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# TOPOLOGICAL REPRESENTATION OF PRECONTACT ALGEBRAS AND A CONNECTED VERSION OF THE STONE DUALITY THEOREM – II*

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*Dedicated to the memory of Prof. Stoyan Nedev*

Abstract. The notions of *extensional* (and other kinds) *3-precontact* and *3-contact spaces* are introduced. Using them, new representation theorems for precontact and contact algebras, satisfying some additional axioms, are proved. They incorporate and strengthen both the discrete and topological representation theorems from [11, 6, 7, 12, 22]. It is shown that there are bijective correspondences between such kinds of algebras and such kinds of spaces. In particular, such a bijective correspondence for the RCC systems of [19] is obtained, strengthening in this way the previous representation theorems from [12, 6]. As applications of the obtained results, we prove several Smirnov-type theorems for different kinds of compact semiregular

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$T_0$-extensions of compact Hausdorff extremally disconnected spaces. Also, for every compact Hausdorff space $X$, we construct a compact semiregular $T_0$-extension $(\varkappa X, \varkappa)$ of $X$ which is characterized as the unique, up to equivalence, C-semiregular extension $(cX, c)$ of $X$ such that $c(X)$ is 2-combinatorially embedded in $cX$; moreover, $\varkappa X$ contains as a dense subspace the absolute $EX$ of $X$.

**1. Introduction.** In this paper, as well as in its first part [9], we give the proofs of the results announced in our paper [8] and obtain many new additional results and some new applications (for example, all results from the last section of this paper were not announced in [8]). We present a common approach both to the discrete and to the non-discrete region-based theory of space. The paper is a continuation of the investigations started in [22] and continued in [6, 7, 12].

In the first part [9] of this paper, we extended to precontact algebras the representation techniques developed in [6, 7] and proved that each precontact algebra can be embedded in a special topological object, called *2-precontact space.* In [9] we also established a bijective correspondence between precontact algebras and 2-precontact spaces. Introducing the notion of *2-contact space* as a specialization of that of 2-precontact space, we showed in [9] that there is a bijective correspondence between contact algebras and 2-contact spaces. In this paper, we show that similar representation theorems hold also for precontact and contact algebras satisfying some additional axioms, namely, for *extensional* (resp., *N-regular; regular; normal*) *precontact* and *contact algebras.* The topological objects that correspond to these algebras are introduced here under the names of *extensional* (resp., *N-regular; regular; normal*) *3-precontact* and *3-contact spaces.* It is shown that there are bijective correspondences between such kinds of algebras and such kinds of spaces. In particular, such a bijective correspondence for the RCC systems of [19] is obtained, strengthening in this way the previous representation theorems from [12, 6]. As applications of the obtained results, we prove several Smirnov-type theorems for different kinds of compact semiregular $T_0$-extensions of compact Hausdorff extremally disconnected spaces. Also, for every compact Hausdorff space $X$ we construct a compact semiregular $T_0$-extension $(\varkappa X, \varkappa)$ of $X$ which is characterized as the unique, up to equivalence, C-semiregular extension $(cX, c)$ of $X$ such that $c(X)$ is 2-combinatorially embedded (in the sense of [3]) in $cX$; moreover, $\varkappa X$ contains as a dense subspace the absolute $EX$ of $X$ (see, e.g., [18] for absolutes). Recall that the notion of C-semiregular space was introduced in [6]; every C-semiregular space is a compact semiregular $T_0$-space ([6]).

The paper is organized as follows. In the preliminary Section 2 we recall some basic facts from the first part of this paper; they are indispensable for our further exposition.

In Sections 3 and 4 we introduce the axiom of extensionality for precontact algebras which generalizes the well-known axiom of extensionality for contact algebras. With its help, we introduce the class of extensional precontact algebras which contains as subclasses some well-known systems as, for example, RCC systems from [19]. We modify our representation constructions used in the first part of this paper in order to obtain topological representation theorems for extensional precontact algebras. The notions of extensional 3-precontact space and extensional 3-contact space are introduced here and it is proved that there exists a bijective correspondence between the class of all, up to isomorphism, extensional precontact algebras (resp., extensional contact algebras) and the class of all, up to isomorphism, extensional 3-precontact spaces (resp., extensional 3-contact spaces). This is a generalization of a similar result about complete extensional contact algebras obtained in [6].

In Sections 5 and 6 we introduce the notions of N-regular (resp., regular; normal) precontact algebra and extend the results obtained in the previous two sections to these kinds of precontact and contact algebras.

The technique developed in the Sections 3,4,5,6 permits us to obtain some results about the extensions of topological spaces which were not presented in the paper [8]. As it is well known, the classical Compactification Theorem of Ju. M. Smirnov [20] says that there exists an isomorphism between the ordered set of all, up to equivalence, Hausdorff compactifications of a Tychonoff space $X$, and the ordered set of all proximities on the space $X$. Now, in Section 7, we obtain several Smirnov-type theorems with which we describe different kinds of compact semiregular $T_0$-extensions of compact Hausdorff extremally disconnected spaces. These new theorems are the Theorems 7.7, 7.9, 7.11, 7.13. For proving them we use a new result about the structure of C-semiregular spaces (see Theorem 7.15). We also construct here the extension $(\varkappa X, \varkappa)$ mentioned above (see Corollary 7.16); a similar extension, denoted again by $(\varkappa X, \varkappa)$, is constructed not only for compact Hausdorff spaces but also for C-weakly regular spaces, CN-regular spaces and C-regular spaces (all these notions were introduced in [6]); in these cases the characterization of the extension $(\varkappa X, \varkappa)$ is slightly changed: we now use the notion of "open combinatorial embedding" (in the sense of [17]) instead of the notion of "2-combinatorial embedding" (see Theorem 7.15). Finally, taking a compact Hausdorff extremally disconnected space $Y$, we describe all compact Hausdorff spaces $X$ for which $Y = EX$ (see Proposition 7.14).

Section 7 contains our concluding remarks.

We now fix the notations.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct.

If $(X,\tau)$ is a topological space and $M$ is a subset of $X$, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of $M$ in $(X,\tau)$ and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of $M$ in $(X,\tau)$.

If $X$ is a topological space, we denote by $CO(X)$ the set of all clopen (= closed and open) subsets of $X$. Obviously, $(CO(X),\cup,\cap,\setminus,\emptyset,X)$ is a Boolean algebra. Also, we denote by $RC(X)$ the set of all regular closed subsets of $X$ (recall that a subset $F$ of $X$ is said to be *regular closed* if $F=\mathrm{cl}(\mathrm{int}(F))$).

Recall that a topological space $X$ is said to be: (a) *semiregular* if $RC(X)$ is a closed base for $X$, (b) *connected* if the only clopen subsets of $X$ are $X$ and the empty set, (c) *extremally disconnected* if the closure of every open subset of $X$ is open.

The closed maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is *perfect* if it is compact (i.e. point inverses are compact sets) and closed. A continuous map $f:X\longrightarrow Y$ is *irreducible* if $f(X)=Y$ and for each proper closed subset $A$ of $X$, $f(A)\neq Y$.

As usual, by a *Stone space*, we mean a compact Hausdorff zero-dimensional space.

If $X$ is a topological space, then by a *closed relation on* $X$ we mean a relation on $X$ which is a closed subset of the space $X\times X$.

Recall as well the following definition: if $(A,\leq)$ is a poset and $B\subseteq A$ then $B$ is said to be a *dense subset of* $A$ if for any $a\in A\setminus\{0\}$ there exists $b\in B\setminus\{0\}$ such that $b\leq a$; when $(B,\leq_1)$ is a poset and $f:A\longrightarrow B$ is a map, then we will say that $f$ is a *dense map* if $f(A)$ is a dense subset of $(B,\leq_1)$.

*The set of all ultrafilters of a Boolean algebra $B$ will be denoted by* $\mathrm{Ult}(B)$.

If $X$ is a set, we denote by $2^X$ the power set of $X$.

The main reference book for all undefined here topological notions is [13].

## 2. Preliminaries.

As it was already mentioned in the Introduction, in this section we recall some definitions and results from the first part [9] of this paper; they are indispensable for our further exposition.

**Definition 2.1.** *An algebraic system $\underline{B}=(B,C)$ is called a* precontact algebra ([11]) *(abbreviated as PCA) if the following holds:*

- $B=(B,0,1,+,\cdot,*)$ *is a Boolean algebra (where the complement is denoted by "$*$");*

- *$C$ is a binary relation on $B$ (called a* precontact relation*) satisfying the following axioms:*

(C0) *If $aCb$ then $a \neq 0$ and $b \neq 0$;*

(C+) *$aC(b+c)$ iff $aCb$ or $aCc$; $(a+b)Cc$ iff $aCc$ or $bCc$.*

*A precontact algebra $(B, C)$ is said to be* complete *if the Boolean algebra $B$ is complete. Two precontact algebras $\underline{B} = (B, C)$ and $\underline{B_1} = (B_1, C_1)$ are said to be* PCA-isomorphic *(or, simply,* isomorphic*) if there exists a* PCA-isomorphism *between them, i.e., a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for every $a, b \in B$, $aCb$ iff $\varphi(a)C_1\varphi(b)$.*

*The complement of the relation $C$ is denoted by $(-C)$.*

*For any PCA $(B, C)$, we define a binary relation "$\ll_C$" on $B$ (called* non-tangential inclusion*) by*

$$a \ll_C b \leftrightarrow a(-C)b^*.$$

*Sometimes we will write simply "$\ll$" instead of "$\ll_C$".*

*We will also consider precontact algebras satisfying some additional axioms:*

(Cref) *If $a \neq 0$ then $aCa$ (reflexivity axiom);*

(Csym) *If $aCb$ then $bCa$ (symmetry axiom);*

(Ctr) *If $a \ll_C c$ then $(\exists b)(a \ll_C b \ll_C c)$ (transitivity axiom);*

(Ccon) *If $a \neq 0, 1$ then $aCa^*$ or $a^*Ca$ (connectedness axiom).*

*A precontact algebra $(B, C)$ is called a* contact algebra ([6]) *(and $C$ is called a* contact relation*) if it satisfies the axioms* (Cref) *and* (Csym)*. We say that two contact algebras are* CA-isomorphic *if they are PCA-isomorphic; also, a PCA-isomorphism between two contact algebras will be called a* CA-isomorphism.

*A precontact algebra $(B, C)$ is called* connected *if it satisfies the axiom* (Ccon).

The following lemma says that in every precontact algebra we can define a contact relation.

**Lemma 2.2** ([8])**.** *Let $(B, C)$ be a precontact algebra. Define*

$$aC^{\#}b \iff ((aCb) \vee (bCa) \vee (a \cdot b \neq 0)).$$

*Then $C^{\#}$ is a contact relation on $B$ and hence $(B, C^{\#})$ is a contact algebra.*

**Remark 2.3** ([8]). We will also consider precontact algebras satisfying the following variant of the transitivity axiom (Ctr):

(Ctr#) If $a \ll_{C\#} c$ then $(\exists b)(a \ll_{C\#} b \ll_{C\#} c)$.

The axiom (Ctr#) is known as the "Interpolation axiom".

A contact algebra $(B, C)$ is called a *normal contact algebra* ([4, 14]) if it satisfies the axiom (Ctr#) and the following one:

(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

The notion of a normal contact algebra was introduced by Fedorchuk [14] (under the name of "*Boolean $\delta$-algebra*") as an equivalent expression of the notion of a *compingent Boolean algebra* of de Vries [4] (see its definition below). We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras and naturally arise in normal Hausdorff spaces.

The relations $C$ and $\ll$ are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were introduced (under the name of *compingent Boolean algebras*) by de Vries in [4]) as a pair of a Boolean algebra $B = (B, 0, 1, +, \cdot, ^*)$ and a binary relation $\ll$ on $B$ subject to the following axioms:

($\ll$1) $a \ll b$ implies $a \leq b$;
($\ll$2) $0 \ll 0$;
($\ll$3) $a \leq b \ll c \leq t$ implies $a \ll t$;
($\ll$4) ($a \ll b$ and $a \ll c$) implies $a \ll b \cdot c$;
($\ll$5) If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;
($\ll$6) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;
($\ll$7) $a \ll b$ implies $b^* \ll a^*$.

Note that if $0 \neq 1$ then the axiom ($\ll$2) follows from the axioms ($\ll$3), ($\ll$4), ($\ll$6) and ($\ll$7).

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra $B$ and a binary relation $\ll$ on $B$ subject to the axioms ($\ll$1)-($\ll$4) and ($\ll$7); then, clearly, the relation $\ll$ satisfies also the axioms

($\ll$2') $1 \ll 1$;
($\ll$4') ($a \ll c$ and $b \ll c$) implies $(a + b) \ll c$.

It is not difficult to see that precontact algebras could be equivalently defined as a pair of a Boolean algebra $B$ and a binary relation $\ll$ on $B$ subject to the axioms ($\ll$2), ($\ll$2'), ($\ll$3), ($\ll$4) and ($\ll$4').

