristics of 2 onto itself and keeping harmonic relations invariant is a semi-linear transformation induced by an automorphism or an anti-automorphism, was thus derived."

This motivated us to the present paper takes up the special case of an alternative ring.

We investigate the problem of when a Jordan semi-homomorphism on an alternative division ring must be a isomorphism or an anti-isomorphism. We extended Hua's result following:

Theorem ([2])**.** *Let $\mathcal{K}$ be a sfield. Every semi-automorphism of $\mathcal{K}$, that is, a mapping $\sigma : \mathcal{K} \longrightarrow \mathcal{K}$ onto itself satisfying*

$$\sigma(a + b) = \sigma(a) + \sigma(b),$$

$$\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$$

and $\sigma(1) = 1$, is either an automorphism or an anti-automorphism.

The main purpose of this paper is to extend the Hua's result for an alternative division ring. We will prove the following main result:

Theorem 2.1. *Let be $\mathcal{R}$ and $\mathcal{R}'$ arbitrary alternative division rings. Consider a Jordan semi-automorphism (M, M^*) satisfying:*

$$(*) \qquad M\Big((xM^*(y))z + z(M^*(y)x)\Big) = \Big(M(x)y\Big)M(z) + M(z)\Big(yM(x)\Big)$$

and

$$(**) \qquad M^*\Big((yM(x))z + z(M(x)y)\Big) = \Big(M^*(y)x\Big)M^*(z) + M^*(z)\Big(xM^*(y)\Big).$$

Then (M, M^*) *is either an isomorphism or an anti-isomorphism.*

We will prove this Theorem by checking some Lemmas. These Lemmas have the same hypotheses of Theorem 2.1 and them are generalizations of Hua's results for class of alternative division rings.

Lemma 2.1. *The Jordan semi-homomorphism* (M, M^*) *of* $\mathcal{R} \times \mathcal{R}'$ *is injective.*

Proof. First, let us show the injectivity of M. Let $a, b \in \mathcal{R}$ and suppose $M(a) = M(b)$. Hence

$$a = 1a1 = M^*(1'M(a)1') = M^*(1'M(b)1') = 1b1 = b.$$

and so $a = b$. Now, let $x, y \in \mathcal{R}'$ and suppose $M^*(x) = M^*(y)$. Hence

$$x = 1'x1' = M(1M^*(x)1) = M(1M^*(y)1) = 1'y1' = y.$$

and so $x = y$. □

Lemma 2.2. *We have:*

$$M(x^{-1}) = M(x)^{-1} \quad \textit{and} \quad M^*(y^{-1}) = M^*(y)^{-1},$$

for all $x \in \mathcal{R}$ *and* $y \in \mathcal{R}'$.

Proof. For all $x \in \mathcal{R}$,

$$1' = M(1) = M(x1x^{-1}) = M(x)1'M(x^{-1}) = M(x)M(x^{-1}).$$

Analogously we have $1' = M(x^{-1})M(x)$. So $M(x^{-1}) = M(x)^{-1}$. Similarly we get $M^*(y^{-1}) = M^*(y)^{-1}$. □

Lemma 2.3. *We have:*
(i) $M(ab + ba) = M(a)M(b) + M(b)M(a)$;
(ii) $M^*(yz + zy) = M^*(y)M^*(z) + M^*(z)M^*(y)$,
for all $a, b \in \mathcal{R}$ *and* $y, z \in \mathcal{R}'$.

Proof. Replacing x by $a+b$ and $M^*(y)$ by 1 in

$$M(xM^*(y)x) = M(x)yM(x)$$

we have

$$M((a+b)^2) = M((a+b)1(a+b)) = M(a+b)1'M(a+b) = (M(a+b))^2$$

which implies $M(ab+ba) = M(a)M(b)+M(b)M(a)$. Similarly we obtain (ii). □

Lemma 2.4. *We have:*
(i) $M(ab) = M(a)M(b)$ or $M(b)M(a)$;
(ii) $M^*(yz) = M^*(y)M^*(z)$ or $M^*(z)M^*(y)$,
for all $a, b \in \mathcal{R}$ and $y, z \in \mathcal{R}'$.