It is easy to see that axiom (C6) can be stated equivalently in the form of ($\ll$6).

We will recall that every topological space generates canonically a contact algebra.

Let $X$ be a topological space and let us equip $RC(X)$ with the following Boolean operations and *contact relation* $C_X$ :

- $F + G = F \cup G$;
- $F^* = \mathrm{cl}(X \setminus F)$;
- $F \cdot G = \mathrm{cl}(\mathrm{int}(F \cap G))(= (F^* \cup G^*)^*)$;
- $0 = \emptyset$, $1 = X$;
- $FC_XG$ iff $F \cap G \neq \emptyset$.

It is a well-known fact that if $X$ is a topological space, then

$$(RC(X), C_X) = (RC(X), 0, 1, +, \cdot, *, C_X)$$

is a contact algebra (see [4]).

The contact algebras of the type $(RC(X), C_X)$, where $X$ is a topological space, are called *standard contact algebras*.

We will use the following notation.

**Notation 2.4.** *Let $(X, \mathcal{T})$ be a topological space, $X_0$ be a subspace of $X$, $x \in X$ and $B$ be a subalgebra of the Boolean algebra $(RC(X), +, \cdot, *, \emptyset, X)$. We put*

(1) $$\sigma_x^B = \{F \in B \mid x \in F\}; \quad \Gamma_{x,X_0} = \{F \in CO(X_0) \mid x \in cl_X(F)\}.$$

*When $B = RC(X)$, we will often write simply $\sigma_x$ instead of $\sigma_x^B$; in this case we will sometimes use the notation $\sigma_x^X$ as well.*

Recall that *the Stone space $S(A)$ of a Boolean algebra $A$* is the set $X = \mathrm{Ult}(A)$ endowed with a topology $\mathcal{T}$ having as a closed base the family $\{s_A(a) \mid a \in A\}$, where

(2) $$s_A(a) = \{u \in X \mid a \in u\},$$

for every $a \in A$; then

(3) $$S(A) = (X, \mathcal{T})$$

is a compact Hausdorff zero-dimensional space, $s_A(A) = CO(X)$ and *the Stone map*

(4) $$s_A : A \longrightarrow CO(X), \quad a \mapsto s_A(a),$$

is a Boolean isomorphism; also, the family $\{s_A(a) \mid a \in A\}$ is an open base for $(X, \mathcal{T})$. Further, for every Stone space $X$ and for every $x \in X$, we set

$$u_x = \{P \in CO(X) \mid x \in P\} \tag{5}$$

(sometimes we will also write $u_x^X$ instead of $u_x$). Then $u_x \in \mathrm{Ult}(CO(X))$ and the map

$$f : X \longrightarrow S(CO(X)), \quad x \mapsto u_x,$$

is a homeomorphism.

**Definition 2.5** ([8])**.** Let $\underline{B} = (B, C)$ be a precontact algebra. A non-empty subset $\Gamma$ of $B$ is called a *clan* if it satisfies the following conditions:

(Clan1) $0 \notin \Gamma$;

(Clan2) If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$;

(Clan3) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$;

(Clan4) If $a, b \in \Gamma$ then $aC^{\#}b$.

A clan $\Gamma$ in $\underline{B}$ is called a *maximal clan* in $\underline{B}$ if it is maximal among all clans in $\underline{B}$ with respect to set-inclusion.

The set of all clans (resp., maximal clans) of a precontact algebra $\underline{B}$ is denoted by $\mathrm{Clans}(\underline{B})$ (resp., $\mathrm{MClans}(\underline{B})$).

The following lemma is obvious:

**Lemma 2.6** ([6, 8])**.** *Let $\underline{B} = (B, C)$ be a precontact algebra. Each ultrafilter of $B$ is a clan in $\underline{B}$ and hence* $\mathrm{Ult}(B) \subseteq \mathrm{Clans}(\underline{B})$.

As it was proved in [6], for every topological space $X$ and every $x \in X$, $\sigma_x$ is a clan in $(RC(X), C_X)$. Also, a clan $\sigma$ in $(RC(X), C_X)$ is called a *point-clan* ([6]) if there exists $x \in X$ such that $\sigma = \sigma_x$.

Let us recall that by an *adjacency space* (see [15] and [11]) we mean a relational system $(W, R)$, where $W$ is a non-empty set whose elements are called *cells*, and $R$ is a binary relation on $W$ called the *adjacency relation*; the subsets of $W$ are called *regions*.

The reflexive and symmetric closure $R^{\flat}$ of $R$ is defined as follows:

$$xR^{\flat}y \iff ((xRy) \vee (yRx) \vee (x = y)). \tag{6}$$

A *precontact relation* $C_R$ between the regions of an adjacency space $(W, R)$ is defined as follows: for every $M, N \subseteq W$,

$$MC_RN \text{ iff } (\exists x \in M)(\exists y \in N)(xRy). \tag{7}$$

**Proposition 2.7** ([11]). *Let $(W, R)$ be an adjacency space and let $2^W$ be the Boolean algebra of all subsets of $W$. Then:*

(a) *$(2^W, C_R)$ is a precontact algebra;*

(b) *$(2^W, C_R)$ is a contact algebra iff $R$ is a reflexive and symmetric relation on $W$. If $R$ is a reflexive and symmetric relation on $W$ then $C_R$ coincides with $(C_R)^{\#}$ and $C_{R^\flat}$;*

(c) *$C_R$ satisfies the axiom* (Ctr) *iff $R$ is a transitive relation on $W$;*

(d) *$C_R$ satisfies the axiom* (Ccon) *iff $R$ is a connected relation on $W$ (which means that if $x, y \in W$ and $x \neq y$ then there is an $R$-path from $x$ to $y$ or from $y$ to $x$).*

Clearly, Proposition 2.7(a) implies that if $B$ is a Boolean subalgebra of the Boolean algebra $2^W$, then $(B, C_R)$ is also a precontact algebra (here (and further on), for simplicity, we denote again by $C_R$ the restriction of the relation $C_R$ to $B$).

**Definition 2.8** ([8]). Let $X$ be a non-empty topological space and $R$ be a binary relation on $X$. Then the pair $(CO(X), C_R)$ (see (7) for $C_R$) is a precontact algebra (by Proposition 2.7(a)), called the *canonical precontact algebra of the relational system* $(X, R)$.

**Definition 2.9** ([8]). Let $\underline{B} = (B, C)$ be a precontact algebra and let $U_1, U_2$ be ultrafilters of $B$. We set

(8) $$U_1 R_{\underline{B}} U_2 \text{ iff } (\forall a \in U_1)(\forall b \in U_2)(aCb) \quad (\text{i.e., iff } U_1 \times U_2 \subseteq C).$$

The relational system $(\mathrm{Ult}(B), R_{\underline{B}})$ is called the *canonical adjacency space of* $\underline{B}$.

We say that $U_1, U_2$ are *connected* iff $U_1(R_{\underline{B}})^\flat U_2$ (see (6) for the notation $R^\flat$).

The next lemma is obvious.

**Lemma 2.10** ([8]). *Let $\underline{B} = (B, C)$ be a precontact algebra and let $I$ be a set of connected ultrafilters. Then the union $\Gamma = \bigcup\{U \mid U \in I\}$ is a clan.*

**Lemma 2.11** (Ultrafilter and clan characterizations of precontact and contact relations. [6, 11]). *Let $\underline{B} = (B, C)$ be a precontact algebra and $(\mathrm{Ult}(B), R_{\underline{B}})$ be the canonical adjacency space of $\underline{B}$. Then the following is true for any $a, b \in B$:*

(a) *$aCb$ iff $(\exists U_1, U_2 \in \mathrm{Ult}(B))((a \in U_1) \wedge (b \in U_2) \wedge (U_1 R_{\underline{B}} U_2))$;*

(b) $aC^{\#}b$ *iff* $(\exists U_1, U_2 \in \mathrm{Ult}(B))((a \in U_1) \wedge (b \in U_2) \wedge (U_1 R^{\flat}_{(B,C)} U_2))$;

(c) $aC^{\#}b$ *iff* $(\exists \Gamma \in \mathrm{Clans}(\underline{B}))(a, b \in \Gamma)$ *iff* $(\exists \Gamma \in \mathrm{MClans}(\underline{B}))(a, b \in \Gamma)$;

(d) $R_{\underline{B}}$ *is a reflexive relation iff* $\underline{B}$ *satisfies the axiom* (Cref);

(e) $R_{\underline{B}}$ *is a symmetric relation iff* $\underline{B}$ *satisfies the axiom* (Csym);

(f) $R_{\underline{B}}$ *is a transitive relation iff* $\underline{B}$ *satisfies the axiom* (Ctr).

**Definition 2.12** ([8])**.** An adjacency space $(X, R)$ is called a *topological adjacency space* (abbreviated as TAS) if $X$ is a topological space and $R$ is a closed relation on $X$. When $X$ is a compact Hausdorff zero-dimensional space (i.e., when $X$ is a *Stone space*), we say that the topological adjacency space $(X, R)$ is a *Stone adjacency space*.

Two topological adjacency spaces $(X, R)$ and $(X_1, R_1)$ are said to be *TAS-isomorphic* if there exists a homeomorphism $f : X \longrightarrow X_1$ such that, for every $x, y \in X$, $xRy$ iff $f(x)R_1 f(y)$.

When $\underline{B} = (B, C)$ is a precontact algebra, the pair $(S(B), R_{\underline{B}})$ is said to be *the canonical Stone adjacency space of* $\underline{B}$.

**Theorem 2.13** ([8])**.** (a) *Each PCA* $\underline{B} = (B, C)$ *is isomorphic to the canonical precontact algebra* $(CO(X, \mathcal{T}), C_{R_{\underline{B}}})$ *of the Stone adjacency space* $((X, \mathcal{T}), R_{\underline{B}})$*, where* $(X, \mathcal{T}) = S(B)$ *and for every* $u, v \in X$, $uR_{\underline{B}}v \iff u \times v \subseteq C$*; the isomorphism between them is just the Stone map* $s_B : B \longrightarrow CO(X, \mathcal{T})$*, i.e.*

(9) $s_{(B,C)} : (B, C) \longrightarrow (CO(X, \mathcal{T}), C_{R_{\underline{B}}}),\quad b \mapsto s_B(b),$ *is a PCA-isomorphism.*

*Moreover, the relation $C$ satisfies the axiom (Cref) (resp., (Csym); (Ctr)) iff the relation* $R_{\underline{B}}$ *is reflexive (resp., symmetric; transitive).*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, precontact algebras and the class of all, up to TAS-isomorphism, Stone adjacency spaces* $(X, R)$*; namely, for each precontact algebra* $\underline{B} = (B, C)$*, the PCA-isomorphism class* $[\underline{B}]$ *of* $\underline{B}$ *corresponds to the TAS-isomorphism class of the canonical Stone adjacency space* $(S(B), R_{\underline{B}})$ *of* $\underline{B}$*, and for each Stone adjacency space* $(X, R)$*, the TAS-isomorphism class* $[(X, R)]$ *of* $(X, R)$ *corresponds to the PCA-isomorphism class of the canonical precontact algebra* $(CO(X), C_R)$ *of* $(X, R)$ *(see* (7) *for* $C_R$*).*

**Definition 2.14** ([8])**.** ($a$) *Let $X$ be a topological space and $X_0$ be a dense subspace of $X$. Then the pair $(X, X_0)$ is called a* topological pair.

($b$) *Let $(X, X_0)$ be a topological pair. Then we set*

$$RC(X, X_0) = \{cl_X(A) \mid A \in CO(X_0)\}. \tag{10}$$

**Lemma 2.15** ([8]). *Let $(X, X_0)$ be a topological pair. Then:*

*(a) $RC(X, X_0) \subseteq RC(X)$;*

*(b) The set $RC(X, X_0)$ with the standard Boolean operations on the regular closed subsets of $X$ is a Boolean subalgebra of $RC(X)$;*

*(c) $RC(X, X_0)$ is isomorphic to the Boolean algebra $CO(X_0)$;*

*(d) the sets $RC(X)$ and $RC(X, X_0)$ coincide iff $X_0$ is an extremally disconnected space.*

*(e) If $C_{(X,X_0)}$ is the restriction of the contact relation $C_X$ to $RC(X, X_0)$, then*

$$(RC(X, X_0), C_{(X,X_0)})$$

*is a contact subalgebra of $(RC(X), C_X)$.*

**Definition 2.16** (2-Precontact spaces [8]). *A triple $\underline{X} = (X, X_0, R)$ is called a* 2-precontact space *(abbreviated as PCS) if the following conditions are satisfied:*

(PCS1) $(X, X_0)$ *is a topological pair and $X$ is a $T_0$-space;*

(PCS2) $(X_0, R)$ *is a Stone adjacency space;*

(PCS3) $RC(X, X_0)$ *is a closed base for $X$;*

(PCS4) *For every $F, G \in CO(X_0)$, $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$ implies that $F(C_R)^{\#}G$ (see (7) for $C_R$);*

(PCS5) *If $\Gamma \in \mathrm{Clans}(CO(X_0), C_R)$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$ (see (1) for $\Gamma_{x,X_0}$).*

**Definition 2.17** ([8]). *Let $\underline{X} = (X, X_0, R)$ be a 2-precontact space. Define, for every $F, G \in RC(X, X_0)$,*

$$F \; C_{\underline{X}} \; G \iff ((\exists x \in F \cap X_0)(\exists y \in G \cap X_0)(xRy)).$$

*Then the precontact algebra*

$$\underline{B}(\underline{X}) = (RC(X, X_0), C_{\underline{X}})$$

*is said to be the* canonical precontact algebra of $\underline{X}$.

**Definition 2.18** ([8]). *A 2-precontact space $\underline{X} = (X, X_0, R)$ is called* reflexive *(resp.,* symmetric*;* transitive*) if the relation $R$ is reflexive (resp., symmetric; transitive); $\underline{X}$ is called* connected *if the space $X$ is connected.*

**Definition 2.19** ([8]). *Let* $\underline{X} = (X, X_0, R)$ *and* $\widehat{\underline{X}} = (\widehat{X}, \widehat{X}_0, \widehat{R})$ *be two 2-precontact spaces. We say that* $\underline{X}$ *and* $\widehat{\underline{X}}$ *are* PCS-isomorphic *(or, simply,* isomorphic*) if there exists a homeomorphism* $f : X \longrightarrow \widehat{X}$ *such that:*

(ISO1) $f(X_0) = \widehat{X}_0$*; and*

(ISO2) $(\forall x, y \in X_0)(xRy \leftrightarrow f(x)\widehat{R}f(y))$.

**Proposition 2.20** ([8]). (*a*) *Let* $(X, X_0, R)$ *be a 2-precontact space. Then* $X$ *is a semiregular space and, for every* $F, G \in CO(X_0)$,

$$cl_X(F) \cap cl_X(G) \neq \emptyset \textit{ iff } F(C_R)^{\#}G. \tag{11}$$

(*b*) *Let* $\underline{X} = (X, X_0, R)$ *and* $\widehat{\underline{X}} = (\widehat{X}, \widehat{X}_0, \widehat{R})$ *be two isomorphic 2-precontact spaces. Then the corresponding canonical precontact algebras* $\underline{B}(\underline{X})$ *and* $\underline{B}(\widehat{\underline{X}})$ *are PCA-isomorphic.*

**Definition 2.21** ([8]). *Let* $\underline{B} = (B, C)$ *be a precontact algebra. We associate with* $\underline{B}$ *a 2-precontact space*

$$\underline{X}(\underline{B}) = (X, X_0, R),$$

*called the* canonical 2-precontact space of $\underline{B}$*, as follows:*

- $X = \mathrm{Clans}(\underline{B})$ *and* $X_0 = \mathrm{Ult}(B)$*;*
- *The topology* $\mathcal{T}$ *on the set* $X$ *is defined in the following way: the family*

  $$\{g_B(a) \mid a \in B\},$$

  *where, for any* $a \in B$,

  $$g_B(a) = \{\Gamma \in X \mid a \in \Gamma\}, \tag{12}$$

  *is a closed base for* $\mathcal{T}$*. The topology on* $X_0$ *is the subspace topology induced by* $(X, \mathcal{T})$.
- $R = R_{\underline{B}}$ *(see* (8) *for the notation* $R_{\underline{B}}$*), i.e.* $(X_0, R)$ *is the canonical adjacency space of* $\underline{B}$.

**Remark 2.22** ([9]). Note that, in the notation of Definition 2.21, setting, for every $a \in B$,

$$g_0^B(a) = g_B(a) \cap X_0,$$

we obtain that the family $\{g_0^B(a) \mid a \in B\}$ is a closed base for $X_0$ and

$$g_0^B(a) = s_B(a),$$

where $s_B : B \longrightarrow CO(X_0)$ is the Stone map.

As it is shown in [9], if $\underline{B} = (B, C)$ is a precontact algebra and if $\underline{X}(\underline{B}) = (X, X_0, R)$ is the canonical 2-precontact space of $\underline{B}$ then, for every $a \in B$, we have that

$$g_B(a) = \mathrm{cl}_X(g_0^B(a)); \tag{13}$$

also, (9) implies that

$$s_{(B,C)} : (B, C) \longrightarrow (CO(X_0), C_R), \quad b \mapsto s_B(b), \quad \text{is a PCA-isomorphism.} \tag{14}$$

**Proposition 2.23** ([9])**.** *Let $\underline{B} = (B, C)$ be a precontact algebra. Then the canonical 2-precontact space $\underline{X}(\underline{B}) = (X, X_0, R)$ of $\underline{B}$ defined above is indeed a 2-precontact space.*

**Theorem 2.24** (Representation theorem for precontact algebras)**.** [8], [9, Theorem 6.1]

(a) *Let $\underline{B} = (B, C)$ be a precontact algebra and let $\underline{X}(\underline{B}) = (X, X_0, R)$ be the canonical 2-precontact space of $\underline{B}$. Then the function $g_B : (B, C) \longrightarrow 2^X$, defined in* (12)*, is a PCA-isomorphism between $(B, C)$ and the canonical precontact algebra $(RC(X, X_0), C_{\underline{X}(\underline{B})})$ of $\underline{X}(\underline{B})$. The same function $g_B$ is a PCA-isomorphism between contact algebras $(B, C^{\#})$ and $(RC(X, X_0), C_{(X,X_0)})$ (see Lemma* 2.15*(e) for $C_{(X,X_0)}$). The sets $RC(X)$ and $RC(X, X_0)$ coincide iff the precontact algebra $\underline{B}$ is complete. The algebra $\underline{B}$ satisfies the axiom* (Cref) *(resp., $(Csym)$; $(Ctr)$) iff the 2-precontact space $\underline{X}(\underline{B})$ is reflexive (resp., symmetric; transitive). The algebra $\underline{B}$ is connected iff $\underline{X}(\underline{B})$ is connected.*

(b) *There exists a bijective correspondence $\Phi_2$ between the class of all, up to PCA-isomorphism, (connected) precontact algebras and the class of all, up to PCS-isomorphism, (connected) 2-precontact spaces; namely, for every precontact algebra $\underline{B}$, the PCA-isomorphism class $[\underline{B}]$ of $\underline{B}$ corresponds to the PCS-isomorphism class*

$$\Phi_2([\underline{B}]) = [\underline{X}(\underline{B})]$$

*of the canonical 2-precontact space $\underline{X}(\underline{B})$ of $\underline{B}$. Also, setting $\Psi_2 = (\Phi_2)^{-1}$, we have that for every 2-precontact space $\underline{X}$, the PCS-isomorphism class $[\underline{X}]$ of $\underline{X}$ corresponds to the PCA-isomorphism class*

$$\Psi_2([\underline{X}]) = [\underline{B}(\underline{X})]$$

*of the canonical precontact algebra $\underline{B}(\underline{X})$ of $\underline{X}$.*

**Proposition 2.25** ([8]). *Let $X_0$ be a subspace of a topological space $X$. For every $F, G \in CO(X_0)$, set*

$$F\delta_{(X,X_0)}G \text{ iff } cl_X(F) \cap cl_X(G) \neq \emptyset. \tag{15}$$

*Then $(CO(X_0), \delta_{(X,X_0)})$ is a contact algebra.*

**Definition 2.26** (2-Contact spaces [8]). *A topological pair $(X, X_0)$ is called a* 2-contact space *(abbreviated as CS) if the following conditions are satisfied:*

(CS1) *$X$ is a $T_0$-space;*

(CS2) *$X_0$ is a compact Hausdorff zero-dimensional space;*

(CS3) *$RC(X, X_0)$ is a closed base for $X$;*

(CS4) *If $\Gamma \in \mathrm{Clans}(CO(X_0), \delta_{(X,X_0)})$ (see (15) for the notation $\delta_{(X,X_0)}$) then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$ (see (1) for $\Gamma_{x,X_0}$).*

*A 2-contact space $(X, X_0)$ is called* connected *if the space $X$ is connected.*

**Definition 2.27** ([8]). *Let $(X, X_0)$ be a 2-contact space. Then the contact algebra*

$$\underline{B}^c(X, X_0) = (RC(X, X_0), C_{(X,X_0)})$$

*(see Lemma 2.15(a) for the notation $C_{(X,X_0)}$) is said to be the* canonical contact algebra of the 2-contact space $(X, X_0)$.

**Definition 2.28** ([8]). *Let $\underline{B} = (B, C)$ be a contact algebra, $X = \mathrm{Clans}(B, C)$, $X_0 = \mathrm{Ult}(B)$ and $\mathfrak{T}$ be the topology on $X$ described in Definition 2.21. Take the subspace topology on $X_0$. Then the pair*

$$\underline{X}^c(\underline{B}) = (X, X_0)$$

*is called the* canonical 2-contact space of the contact algebra $(B, C)$.

**Definition 2.29** ([8]). *Let $(X, X_0)$ and $(\widehat{X}, \widehat{X}_0)$ be two 2-contact spaces. We say that $(X, X_0)$ and $(\widehat{X}, \widehat{X}_0)$ are* CS-isomorphic *(or, simply,* isomorphic*) if there exists a homeomorphism $f : X \longrightarrow \widehat{X}$ such that $f(X_0) = \widehat{X}_0$.*

**Lemma 2.30** ([9]). *For every 2-contact space $(X, X_0)$ there exists a unique reflexive and symmetric binary relation $R$ on $X_0$ such that $(X, X_0, R)$ is a 2-precontact space. The relation $R$ is defined by the formula*

$$xRy \iff ((\forall F \in u_x)(\forall G \in u_y)(\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset)), \tag{16}$$

*where* $x, y \in X_0$. *Also, for every* $F, G \in CO(X_0)$,

$$\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset \iff FC_R G. \tag{17}$$

**Theorem 2.31** (New representation theorem for CAs [8]).

(a) *Let* $(B, C)$ *be a contact algebra and let* $(X, X_0)$ *be the canonical 2-contact space of* $(B, C)$ *(see Definition* 2.28*). Then the function* $g_B : B \longrightarrow 2^X$*, defined in* (12)*, is a CA-isomorphism between the algebra* $(B, C)$ *and the canonical contact algebra* $(RC(X, X_0), C_{(X,X_0)})$ *of* $(X, X_0)$*. The sets* $RC(X, X_0)$ *and* $RC(X)$ *coincide iff the contact algebra* $(B, C)$ *is complete. The contact algebra* $(B, C)$ *is connected iff the 2-contact space* $(X, X_0)$ *is connected.*

(b) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) contact algebras and the class of all, up to CS-isomorphism, (connected) 2-contact spaces; namely, for every CA* $\underline{B}$*, the CA-isomorphism class* $[\underline{B}]$ *of* $\underline{B}$ *corresponds to the CS-isomorphism class* $[\underline{X}^c(\underline{B})]$ *of the canonical 2-contact space* $\underline{X}^c(\underline{B})$ *of* $\underline{B}$*, and for every 2-contact space* $(X, X_0)$*, the CS-isomorphism class* $[(X, X_0)]$ *of* $(X, X_0)$ *corresponds to the CA-isomorphism class* $[\underline{B}^c(X, X_0)]$ *of the canonical contact algebra* $\underline{B}^c(X, X_0)$ *of* $(X, X_0)$*.*

**Definition 2.32** ([6]). *A semiregular* $T_0$*-space* $(X, \mathcal{T})$ *is said to be* C-semiregular *if for every clan* $\Gamma$ *in* $(RC(X), C_X)$ *there exists a point* $x \in X$ *such that* $\Gamma = \sigma_x$ *(see* (1) *for* $\sigma_x$*), i.e., if every clan in* $(RC(X), C_X)$ *is a point-clan.*

The next assertion was stated in [6] but it was left without proof there. It was proved in [9].

**Proposition 2.33** ([6, Fact 4.1]). *Every C-semiregular space* $X$ *is a compact space.*

**Lemma 2.34** ([9]). *If* $(X, X_0)$ *is a 2-contact space and* $X_0$ *is extremally disconnected, then* $X$ *is C-semiregular.*

**3. Precontact algebras with the axiom of extensionality.** Let us start with recalling the following well known statement (see, e.g., [2, p. 271]).

**Lemma 3.1.** *Let* $X$ *be a dense subspace of a topological space* $Y$*. Then the functions*

$$r_{X,Y} : RC(Y) \longrightarrow RC(X),\ F \mapsto F \cap X,$$

*and*

$$e_{X,Y} : RC(X) \longrightarrow RC(Y),\ G \mapsto \mathrm{cl}_Y(G),$$

*are Boolean isomorphisms between Boolean algebras $RC(X)$ and $RC(Y)$, and $e_{X,Y} \circ r_{X,Y} = id_{RC(Y)}$, $r_{X,Y} \circ e_{X,Y} = id_{RC(X)}$. (If $X$ and $Y$ are clear from the context, we will sometimes write $r$ (resp., $e$) instead of $r_{X,Y}$ (resp., $e_{X,Y}$).)*

**Definition 3.2.** (*a*) *Let $X$ be a topological space and $X_0, X_1$ be dense subspaces of $X$. Then the triple $(X, X_0, X_1)$ is called a* topological triple.