Proof. We show that $[M(ab) - M(a)M(b)][M(ab) - M(b)M(a)] = 0$. In fact,

$$\begin{aligned} & [M(ab) - M(a)M(b)][M(ab) - M(b)M(a)] \\ = \; & (M(ab))^2 + (M(a)M(b))(M(b)M(a)) \\ - \; & M(ab)(M(b)M(a)) - (M(a)M(b))M(ab) \\ = \; & M((ab)^2) + M(a)(M(b)^2)M(a) \\ - \; & M((ab)(ba) + (ab)(ab)) \\ = \; & M((ab)^2) + M(a)M(b^2)M(a) - M(ab^2a + (ab)^2) \\ = \; & M((ab)^2 + ab^2a - (ab^2a + (ab)^2)) \\ = \; & M(0) \\ = \; & 0, \end{aligned}$$

where we use the condition $(*)$ of the Theorem 2.1 and the identity de Moufang. As $\mathbb{R}$ is an alternative division ring follows that $M(ab) = M(a)M(b)$ or $M(ba) = M(b)M(a)$. Similarly we get (ii). □

Lemma 2.5. *We have:*
(i) if we have a pair of elements $a, b \in \mathcal{R}$ such that

$$M(ab) = M(b)M(a) \neq M(a)M(b),$$

then

$$M(ac) = M(c)M(a),\ \textit{for all } c \in \mathcal{R},$$

and

$$M(db) = M(b)M(d),\ \textit{for all } d \in \mathcal{R};$$

(*ii*) *if we have a pair of elements* $t, u \in \mathcal{R}$ *such that*

$$M^*(tu) = M^*(u)M^*(t) \neq M^*(t)M^*(u),$$

then

$$M^*(tv) = M^*(v)M^*(t),\ \textit{for all } v \in \mathcal{R}'$$

and

$$M^*(wu) = M^*(u)M^*(w),\ \textit{for all } w \in \mathcal{R}'.$$

Proof. The idea of the proof of this Lemma is the same as in [2] but for a better clarity of the article we aim here. Suppose, by Lemma 2.4, that $M(ac) = M(a)M(c) \neq M(c)M(a)$. We get the follows identity

$$M(a)M(c) + M(b)M(a) = M(ac) + M(ab) = M(a(c + b)).$$

Now by Lemma 2.4 and additivity of M we have

$$M(a(c + b)) = M(a)M(c + b) = M(a)M(c) + M(a)M(b)$$

or

$$M(a(c + b)) = M(c + b)M(a) = M(c)M(a) + M(b)M(a),$$

but this implies $M(b)M(a) = M(a)M(b)$ or $M(a)M(c) = M(c)M(a)$ which is a contradiction. Therefore $M(ac) = M(c)M(a)$, for all $c \in \mathbb{R}$. And similarly we prove that, for any d, $M(db) = M(b)M(d)$.

Analogously we prove (*ii*). □

We are ready to prove our main theorem.

Suppose M is not an isomorphism, let show that M is an anti-isomorphism. As M is not an isomorphism there are $a, b \in \mathbb{R}$ such that

$$M(ab) = M(b)M(a) \neq M(a)M(b).$$

We want to show that $M(dc) = M(c)M(d)$ for all $c, d \in \mathbb{R}$. Let us suppose by contradiction that $M(dc) = M(d)M(c) \neq M(c)M(d)$, by Lemma 2.4. By the same argument used in the proof of Lemma 2.5 we get $M(ac) = M(a)M(c)$ and $M(db) = M(d)M(b)$. Now as made in [2] we have the following identity

$$M(b)M(a) + M(ac) + M(db) + M(d)M(c) = M((a+d)(b+c)).$$

But

$$M((a+d)(b+c)) = M(a+d)M(b+c)$$

or

$$M((a+d)(b+c)) = M(b+c)M(a+d),$$

by Lemma 2.4 this contradicts

$$M(b)M(a) \neq M(a)M(b)$$

or

$$M(d)M(c) \neq M(c)M(d),$$

by additivity of M. With a similar reasoning we obtain the same result for M^*. This proves our Theorem 2.1.

It follows as a consequence the Hua's result.

Corollary 2.1. *Let $\mathcal{K}$ be a sfield. Every semi-automorphism of $\mathcal{K}$, that is, a mapping $\sigma : \mathcal{K} \longrightarrow \mathcal{K}$ onto itself satisfying*

$$\sigma(a+b) = \sigma(a) + \sigma(b),$$

$$\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$$

and $\sigma(1) = 1$, is either an automorphism or an anti-automorphism.

Proof. Just observe that σ satisfies the condition $(*)$ and σ^{-1} satisfies the condition $(**)$ of the Theorem 2.1. In fact, linearizing $\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$ we get,

$$\sigma((a\sigma^{-1}(b))c + c(\sigma^{-1}(b)a)) = (\sigma(a)b)\sigma(c) + \sigma(c)(b\sigma(a)).$$

Similarly we show

$$\sigma^{-1}((a\sigma(b))c + c(\sigma(b)a)) = (\sigma^{-1}(a)b)\sigma^{-1}(c) + \sigma^{-1}(c)(b\sigma^{-1}(a)).$$

Therefore, by Theorem 2.1, (σ, σ^{-1}) is either an automorphism or anti-automorphism. $\square$

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Technological Federal University of Paraná
Professora Laura Pacheco Bastos Avenue, 800
85053-510 Guarapuava, Brazil
e-mail: brunoferreira@utfpr.edu.br
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