(*b*) *Let $(X, X_0, X_1)$ be a topological triple. Then we set*

$$RC(X, X_0, X_1) = \{X_1 \cap cl_X(A) \mid A \in CO(X_0)\}. \tag{18}$$

**Lemma 3.3.** *Let $(X, X_0, X_1)$ be a topological triple. Then:*

(*a*) *$RC(X, X_0, X_1) \subseteq RC(X_1)$;*

(*b*) *the set $RC(X, X_0, X_1)$ with the standard Boolean operations on the regular closed subsets of $X_1$ is a Boolean subalgebra of the Boolean algebra $RC(X_1)$;*

(*c*) *the Boolean algebra $CO(X_0)$ is isomorphic to $RC(X, X_0, X_1)$;*

(*d*) *the sets $RC(X, X_0, X_1)$ and $RC(X_1)$ coincide iff $X_0$ is an extremally disconnected space.*

(*e*) *Let us denote by*

$$C_{(X,X_0,X_1)}$$

*the restriction of the contact relation $C_{X_1}$ to $RC(X, X_0, X_1)$. Then*

$$(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$$

*is a contact subalgebra of $(RC(X_1), C_{X_1})$.*

P r o o f. Let us denote by $\mathfrak{T}$ the topology of the space $X$. Using Lemma 3.1 and its notation, we obtain that

$$r_{X_1,X}(RC(X, X_0)) = RC(X, X_0, X_1)$$

and $(r_{X_1,X})|_{RC(X,X_0)} : RC(X, X_0) \longrightarrow RC(X, X_0, X_1)$ is a Boolean isomorphism. The first three assertions follow from this fact. Also, it is easy to see that the sets $RC(X_1)$ and $RC(X, X_0, X_1)$ coincide iff $X_0$ is an extremally disconnected space.

Now, it becomes obvious that $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$ is a contact subalgebra of $(RC(X_1), C_{X_1})$. $\square$

The axiom of extensionality is one of the most interesting axioms for contact algebras. Since $C^{\#}$ is a contact relation for every precontact relation $C$, we formulate it for the relation $C^{\#}$:

$(\forall c)(aC^{\#}c \leftrightarrow bC^{\#}c) \rightarrow (a = b)$.

The axiom of extensionality has several equivalent formulations (see, e.g., [6]); the shortest one is the following: $(\forall a \neq 1)(\exists b \neq 0)(a(-C^{\#})b)$, or, equivalently,

(Cext) $(\forall a \neq 1)(\exists b \neq 0)(a(-C)b$ and $b(-C)a$ and $a \cdot b = 0)$.

Note that the class of extensional precontact algebras contains some well-known systems as, for example, RCC system from [19].

In this section we will modify the representation theory for precontact algebras developed in [8, 9] in order to obtain a similar theory for extensional precontact algebras.

**Definition 3.4.** *A precontact algebra $(B, C)$ which satisfies the axiom* (Cext) *is said to be an* extensional precontact algebra *(abbreviated as EPA). The abbreviation for "extensional contact algebra" will be "ECA".*

The following notion was introduced by A. V. Arhangel'skiĭ (and, independently, in [5]): a topological space $(X, \tau)$ is said to be *$\pi$-regular* if for each non-empty $U \in \tau$ there exists a non-empty $V \in \tau$ such that $\mathrm{cl}(V) \subseteq U$. The semiregular $\pi$-regular spaces are called *weakly regular* ([12]).

**Proposition 3.5** ([12]). *If $X$ is a weakly regular space, then $(RC(X), C_X)$ is an ECA.*

The next representation theorem for extensional contact algebras was proved by I. Düntsch and M. Winter [12] (see also [6, Theorem 5.1(a)]):

**Theorem 3.6** ([12]). *For each extensional contact algebra $\underline{B} = (B, C)$ there exists a dense embedding $g_B$ of $B$ into a standard extensional contact algebra $(RC(X, \tau), C_X)$, where $(X, \tau)$ is a compact weakly regular $T_1$-space. The algebra $B$ is connected iff the space $X$ is connected. When $B$ is complete then the embedding $g_B$ becomes an isomorphism between contact algebras $(B, C)$ and $(RC(X), C_X)$.*

Now we are going to strengthen Theorem 3.6 in two directions. The first one is that we will prove a topological representation theorem not only for ECAs but also for extensional precontact algebras. The second one is that in both cases – for ECAs and for EPAs – we will not only find a topological object in whose canonical algebra our algebras can be densely embedded (as it is for ECAs in Theorem 3.6) but we will find topological objects which are in bijective correspondence (up to isomorphisms) with our algebras. Such topological representation theorems for PCAs and CAs were proved in [9]. There the corresponding topological objects were 2-precontact spaces and 2-contact spaces. Since the class of EPAs (resp., ECAs) is a proper subclass of the class of PCAs (resp., CAs), we have to find some subclasses of the classes of 2-precontact spaces and 2-contact spaces in order to obtain a bijective correspondence between EPAs (resp., ECAs) and the corresponding new topological objects. In this section we will fulfill this plan for EPAs, and in the next one - for ECAs. Let us start with the definition of 3-precontact spaces. They will be our topological objects which will be in a bijective correspondence with EPAs.

**Definition 3.7.** *A quadruple* $\underline{X} = (X, X_0, X_1, R)$ *is called an* extensional 3-precontact space *(abbreviated as EPS) if it satisfies the following conditions:*

(EPS1) $(X, X_0, R)$ *is a 2-precontact space;*

(EPS2) $X_1$ *is a dense subspace of* $X$*;*

(EPS3) $X_1$ *is a weakly regular* $T_1$*-space;*

(EPS4) *If* $x \in X_1$ *then the set* $\Gamma_{x,X_0}$ *(see* (1) *for this notation) is a maximal clan in the precontact algebra* $(CO(X_0), C_R)$ *(see* (7) *for the notation* $C_R$*); conversely, for every maximal clan* $\Gamma$ *in* $(CO(X_0), C_R)$ *there exists a point* $x \in X_1$ *such that* $\Gamma = \Gamma_{x,X_0}$.

**Definition 3.8.** *Let* $\underline{X} = (X, X_0, X_1, R)$ *be an extensional 3-precontact space. Define, for every* $F, G \in RC(X, X_0, X_1)$,

$$FC_{\underline{X}}G \text{ iff there exist } x \in cl_X(F) \cap X_0 \text{ and } y \in cl_X(G) \cap X_0 \text{ such that } xRy.$$

*Then the precontact algebra*

$$\underline{B}(\underline{X}) = (RC(X, X_0, X_1), C_{\underline{X}})$$

*is called* the canonical extensional precontact algebra of the extensional 3-precontact space $\underline{X}$.

**Definition 3.9.** *An extensional 3-precontact space* $\underline{X} = (X, X_0, X_1, R)$ *is called* reflexive *(resp.,* symmetric*;* transitive*) if the relation* $R$ *is reflexive (resp., symmetric; transitive);* $\underline{X}$ *is called* connected *if the space* $X_1$ *is connected.*

**Definition 3.10.** *Let* $\underline{X} = (X, X_0, X_1, R)$ *and* $\underline{\widehat{X}} = (\widehat{X}, \widehat{X}_0, \widehat{X}_1, \widehat{R})$ *be two extensional 3-precontact spaces. We say that* $\underline{X}$ *and* $\underline{\widehat{X}}$ *are* EPS-isomorphic *(or, simply,* isomorphic*) if there exists a homeomorphism* $f : X \longrightarrow \widehat{X}$ *such that:*

(ISOE1) $f(X_0) = \widehat{X}_0$;

(ISOE2) $f(X_1) = \widehat{X}_1$; *and*

(ISOE3) $(\forall x, y \in X_0)(xRy \leftrightarrow f(x)\widehat{R}f(y))$.

**Remark 3.11.** It is easy to see that the canonical extensional precontact algebra of a 3-precontact space, defined in Definition 3.8, is indeed a precontact algebra. The fact that it is an extensional precontact algebra will be established below in Corollary 3.15.

**Proposition 3.12.** *Let $(X, X_0, X_1, R)$ be an extensional 3-precontact space. Then, for every $F, G \in CO(X_0)$, we have that*

$$cl_X(F) \cap cl_X(G) \neq \emptyset \iff cl_X(F) \cap cl_X(G) \cap X_1 \neq \emptyset. \tag{19}$$

P r o o f. Let $F, G \in CO(X_0)$ and $cl_X(F) \cap cl_X(G) \neq \emptyset$. Then, by the axioms (EPS1) and (PCS4), $F(C_R)^{\#}G$. Hence, by Proposition 2.11(c), there exists a maximal clan $\Gamma$ in $(CO(X_0), C_R)$ such that $F, G \in \Gamma$. The axiom (EPS4) implies that there exists $x \in X_1$ such that $\Gamma = \Gamma_{x,X_0}$. Thus $x \in cl_X(F) \cap cl_X(G)$. Therefore, $cl_X(F) \cap cl_X(G) \cap X_1 \neq \emptyset$. The converse implication is obvious. □

We will need the following well-known fact:

**Fact 3.13.** *Let $U$ be an open subset of a topological space $X$ and $A$ be a dense subset of $X$. Then $\mathrm{cl}_X(U) = \mathrm{cl}_X(A \cap \mathrm{cl}_X(U))$.*

**Proposition 3.14.** *Let $\underline{X} = (X, X_0, X_1, R)$ be an extensional 3-precontact space. Then, for every $F, G \in RC(X, X_0, X_1)$, we have that $F(C_{\underline{X}})^{\#}G \iff F \cap G \neq \emptyset \iff FC_{X_1}G$; if $F = X_1 \cap \mathrm{cl}_X(A_1)$ and $G = X_1 \cap \mathrm{cl}_X(A_2)$, where $A_1, A_2 \in CO(X_0)$, then $FC_{\underline{X}}G \iff A_1C_RA_2$.*

P r o o f. Let $F, G \in RC(X, X_0, X_1)$, $F = X_1 \cap \mathrm{cl}_X(A_1)$ and $G = X_1 \cap \mathrm{cl}_X(A_2)$, where $A_1, A_2 \in CO(X_0)$. Then, using Fact 3.13 and Lemma 3.1, we obtain that $X_0 \cap \mathrm{cl}_X(F) = X_0 \cap \mathrm{cl}_X(X_1 \cap \mathrm{cl}_X(A_1)) = X_0 \cap \mathrm{cl}_X(A_1) = A_1$ and, analogously, $X_0 \cap \mathrm{cl}_X(G) = A_2$. This implies that $FC_{\underline{X}}G \iff A_1C_RA_2$. Further, by Lemma 3.3(c), we have that the function

$$r_1 : CO(X_0) \longrightarrow RC(X, X_0, X_1),\ A \mapsto X_1 \cap \mathrm{cl}_X(A),$$

is a Boolean isomorphism. Therefore, for every $A', A'' \in CO(X_0)$, we have that

$$A' \cdot A'' \neq \emptyset \iff r_1(A') \cdot r_1(A'') \neq \emptyset.$$

Thus we obtain that

$$F(C_{\underline{X}})^{\#}G \iff A_1(C_R)^{\#}A_2. \tag{20}$$

Now, Proposition 2.20 and Proposition 3.12 imply that $F(C_{\underline{X}})^{\#}G \iff A_1(C_R)^{\#}A_2 \iff \mathrm{cl}_X(A_1)\cap\mathrm{cl}_X(A_2)\neq\emptyset \iff X_1\cap\mathrm{cl}_X(A_1)\cap\mathrm{cl}_X(A_2)\neq\emptyset \iff F\cap G\neq\emptyset \iff FC_{X_1}G$. □

**Corollary 3.15.** *The canonical extensional precontact algebra of an extensional 3-precontact space $\underline{X}=(X,X_0,X_1,R)$, defined in Definition 3.8, is indeed an extensional precontact algebra. Also, the Boolean algebra $RC(X,X_0,X_1)$ is a dense subset of the Boolean algebra $RC(X_1)$.*

Proof. In Remark 3.11 we have already noted that the canonical extensional precontact algebra $\underline{B}(\underline{X})$ of $\underline{X}$ is a precontact algebra. We will now show that it is an extensional precontact algebra. We only need to check that it satisfies axiom (Cext). Let $F\in RC(X,X_0,X_1)$ and $F\neq X_1$. Then $X_1\setminus F$ is a non-empty open subset of $X_1$. Since $X_1$ is a weakly regular space, there exists $F_1\in RC(X_1)$ such that $F_1\neq\emptyset$ and $F_1\subseteq X_1\setminus F$. Then the set $A_1=\mathrm{cl}_X(F_1)\cap X_0$ is a non-empty regular closed subset of $X_0$ and $\mathrm{cl}_X(A_1)=\mathrm{cl}_X(F_1)$ (see Fact 3.13). Hence there exists a non-empty clopen subset $A_2$ of $X_0$ such that $A_2\subseteq A_1$. Set $G=\mathrm{cl}_X(A_2)\cap X_1$. Then $G$ is a non-empty subset of $F_1$ and $G\in RC(X,X_0,X_1)$. (Note that in such a way we have shown that $RC(X,X_0,X_1)$ is a dense subset of $RC(X_1)$.) Clearly, $F\cap G=\emptyset$. Thus, by Proposition 3.14, we obtain that $F(-((C_{\underline{X}})^{\#}))G$. Therefore $\underline{B}(\underline{X})$ is an extensional precontact algebra. □

We will now associate with each extensional precontact algebra an extensional 3-precontact space.

**Definition 3.16.** *Let $\underline{B}=(B,C)$ be an extensional precontact algebra. We associate with $\underline{B}$ an extensional 3-precontact space*

$$\underline{X}(\underline{B})=(X,X_0,X_1,R)$$

*called the* canonical extensional 3-precontact space of $\underline{B}$, *where $(X,X_0,R)$ is the canonical 2-precontact space of the precontact algebra $(B,C)$ (see Definition 2.21), $X_1=\mathrm{MClans}(B,C)$ (thus $X_1\subseteq X$) and on $X_1$ the subspace topology is taken.*

We will need the following assertion which follows immediately from [6, Proposition 3.5]:

**Proposition 3.17.** *Let $\underline{B}=(B,C)$ be an extensional precontact algebra and $a\in B\setminus\{1\}$. Then there exists a maximal clan in $\underline{B}$ which does not contain $a$.*

**Proposition 3.18.** *Let $\underline{B}=(B,C)$ be an extensional precontact algebra. Then the canonical extensional 3-precontact space $\underline{X}(\underline{B})=(X,X_0,X_1,R)$ of $\underline{B}$ is indeed an extensional 3-precontact space.*

Proof. By Proposition 2.23, $(X, X_0, R)$ is a 2-precontact space. Hence, the axiom (EPS1) is satisfied. Further, by Definition 2.21, the family $\{g_B(a) \mid a \in B\}$, where $g_B(a) = \{\Gamma \in X \mid a \in \Gamma\}$, is a closed base for a topology $\tau$ on $X$. Hence the family $\{h_B(a) \mid a \in B\}$, where $h_B(a) = X \setminus g_B(a) = \{\Gamma \in X \mid a \notin \Gamma\}$, is an open base for $X$. Let $a \in B \setminus \{1\}$. Then, by Proposition 3.17, there exists $x \in X_1$ such that $a \notin x$. We obtain that $x \in h_B(a) \cap X_1$. Therefore, $X_1$ is a dense subspace of $X$. Hence, the axiom (EPS2) is satisfied.

We are now going to check the axiom (EPS3). Set, for every $b \in B$,

$$g_1^B(b) = g_B(b) \cap X_1. \tag{21}$$

Then $\{g_1^B(a) \mid a \in B\}$ is a closed base for the topology of $X_1$ and, for every $a \in B$, $g_1^B(a) = \{\Gamma \in \mathrm{MClans}(B, C^{\#}) \mid a \in \Gamma\}$. Note that, according to Definition 3.4, the fact that $\underline{B}$ is an extensional precontact algebra implies that the contact algebra $(B, C^{\#})$ is extensional. Hence, by the Representation Theorem of I. Düntsch and M. Winter [12] (= Theorem 3.6 here) (see also [6, Theorem 5.1(i) and Remark 5.2]) and its proof, the space $X_1$ is a (compact) weakly regular $T_1$-space. Therefore, the axiom (EPS3) is satisfied.

Finally, we will check the axiom (EPS4). Recall that for every $a \in B$, $g_0^B(a) = s_B(a)$ (see Definition 2.21). Let $x \in X_1$, i.e. $x \in \mathrm{MClans}(B, C)$. By Definition 2.21, we have that $R = R_{\underline{B}}$; hence, by Theorem 2.13 and Definition 2.21, $s_B(x) = \{s_B(a) \mid a \in x\}$ is a maximal clan in the precontact algebra $(CO(X_0), C_R)$ (see (14)). We will show that $s_B(x) = \Gamma_{x,X_0}$. Indeed, using (13), we obtain that $s_B(x) = \{s_B(a) \mid a \in x\} = \{s_B(a) \mid a \in B, x \in g_B(a)\} = \{s_B(a) \mid a \in B, x \in \mathrm{cl}_X(s_B(a))\} = \{F \in CO(X_0) \mid x \in \mathrm{cl}_X(F)\} = \Gamma_{x,X_0}$. Hence $\Gamma_{x,X_0}$ is a maximal clan in the precontact algebra $(CO(X_0), C_R)$. So, the first part of the axiom (EPS4) is established.

Let now $\Gamma$ be a maximal clan in the precontact algebra $(CO(X_0), C_R)$. Set $x = (s_B)^{-1}(\Gamma)$, where $s_B : B \longrightarrow CO(X_0)$ is the Stone map. Since, by Theorem 2.13, the same map $s_B$ is an isomorphism between precontact algebras $(B, C)$ and $(CO(X_0), C_R)$, we obtain that $x \in \mathrm{MClans}(B, C)$. Thus $x \in X_1$ and $s_B(x) = \Gamma$. Exactly as above, we obtain that $s_B(x) = \Gamma_{x,X_0}$. Therefore, $\Gamma = \Gamma_{x,X_0}$. Hence, the second part of the axiom (EPS4) is also established.

All this shows that $(X, X_0, X_1, R)$ is an extensional 3-precontact space. □

**Theorem 3.19** (Representation theorem for extensional precontact algebras)**.**

(a) *Let $\underline{B} = (B, C)$ be an extensional precontact algebra and let*

$$\underline{X}(\underline{B}) = (X, X_0, X_1, R)$$

*be the canonical extensional 3-precontact space of* $\underline{B}$*. Then the function* $g_1^B$*, defined in* (21)*, is a PCA-isomorphism between* $(B,C)$ *and the canonical extensional precontact algebra* $(RC(X,X_0,X_1),C_{\underline{X}(\underline{B})})$ *of* $\underline{X}(\underline{B})$*. The same function* $g_1^B$ *is a CA-isomorphism between the CAs* $(RC(X,X_0,X_1),$ $C_{(X,X_0,X_1)})$ *(see Lemma* 3.3 *for the notation* $C_{(X,X_0,X_1)}$*) and* $(B,C^{\#})$*. The algebra* $\underline{B}$ *is complete iff the sets* $RC(X,X_0,X_1)$ *and* $RC(X_1)$ *coincide iff the space* $X_0$ *is extremally disconnected. The algebra* $\underline{B}$ *satisfies the axiom* (Cref) *(resp.,* $(Csym);(Ctr)$*) iff the extensional 3-precontact space* $\underline{X}(\underline{B})$ *is reflexive (resp., symmetric; transitive). The algebra* $\underline{B}$ *is connected iff* $\underline{X}(\underline{B})$ *is connected.*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, (connected) extensional precontact algebras and the class of all, up to EPS-isomorphism, (connected) extensional 3-precontact spaces.*

Proof. (a) By Proposition 3.18, $\underline{X}(\underline{B})=(X,X_0,X_1,R)$ is an extensional 3-precontact space. By [6, Lemma 5.7(iii1)], the function $g_1^B : B \longrightarrow RC(X_1)$ is an injection and, for any $a,b\in B$, $aC^{\#}b$ iff $g_1^B(a)C_{X_1}g_1^B(b)$. Then [6, Lemma 5.3(vi)] implies that $g_1^B : B \longrightarrow RC(X_1)$ is a dense Boolean embedding; moreover, $g_1^B : B \longrightarrow RC(X_1)$ becomes a Boolean isomorphism when $B$ is complete. Hence, $g_1^B : (B,C^{\#}) \longrightarrow (RC(X_1),C_{X_1})$ is a dense CA-embedding which becomes a CA-isomorphism when $B$ is complete. Using (13), we obtain that for every $a\in B$, $g_1^B(a) = X_1\cap g_B(a) = X_1\cap \mathrm{cl}_X(g_0^B(a)) = X_1\cap \mathrm{cl}_X(X_0\cap g_B(a)) = X_1\cap \mathrm{cl}_X(s_B(a))$ (see (2) and Definition 2.21 for the notation $s_B$ and $g_0^B$). Hence, $g_1^B(B)=RC(X,X_0,X_1)$. Therefore,

$$g_1^B : (B,C^{\#}) \longrightarrow (RC(X,X_0,X_1),C_{(X,X_0,X_1)})$$

is a (P)CA-isomorphism and, using Lemma 3.3, we obtain that the algebra $\underline{B}$ is complete iff the sets $RC(X,X_0,X_1)$ and $RC(X_1)$ coincide. Also, (14) and Proposition 3.14 imply that

$$g_1^B : (B,C) \longrightarrow (RC(X,X_0,X_1),C_{\underline{X}(\underline{B})}) \quad \text{is a PCA-isomorphism.} \tag{22}$$

Indeed, it is clear that $g_1^B = r_1\circ s_{(B,C)}$, where

$$s_{(B,C)} : (B,C) \longrightarrow (CO(X_0),C_R), \quad a\mapsto s_B(a),$$

and

$$r_1 : (CO(X_0),C_R) \longrightarrow (RC(X,X_0,X_1),C_{\underline{X}(\underline{B})}), \quad A\mapsto X_1\cap \mathrm{cl}_X(A); \tag{23}$$

now we use the facts that, by (14), $s_{(B,C)}$ is a PCA-isomorphism, and by Proposition 3.14 and Lemma 3.3(c), $r_1$ is a PCA-isomorphism.

Further, the assertion about connectivity follows from [6, Lemma 5.7(iii3)] (or Theorem 3.6 here) and all the rest of the assertions in (a) follows from Theorem 2.24(a).

(b) Let us denote by $\mathcal{EPA}$ the set of all, up to PCA-isomorphism, extensional precontact algebras and by $\mathcal{EPS}$ the set of all, up to EPS-isomorphism, extensional 3-precontact spaces. We will define two correspondences

$$\Phi_3 : \mathcal{EPA} \longrightarrow \mathcal{EPS} \quad \text{and} \quad \Psi_3 : \mathcal{EPS} \longrightarrow \mathcal{EPA}$$

and we will show that their compositions $\Phi_3 \circ \Psi_3$ and $\Psi_3 \circ \Phi_3$ are equal to the corresponding identities. We set, for every extensional precontact algebra $\underline{B} = (B, C)$,

$$\Phi_3([\underline{B}]) = [\underline{X}(\underline{B})],$$

where $\underline{X}(\underline{B})$ is the canonical extensional 3-precontact space of $\underline{B}$ (see Definition 3.16), $[\underline{B}]$ is the class of all extensional precontact algebras which are PCA-isomorphic to the extensional precontact algebra $\underline{B}$, and, analogously, $[\underline{X}(\underline{B})]$ is the class of all extensional 3-precontact spaces which are EPS-isomorphic to the extensional 3-precontact space $\underline{X}(\underline{B})$. Further, for every extensional 3-precontact space $\underline{X} = (X, X_0, X_1, R)$, we set

$$\Psi_3([\underline{X}]) = [\underline{B}(\underline{X})],$$

where $\underline{B}(\underline{X})$ is the canonical extensional precontact algebra of $\underline{X}$ (see Definition 3.8). It is easy to see that the correspondences $\Phi_3$ and $\Psi_3$ are well-defined.

Using (22), we obtain that for every extensional precontact algebra $\underline{B} = (B, C)$, $\Psi_3(\Phi_3([\underline{B}])) = [\underline{B}]$. Thus we obtain that $\Psi_3 \circ \Phi_3 = id_{\mathcal{EPA}}$.

We will now prove that $\Phi_3 \circ \Psi_3 = id_{\mathcal{EPS}}$. Let $\underline{X} = (X, X_0, X_1, R)$ be an extensional 3-precontact space. Set $(B, C) = (CO(X_0), C_R)$; then $(B, C)$ is PCA-isomorphic to the canonical extensional precontact algebra $\underline{B}(\underline{X})$ of $\underline{X}$ (see Definition 3.8 and (23)). Let $(\widehat{X}, \widehat{X}_0, \widehat{X}_1, \widehat{R})$ be the canonical extensional 3-precontact space of $(B, C)$ (see Definition 3.16). Then $\widehat{X} = \mathrm{Clans}(B, C)$, $\widehat{X}_0 = \mathrm{Ult}(B)$, $\widehat{X}_1 = \mathrm{MClans}(B, C)$ and $\widehat{R} = R_{(B,C)}$. For every $x \in X$, set

$$f(x) = \{a \in B \mid x \in \mathrm{cl}_X(a)\} (= \Gamma_{x,X_0}).$$

Since $(X, X_0, R)$ is a 2-precontact space,

$$\Psi_2([(X, X_0, R)]) = [(B, C)] \text{ and } \Phi_2([(B, C)]) = [(\widehat{X}, \widehat{X}_0, \widehat{R})],$$

we obtain, by (the proof of) Theorem 2.24(b), that

$$f : (X, X_0, R) \longrightarrow (\widehat{X}, \widehat{X}_0, \widehat{R}), \quad x \mapsto f(x), \quad \text{is a PCS-isomorphism.}$$

Hence, for showing that the same map $f$ is an EPS-isomorphism between extensional 3-precontact spaces $\underline{X} = (X, X_0, X_1, R)$ and $(\widehat{X}, \widehat{X}_0, \widehat{X}_1, \widehat{R})$, we need only to show that $f(X_1) = \widehat{X}_1$.

Let $x \in X_1$. Then $f(x) = \{F \in CO(X_0) \mid x \in \mathrm{cl}_X(F)\} = \Gamma_{x,X_0}$ and hence, by (EPCS4), $f(x) \in \widehat{X}_1$. So, $f(X_1) \subseteq \widehat{X}_1$. Let $\Gamma \in \widehat{X}_1$. Then, $\Gamma \in \mathrm{MClans}(CO(X_0), C_R)$ and hence, by (EPCS4), there exists $x \in X_1$ such that $\Gamma = \Gamma_{x,X_0}$, i.e. $\Gamma = f(x)$. We have proved that $f(X_1) = \widehat{X}_1$. Therefore

$$f : (X, X_0, X_1, R) \longrightarrow (\widehat{X}, \widehat{X}_0, \widehat{X}_1, \widehat{R}) \quad \text{is an EPS-isomorphism.}$$

Hence $(\Phi_3 \circ \Psi_3)([\underline{X}]) = [\underline{X}]$, i.e. $\Phi_3 \circ \Psi_3 = id_{\mathcal{EPS}}$. Therefore,

$$\Phi_3 : \mathcal{EPA} \longrightarrow \mathcal{EPS} \quad \text{is a bijection.} \tag{24}$$

The statement for connected extensional precontact algebras follows from (24) and (a). □

## 4. Extensional 3-contact spaces.

**Definition 4.1** (Extensional 3-Contact spaces)**.** *A topological triple* $(X, X_0, X_1)$ *is called an* extensional 3-contact space *(abbreviated as 3ECS) if the following conditions are satisfied:*

(3ECS1) $(X, X_0)$ *is a 2-contact space;*

(3ECS2) $X_1$ *is a weakly regular* $T_1$*-space;*

(3ECS3) *If* $x \in X_1$ *then the set* $\Gamma_{x,X_0}$ *(see* (1)*) is a maximal clan in the contact algebra* $(CO(X_0), \delta_{(X,X_0)})$ *(see* (15) *for the notation* $\delta_{(X,X_0)}$*); conversely, for every maximal clan* $\Gamma$ *in* $(CO(X_0), \delta_{(X,X_0)})$ *there exists a point* $x \in X_1$ *such that* $\Gamma = \Gamma_{x,X_0}$.

*An extensional 3-contact space* $(X, X_0, X_1)$ *is called* connected *(resp.,* extremally disconnected*) if the space* $X_1$ *(resp.,* $X_0$*) is connected (resp., extremally disconnected).*

**Definition 4.2.** *Let* $(X, X_0, X_1)$ *be an extensional 3-contact space. Then the contact algebra* $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$ *(see Lemma* 3.3 *for the notation* $C_{(X,X_0,X_1)}$*) is said to be the* canonical extensional contact algebra of the extensional *3*-contact space $(X, X_0, X_1)$.

**Definition 4.3.** *Let $(B, C)$ be an extensional contact algebra and $(X, X_0)$ be the canonical 2-contact space of $(B, C)$. Set $X_1 = \mathrm{MClans}(B, C)$. Then $X_1 \subseteq X$. Take the subspace topology on $X_1$. Then the triple $(X, X_0, X_1)$ is called the* canonical extensional 3-contact space of the extensional contact algebra $(B, C)$.

**Definition 4.4.** *Let $(X, X_0, X_1)$ and $(\widehat{X}, \widehat{X}_0, \widehat{X}_1)$ be two extensional 3-contact spaces. We say that $(X, X_0, X_1)$ and $(\widehat{X}, \widehat{X}_0, \widehat{X}_1)$ are* 3ECS-isomorphic *(or, simply,* isomorphic*) if there exists a homeomorphism $f : X \longrightarrow \widehat{X}$ such that $f(X_0) = \widehat{X}_0$ and $f(X_1) = \widehat{X}_1$.*

**Lemma 4.5.** *For every extensional 3-contact space $(X, X_0, X_1)$ there exists a unique reflexive and symmetric binary relation $R$ on $X_0$ such that $(X, X_0, X_1, R)$ is an extensional 3-precontact space.*

P r o o f. Let $R$ be the reflexive and symmetric binary relation on $X_0$ defined by (16) (see Lemma 2.30). Since $(X, X_0)$ is a 2-contact space, we have, by Lemma 2.30 and (17), that $(X, X_0, R)$ is a 2-precontact space (and, thus, the axiom (EPS1) is fulfilled) and for every $F, G \in CO(X_0)$, $FC_RG \iff \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$; hence,

$$FC_RG \iff F\delta_{(X,X_0)}G. \tag{25}$$

Therefore, the axiom (EPS4) follows from the axiom (3ECS3). Obviously, the axioms (EPS2) and (EPS3) are fulfilled. Hence $(X, X_0, X_1, R)$ is an extensional 3-precontact space. The proof of the uniqueness of the relation $R$ is completely analogous to that given in the proof of Lemma 2.30=[9, Lemma 7.5]. □

**Corollary 4.6.** *Let $(X, X_0, X_1)$ be an extensional 3-contact space. Then:*

*(a) for every $F, G \in CO(X_0)$, we have that* (19) *holds;*

*(b) the canonical extensional CA of the extensional 3-contact space $(X, X_0, X_1)$ is indeed an extensional contact algebra. Also, the Boolean algebra $RC(X, X_0, X_1)$ is a dense subset of the Boolean algebra $RC(X_1)$.*

P r o o f. It follows from Lemma 4.5, Proposition 3.12, Proposition 3.14 and Corollary 3.15. Indeed, the assertion (a) follows immediately from Lemma 4.5 and Proposition 3.12. Further, if $R$ is the unique reflexive and symmetric binary relation on $X_0$ such that $\underline{X} = (X, X_0, X_1, R)$ is an extensional 3-precontact space (see Lemma 4.5), then it is easy to see that $C_{\underline{X}} \equiv (C_{\underline{X}})^{\#}$ and thus $C_{\underline{X}} \equiv C_{(X,X_0,X_1)}$ (by Proposition 3.14). Now, the assertion (b) follows from Corollary 3.15. □

**Proposition 4.7.** *Let $\underline{B} = (B, C)$ be an extensional contact algebra. Then the canonical extensional 3-contact space of $\underline{B}$ is indeed an extensional 3-contact space.*

Proof. Let $(X, X_0, X_1, R)$ be the canonical extensional 3-precontact space of $\underline{B}$. We will show that $(X, X_0, X_1)$ is the canonical extensional 3-contact space of $\underline{B}$ and that it is indeed an extensional 3-contact space. We have, by Proposition 3.18, that $(X, X_0, X_1, R)$ is an extensional 3-precontact space. Hence $(X, X_0, X_1)$ is a topological triple and also the axiom (3ECS2) is fulfilled. Further, $(X, X_0, R)$ is the canonical 2-precontact space of $\underline{B}$ and $X_1 = \mathrm{MClans}(B, C)$, $X_1 \subseteq X$ and $X_1$ is endowed with the subspace topology. By Lemma 2.11(d,e), we have that $R = R_{\underline{B}}$ is a reflexive and symmetric relation on $X_0$. Thus, by [9, Proposition 7.7], $(X, X_0)$ is a 2-contact space. Hence, the axiom (3ECS1) is also fulfilled. For showing that axiom (3ECS3) takes place, it is enough to prove that the relations $C_R$ and $\delta_{(X,X_0)}$ coincide. Note first that, by Proposition 2.7(b), $C_R \equiv (C_R)^{\#}$. Now, using Proposition 3.14, (20) and (19), we obtain that, for every $F, G \in CO(X_0)$, $FC_RG \iff X_1 \cap \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset \iff \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset \iff F\delta_{(X,X_0)}G$. So, we have proved that $(X, X_0, X_1)$ is an extensional 3-contact space. The fact that $(X, X_0, X_1)$ is the canonical extensional 3-contact space of $\underline{B}$ follows from Definition 4.3 and Definition 3.16. □

**Theorem 4.8** (New representation theorem for extensional contact algebras).

(a) *Let $(B, C)$ be an extensional contact algebra and let $(X, X_0, X_1)$ be the canonical extensional 3-contact space of $(B, C)$ (see Definition 4.3). Then the function $g_1^B$, defined in (21), is a CA-isomorphism between $(B, C)$ and the canonical extensional contact algebra $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$ of the extensional 3-contact space $(X, X_0, X_1)$. The sets $RC(X, X_0, X_1)$ and $RC(X_1)$ coincide iff the algebra $(B, C)$ is complete iff the space $X_0$ is extremally disconnected. The contact algebra $(B, C)$ is connected iff the extensional 3-contact space $(X, X_0, X_1)$ is connected.*

(b) *There exists a bijective correspondence $\Phi_3^c$ between the class of all, up to CA-isomorphism, (connected) extensional contact algebras and the class of all, up to 3ECS-isomorphism, (connected) extensional 3-contact spaces. It is defined by the formula*

$$\Phi_3^c([(B, C)]) = [(X, X_0, X_1)],$$

*where $(B, C)$ is an ECA and $(X, X_0, X_1)$ is the canonical extensional 3-*

*contact space of* $(B, C)$*. Also, the correspondence*

$$\Psi_3^c = (\Phi_3^c)^{-1}$$

*is defined by the formula*

$$\Psi_3^c([(X, X_0, X_1)]) = [(B, C)],$$

*where* $(X, X_0, X_1)$ *is an extensional 3-contact space and* $(B, C)$ *is its canonical extensional contact algebra (see Definition* 4.2*).*

(c) *There exists a bijective correspondence* $\Phi_3^{cc}$ *between the class of all, up to CA-isomorphism, (connected) complete extensional contact algebras and the class of all, up to 3ECS-isomorphism, (connected) extremally disconnected extensional 3-contact spaces. The correspondence* $\Phi_3^{cc}$ *is just the restriction of the correspondence* $\Phi_3^c$ *defined in* (b) *here.*

Proof. It follows from Lemma 4.5 and Theorem 3.19 (and is analogous to the proof of Theorem 2.31=[9, Theorem 7.9]). □

We are now going to obtain an assertion from [6] (namely, Theorem 5.1(ii)(the case for complete extensional CAs) there) as a corollary of Theorem 4.8. This assertion concerns the class of C-weakly regular spaces introduced in [6] (see Definition 4.9 below). We start with recalling and proving some preliminary assertions. Then we derive [6, Theorem 5.1(ii)(for complete ECAs)] from Theorem 4.8 (see Corollary 4.14 below).

Let $X$ be a set and $\Gamma$ be a family of subsets of $X$. Recall that the family $\Gamma$ is said to be *fixed* if $\bigcap \Gamma \neq \emptyset$.

**Definition 4.9** ([6]). *A weakly regular* $T_1$*-space* $(X, \tau)$ *is said to be* C-weakly regular *if every maximal clan* $\Gamma$ *in* $(RC(X), C_X)$ *is fixed.*

Since every clan is contained in a maximal clan (see [6, Fact 3.3(iii)]), we obtain that a weakly regular $T_1$-space $(X, \tau)$ is C-weakly regular iff every clan in $(RC(X), C_X)$ is fixed. Using this fact and the proof of Proposition 2.33=[9, Proposition 7.18], we obtain a proof of the following assertion which was stated in [6] without proof.

**Proposition 4.10** ([6, Fact 4.2]). *Every C-weakly regular space* $X$ *is a compact space.*

In [6, Proposition 4.4(i)], we have observed that if $X$ is a topological space then every fixed maximal clan in $(RC(X), C_X)$ is a point-clan. Hence, a weakly regular $T_1$-space $(X,\tau)$ is C-weakly regular iff every maximal clan in $(RC(X),C_X)$ is a point-clan.

**Proposition 4.11.** *If $(X,X_0,X_1)$ is an extremally disconnected extensional 3-contact space, then the space $X_1$ is C-weakly regular.*

P r o o f. Since the space $X_0$ is extremally disconnected, we obtain, by Theorem 4.8, that $(RC(X,X_0,X_1),C_{(X,X_0,X_1)}) = (RC(X_1),C_{X_1})$. By Lemma 4.5, there exists a unique reflexive and symmetric binary relations $R$ on $X_0$ such that $(X,X_0,X_1,R)$ is an extensional 3-precontact space. Then, by Proposition 2.7(b), $C_R = (C_R)^{\#}$. Now, applying (20), (11) and Proposition 3.14, we obtain that the contact algebras $(RC(X_1),C_{X_1})$ and $(RC(X_0),\delta_{(X,X_0)})$ are CA-isomorphic (note that $RC(X_0) \equiv CO(X_0)$ because the space $X_0$ is extremally disconnected). Denote by $\varphi$ the isomorphism between them. Then, for every $F \in RC(X_1)$, we have that $\varphi(F) = X_0 \cap \mathrm{cl}_X(F)$.

Let now $\Gamma \in \mathrm{MClans}(RC(X_1),C_{X_1})$. Then $\varphi(\Gamma) \in \mathrm{MClans}(CO(X_0),\delta_{(X,X_0)})$. By (3ECS3), there exists a point $x \in X_1$ such that $\varphi(\Gamma) = \Gamma_{x,X_0}$. Since $\varphi^{-1}(G) = X_1 \cap \mathrm{cl}_X(G)$, for every $G \in RC(X_0)$, we obtain that $\Gamma = \varphi^{-1}(\Gamma_{x,X_0}) = \{X_1 \cap \mathrm{cl}_X(G) \mid G \in \Gamma_{x,X_0}\}$. Obviously, $x \in X_1 \cap \mathrm{cl}_X(G)$ for every $G \in \Gamma_{x,X_0}$. Thus, $\bigcap \Gamma \neq \emptyset$. Since, by (3ECS2), $X_1$ is a weakly regular $T_1$-space, we obtain that $X_1$ is a C-weakly regular space. □

The next assertion was, in fact, proved in [6], although it was not formulated explicitly there:

**Proposition 4.12.** *A weakly regular $T_1$-space $X$ is C-weakly regular iff $X$ is homeomorphic to the space $(\mathrm{MClans}(RC(X),C_X),\tau)$, where the topology $\tau$ has as a closed base the family $\{g_1^B(F) = \{\Gamma \in \mathrm{MClans}(RC(X),C_X) \mid F \in \Gamma\} \mid F \in RC(X)\}$.*

P r o o f. Use Proposition 3.5, [6, Proposition 3.5 and Definitions 4.3, 4.4] and the proof of (II(ii)) given on page 239 of [6]. □

We will need the following assertion:

**Proposition 4.13.** *If $(X,X_0,X_1)$ and $(X',X_0',X_1')$ are two extremally disconnected extensional 3-contact spaces such that the spaces $X_1$ and $X_1'$ are homeomorphic, then $(X,X_0,X_1)$ and $(X',X_0',X_1')$ are 3ECS-isomorphic.*

P r o o f. Since $X_0$ (resp., $X_0'$) is extremally disconnected, we obtain, arguing as in the beginning of the proof of Proposition 4.11, that the canonical contact algebra of $(X,X_0,X_1)$ (resp., $(X',X_0',X_1')$) is $(B,C) = (RC(X_1),C_{X_1})$ (resp., $(B',C') = (RC(X_1'),C_{X_1'})$). Since the spaces $X_1$ and $X_1'$ are homeomorphic, we obtain that the contact algebras $(B, C)$ and $(B', C')$ are CA-isomorphic. By Theorem 4.8(c), we obtain that the space $(X, X_0, X_1)$ (resp., $(X', X'_0, X'_1)$) is isomorphic to the canonical 3-contact space of $(B, C)$ (resp., $(B', C')$). Thus, the 3-contact spaces $(X, X_0, X_1)$ and $(X', X'_0, X'_1)$ are 3ECS-isomorphic. $\square$

**Corollary 4.14** ([6]). *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, (connected) complete extensional contact algebras and the class of all, up to homeomorphism, (connected) C-weakly regular spaces.*

Proof. By Theorem 4.8(c), there exists a bijective correspondence between the class $\mathcal{CECA}$ of all, up to PCA-isomorphism, complete extensional contact algebras and the class $\mathcal{EDECS}$ of all, up to 3ECS-isomorphism, extremally disconnected extensional 3-contact spaces. We will show that there exists a bijective correspondence between the class $\mathcal{EDECS}$ and the class $\mathcal{CWRS}$ of all, up to homeomorphism, C-weakly regular spaces.

Let us put

$$\Psi'_3([(X, X_0, X_1)]) = [X_1],$$

for every extremally disconnected extensional 3-contact space $(X, X_0, X_1)$. Then, by Proposition 4.11,

$$\Psi'_3 : \mathcal{EDECS} \longrightarrow \mathcal{CWRS}.$$

Using Proposition 4.13, we obtain that $\Psi'_3$ is an injection. For showing that $\Psi'_3$ is a surjection, let $X_1$ be a C-weakly regular space. Then $(B, C) = (RC(X_1), C_{X_1})$ is a complete extensional contact algebra. Let $(X', X'_0, X'_1)$ be the canonical extensional 3-contact space of $(B, C)$ (see Definition 4.3). Then, by Theorem 4.8(a)(c) and its proof, $(X', X'_0, X'_1)$ is an extremally disconnected extensional 3-contact space. Further, Definition 4.3 and Proposition 4.12 show that the spaces $X_1$ and $X'_1$ are homeomorphic. Thus $\Psi'_3([(X', X'_0, X'_1)]) = [X_1]$. Therefore $\Psi'_3$ is a surjection. All this shows that $\Psi'_3$ is a bijection.

The connected case follows now from Theorem 4.8(c). $\square$

## 5. Extensional precontact algebras satisfying some additional axioms.

**Definition 5.1** (Clusters and co-ends). *Let $\underline{B} = (B, C)$ be a precontact algebra.*

• *A clan $\Gamma$ in $\underline{B}$ is called a* cluster *in $\underline{B}$ if it satisfies the following condition:*

(Clust) *If for every $x \in \Gamma$ we have $xC^{\#}y$ then $y \in \Gamma$.*

• *A clan $\Gamma$ in $\underline{B}$ is called a* co-end *in $\underline{B}$ if it satisfies the following condition:*

(Coend) *If* $x \notin \Gamma$ *then there exists a* $y \notin \Gamma$ *such that* $x(-C^{\#})y^*$.

*The set of all clusters (resp., co-ends) in* $\underline{B}$ *is denoted by* $\mathrm{Clust}(\underline{B})$ *(resp.,* $\mathrm{Coend}(\underline{B})$*).*

**Definition 5.2** (Some new axioms for precontact algebras)**.** *Let* $(B, C)$ *be a precontact algebra. We will regard the following axioms for precontact algebras whose analogues for contact algebras were defined in* [6]:

N-regularity Axiom. *If* $xC^{\#}y$ *then there exists a cluster* $\Gamma$ *containing* $x$ *and* $y$.

Regularity Axiom. *If* $xC^{\#}y$ *then there exists a co-end* $\Gamma$ *containing* $x$ *and* $y$.

**Definition 5.3.** *Let* $(B, C)$ *be an extensional precontact algebra. It is called an* N-regular (resp., regular; normal) precontact algebra *if it satisfies the N-regularity axiom (resp., Regularity axiom; the axiom* $(Ctr\#)$*).*

**Definition 5.4.** *A topological space* $(X, \tau)$ *is said to be* strongly s-regular ([16]) *if the family* $RC(X)$ *is a* network *for* $(X, \tau)$ *(i.e., for every point* $x \in X$ *and every open neighborhood* $U$ *of* $x$ *there exists an* $F \in RC(X)$ *such that* $x \in F \subseteq U$*). A topological space* $(X, \tau)$ *is said to be* N-regular ([6]) *if it is semiregular and strongly s-regular. (Note that when we have introduced in* [6] *the notion of "N-regular space", we were not aware that the notion of "strongly s-regular space" was already introduced.)*

Obviously, every regular space is N-regular and every N-regular space is weakly regular.

The next representation theorems for N-regular (resp., regular; normal) contact algebras were proved by us in [6, Theorem 5.3(a), Theorem 5.2(a)]:

**Theorem 5.5** ([6])**.** *For each N-regular (resp., regular) contact algebra* $\underline{B} = (B, C)$ *there exists a dense embedding* $g_B$ *of* $B$ *into a standard N-regular (resp., regular) contact algebra* $(RC(X, \tau), C_X)$*, where* $(X, \tau)$ *is an N-regular* $T_1$*-space (resp., regular* $T_2$*-space). When* $B$ *is complete then:* (a) *the embedding* $g_B$ *becomes an isomorphism between contact algebras* $(B, C)$ *and* $(RC(X), C_X)$*, and* (b) *the algebra* $B$ *is connected iff the space* $X$ *is connected.*

**Theorem 5.6** ([6])**.** *For each normal contact algebra* $\underline{B} = (B, C)$ *there exists a dense embedding* $g_B$ *of* $B$ *into a standard normal contact algebra* $(RC(X, \tau), C_X)$*, where* $(X, \tau)$ *is a compact* $T_2$*-space. The algebra* $B$ *is connected iff the space* $X$ *is connected. When* $B$ *is complete then the embedding* $g_B$ *becomes an isomorphism between contact algebras* $(B, C)$ *and* $(RC(X), C_X)$*.*

**Definition 5.7.** *A quadruple* $\underline{X} = (X, X_0, X_1, R)$ *is called an* N-regular

3-precontact space *(abbreviated as NPS) (resp.,* regular 3-precontact space *(abbreviated as RPS)) if it satisfies the following conditions:*

(3PCS1) $(X, X_0, R)$ *is a 2-precontact space;*

(3PCS2) $X_1$ *is a dense subspace of* $X$*;*

(3PCS3) $X_1$ *is an N-regular* $T_1$*-space (resp., regular* $T_2$*-space);*

(3PCS4) *If* $x \in X_1$ *then the set* $\Gamma_{x,X_0}$ *(see* (1)*) is a cluster (resp., co-end) in the precontact algebra* $(CO(X_0), C_R)$*; conversely, for every cluster (resp., co-end)* $\Gamma$ *in* $(CO(X_0), C_R)$ *there exists a point* $x \in X_1$ *such that* $\Gamma = \Gamma_{x,X_0}$.

(3PCS5) *For every* $F, G \in CO(X_0)$, (19) *holds.*

*A quadruple* $\underline{X} = (X, X_0, X_1, R)$ *is called a* normal 3-precontact space *if it satisfies conditions* (3PCS1), (3PCS2) *and the following two conditions:*

(3PCS3N) $X_1$ *is a compact* $T_2$*-space;*

(3PCS4N) *If* $x \in X_1$ *then the set* $\Gamma_{x,X_0}$ *is a cluster in the precontact algebra* $(CO(X_0), C_R)$*; conversely, for every cluster* $\Gamma$ *in* $(CO(X_0), C_R)$ *there exists a point* $x \in X_1$ *such that* $\Gamma = \Gamma_{x,X_0}$.

*Replacing in Definitions* 3.8, 3.9 *and* 3.10 *the word "extensional" with the word "N-regular" (resp., "regular", "normal"), we introduce the notions of* canonical N-regular (resp., regular; normal) precontact algebra $\underline{B}(\underline{X})$ of an N-regular (resp., regular; normal) 3-precontact space $\underline{X}$ *(using the same definition of the relation* $C_{\underline{X}}$*), as well as the notions of* reflexive *(or* symmetric*;* transitive*;* connected*)* N-regular *(resp.,* regular*;* normal*)* 3-precontact space*, and the notions of* isomorphism *between such spaces.*

**Remark 5.8.** The axiom (3PCS5) is omitted in [8]; this is a misprint.

In what follows, we will need the following assertions from [6]:

**Proposition 5.9.** *Let* $(B, C)$ *be an N-regular (resp., regular) (pre)contact algebra. Then for every* $b \in B \setminus \{1\}$ *there exists a cluster (resp., co-end)* $\Gamma$ *such that* $b \notin \Gamma$.

Proof. It is contained in the proof of [6, Lemma 5.7(iv1)]. For completeness, we will give it here. Let $b \in B \setminus \{1\}$. Then, by the Axiom of Extensionality, there exists an $a \in B \setminus \{0\}$ such that $a(-C^{\#})b$. Since $aC^{\#}a$, the N-regularity (resp., Regularity) Axiom implies that there exists a cluster (resp., a co-end) $\Gamma$ in $(B, C)$ such that $a \in \Gamma$. Since $a(-C^{\#})b$, we obtain that $b \notin \Gamma$. □

**Proposition 5.10** ([6, Proposition 3.1(i)(ii) and Corollary 3.4]). *Let $(B, C)$ be a (pre)contact algebra which satisfies the Interpolation Axiom $(Ctr\#)$). Then:*

*(a) the notions of a maximal clan, cluster and co-end coincide in $(B, C)$;*

*(b) for every $a, b \in B$ such that $aC^{\#}b$, there exists a cluster $\Gamma$ in $(B, C)$ which contains $a$ and $b$.*

**Proposition 5.11.** *Let $(B, C)$ be a normal (pre)contact algebra. Then for every $b \in B \setminus \{1\}$ there exists a cluster $\Gamma$ such that $b \notin \Gamma$.*

Proof. Using Proposition 5.10(b), we can argue as in the proof of Proposition 5.9. $\square$

**Proposition 5.12.** *Let $(X, X_0, X_1, R)$ be a normal 3-precontact space. Then, for every $F, G \in CO(X_0)$,* (19) *holds.*

Proof. Having in mind Proposition 5.10(a), we can repeat the proof of Proposition 3.12 replacing in it the axiom (EPS4) by the axiom (3PCS4N). $\square$

**Proposition 5.13.** *Let $\underline{X} = (X, X_0, X_1, R)$ be an N-regular (resp., regular; normal) 3-precontact space. Then, for every $F, G \in RC(X, X_0, X_1)$, we have that $F(C_{\underline{X}})^{\#}G \iff F \cap G \neq \emptyset \iff FC_{X_1}G$; if $F = X_1 \cap \mathrm{cl}_X(A_1)$ and $G = X_1 \cap \mathrm{cl}_X(A_2)$, where $A_1, A_2 \in CO(X_0)$, then $FC_{\underline{X}}G \iff A_1C_RA_2$.*

Proof. The only difference with the proof of Proposition 3.14 is that in it we now replace Proposition 3.12 by the axiom (3PCS5) or by Proposition 5.12. $\square$

**Proposition 5.14.** *The canonical N-regular (resp., regular; normal) precontact algebra of an N-regular (resp., regular; normal) 3-precontact space $\underline{X} = (X, X_0, X_1, R)$, defined in Definition* 5.7*, is indeed an N-regular (resp., regular; normal) precontact algebra. Also, the Boolean algebra $RC(X, X_0, X_1)$ is a dense subset of the Boolean algebra $RC(X_1)$.*

Proof. Let $\underline{X} = (X, X_0, X_1, R)$ be an N-regular (resp., regular; normal) 3-precontact space. Since every N-regular (resp., regular; compact $T_2$) space is weakly regular, repeating the proof of Corollary 3.15 and replacing in it Proposition 3.14 with Proposition 5.13, we obtain that $RC(X, X_0, X_1)$ is a dense subset of the Boolean algebra $RC(X_1)$ and the canonical N-regular (resp., regular; normal) precontact algebra $\underline{B}(\underline{X})$ of $\underline{X}$ is an extensional precontact algebra. We will show that it is an N-regular (resp., regular; normal) precontact algebra.

Let first $\underline{X} = (X, X_0, X_1, R)$ be an N-regular 3-precontact space. Let $F, G \in RC(X, X_0, X_1)$ and $FC_{\underline{X}}G$. Then $F(C_{\underline{X}})^{\#}G$ and hence, by Proposition

5.13, we obtain that $F \cap G \neq \emptyset$. Let $x \in F \cap G$. Then $x \in X_1$ and, by the axiom (3PCS4), the set $\Gamma_{x,X_0}$ is a cluster in the precontact algebra $(CO(X_0), C_R)$. Using Lemma 3.3(c) and Proposition 5.13, we obtain that the map

$$r_1 : (CO(X_0), C_R) \longrightarrow (RC(X, X_0, X_1), C_{\underline{X}}), \quad A \mapsto X_1 \cap \mathrm{cl}_X(A),$$

is a PCA-isomorphism. Hence, $\Gamma = r_1(\Gamma_{x,X_0})$ is a cluster in $\underline{B}(\underline{X})$. Since, obviously, $F, G \in \Gamma$, we obtain that the N-regularity axiom is fulfilled in $\underline{B}(\underline{X})$.

The proof of the case when $\underline{X}$ is a regular 3-precontact space is completely analogous.

Let now $\underline{X} = (X, X_0, X_1, R)$ be a normal 3-precontact space. Then, by Proposition 5.13, we have that for every $F, G \in RC(X, X_0, X_1)$, $F(C_{\underline{X}})^{\#}G \iff FC_{X_1}G$. The space $X_1$ is compact Hausdorff and thus it is a normal space. Since, for every $F, G \in RC(X, X_0, X_1)$, we have that $F \ll_{C_{X_1}} G \iff F \subseteq \mathrm{int}_{X_1}(G)$, the normality of $X_1$ implies readily that $\underline{B}(\underline{X})$ satisfies the axiom (Ctr#). Hence $\underline{B}(\underline{X})$ is a normal PCA. □

**Definition 5.15.** *Let $\underline{B} = (B, C)$ be an N-regular (resp., regular; normal) precontact algebra. We associate with $\underline{B}$ an N-regular (resp., regular; normal) 3-precontact space $\underline{X}(\underline{B}) = (X, X_0, X_1, R)$ called the* canonical N-regular (resp., regular; normal) 3-precontact space of $\underline{B}$. *Namely, let $(X, X_0, R)$ be the canonical 2-precontact space of the precontact algebra $(B, C)$; we set $X_1 = \mathrm{Clust}(B, C)$ (resp., $X_1 = \mathrm{Coend}(B, C)$; $X_1 = \mathrm{Clust}(B, C)$); then $X_1 \subseteq X$ and we take the subspace topology on $X_1$. (See Definition* 5.1 *for the notation* $\mathrm{Clust}(B, C)$ *and* $\mathrm{Coend}(B, C)$*.)*

**Proposition 5.16.** *Let $\underline{B} = (B, C)$ be an N-regular (resp., regular; normal) precontact algebra. Then the canonical N-regular (resp., regular; normal) 3-precontact space $\underline{X}(\underline{B}) = (X, X_0, X_1, R)$ of $\underline{B}$ is indeed an N-regular (resp., regular; normal) 3-precontact space.*

P r o o f. Let $\underline{B} = (B, C)$ be an N-regular precontact algebra. Then, by Definition 5.15 and Proposition 2.23, the axiom (3PCS1) is satisfied. The proof that the axiom (3PCS2) (resp., (3PCS3)) is satisfied is almost identical with that one, given in the proof of Proposition 3.18, that the axiom (EPS2) (resp., (EPS3)) is satisfied. The only difference is that now we use Proposition 5.9 (resp., Theorem 5.5) instead of Proposition 3.17 (resp., Theorem 3.6). The situation is the same with the axiom (3PCS4): the only difference is that now we speak about clusters instead of maximal clans. So, it remains to prove the axiom (3PCS5). Let $F, G \in CO(X_0)$ and $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$. Since $(X, X_0, R)$ is a 2-precontact space, it satisfies the axiom (PCS4); therefore, we obtain that $F(C_R)^{\#}G$. There exist $a, b \in B$ such that $F = s_B(a)$ and $G = s_B(b)$. Now, by Theorem 2.13(a), we obtain that $a(C^{\#})b$. Thus, by the N-regularity Axiom, there exists a cluster $\Gamma$ in $(B, C)$ containing $a$ and $b$. Then, obviously, $\Gamma \in X_1 \cap g_B(a) \cap g_B(b)$. Since $s_B(c) = g_0^B(c)$, for every $c \in B$ (see Remark 2.22), using (13), we obtain that $X_1 \cap \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$. The converse implication is obvious. Hence, the axiom (3PCS5) is satisfied as well. Therefore $\underline{X}(\underline{B}) = (X, X_0, X_1, R)$ is indeed an N-regular 3-precontact space.

The proof for the regular precontact algebras is analogous. The same is true for the normal precontact algebras but in this case we have to use Proposition 5.11 and Theorem 5.6; also, we don't need to check the axiom (3PCS5). □

**Theorem 5.17** (Representation theorem for N-regular and regular precontact algebras)**.**

(a) *Let $\underline{B} = (B, C)$ be an N-regular (resp., regular) precontact algebra and let*

$$\underline{X}(\underline{B}) = (X, X_0, X_1, R)$$

*be the canonical N-regular (resp., regular) 3-precontact space of $\underline{B}$. Then the function $g_1^B$, defined in* (21), *is a PCA-isomorphism between* $(B, C)$ *and the canonical N-regular (resp., regular) precontact algebra*

$$(RC(X, X_0, X_1), C_{\underline{X}(\underline{B})})$$

*of $\underline{X}(\underline{B})$. The same function $g_1^B$ is a PCA-isomorphism between the contact algebras $(B, C^{\#})$ and $(RC(X, X_0, X_1), C_{(X, X_0, X_1)})$. The sets $RC(X, X_0, X_1)$ and $RC(X_1)$ coincide iff the algebra $\underline{B}$ is complete iff the space $X_0$ is extremally disconnected. The algebra $\underline{B}$ satisfies the axiom* (Cref) *(resp.,* $(Csym)$; $(Ctr)$*) iff the N-regular (resp., regular) 3-precontact space $\underline{X}(\underline{B})$ is reflexive (resp., symmetric; transitive). If the algebra $\underline{B}$ is complete then it is connected iff $\underline{X}(\underline{B})$ is connected.*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, N-regular (resp., regular) precontact algebras and the class of all, up to respective isomorphism, N-regular (resp., regular) 3-precontact spaces.*

(c) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, (connected) complete N-regular (resp., regular) precontact algebras and the class of all, up to respective isomorphism, (connected) extremally disconnected N-regular (resp., regular) 3-precontact spaces.*

P r o o f. The proof of this theorem is completely analogous to that of Theorem 3.19. The only difference is that instead of Proposition 3.18, [6, Lemma 5.7(iii1)], Proposition 3.14, Theorem 3.6, we have now to use in it, respectively, Proposition 5.16, [6, Lemma 5.7(iv1)] (resp., [6, Lemma 5.7(v1)]), Proposition 5.13, Theorem 5.5. □

**Theorem 5.18** (Representation theorem for normal precontact algebras)**.**

(a) *Let $\underline{B} = (B, C)$ be a normal precontact algebra and let*

$$\underline{X}(\underline{B}) = (X, X_0, X_1, R)$$

*be the canonical normal 3-precontact space of $\underline{B}$. Then the function $g_1^B$, defined in* (21)*, is a PCA-isomorphism between $(B, C)$ and the canonical normal precontact algebra*

$$(RC(X, X_0, X_1), C_{\underline{X}(\underline{B})})$$

*of $\underline{X}(\underline{B})$. The same function $g_1^B$ is a PCA-isomorphism between the contact algebras $(B, C^{\#})$ and $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$. The sets $RC(X, X_0, X_1)$ and $RC(X_1)$ coincide iff the algebra $\underline{B}$ is complete iff the space $X_0$ is extremally disconnected. The algebra $\underline{B}$ satisfies the axiom* (Cref) *(resp.,* (Csym)*;* (Ctr)*) iff the normal 3-precontact space $\underline{X}(\underline{B})$ is reflexive (resp., symmetric; transitive). The algebra $\underline{B}$ is connected iff $\underline{X}(\underline{B})$ is connected.*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, (connected) normal precontact algebras and the class of all, up to respective isomorphism, (connected) normal 3-precontact spaces.*

P r o o f. The proof of this theorem is completely analogous to that of Theorem 3.19. The only difference is that instead of Proposition 3.18, [6, Lemma 5.7(iii1)], Proposition 3.14, Theorem 3.6, we have now to use in it, respectively, Proposition 5.16, Proposition 5.10(a), [6, Lemma 5.7(iii1)], Proposition 5.13, Theorem 5.6. □

## 6. N-regular, regular and normal 3-contact spaces.

**Definition 6.1** (N-regular, regular and normal 3-contact spaces)**.** *A topological triple $(X, X_0, X_1)$ is called an* N-regular (resp., regular) 3-contact space *if the following conditions are satisfied:*

(3–1) *$(X, X_0)$ is a 2-contact space;*

(3−2) *$X_1$ is an N-regular $T_1$-space (resp., regular $T_2$-space);*

(3−3) *If $x \in X_1$ then the set $\Gamma_{x,X_0}$ (see* (1)*) is a cluster (resp., co-end) in the contact algebra $(CO(X_0), \delta_{(X,X_0)})$; conversely, for every cluster (resp., co-end) $\Gamma$ in $(CO(X_0), \delta_{(X,X_0)})$ there exists a point $x \in X_1$ such that $\Gamma = \Gamma_{x,X_0}$;*

$(3-4)$ *For every $F, G \in CO(X_0)$,* (19) *holds.*

*A topological triple $(X, X_0, X_1)$ is called a* normal 3-contact space *if it satisfies axiom* (3−1) *and the following two conditions:*

(3−2$N$) *$X_1$ is a compact Hausdorff space;*

(3−3$N$) *If $x \in X_1$ then the set $\Gamma_{x,X_0}$ is a cluster in the CA $(CO(X_0), \delta_{(X,X_0)})$; conversely, for every cluster $\Gamma$ in $(CO(X_0), \delta_{(X,X_0)})$ there exists a point $x \in X_1$ such that $\Gamma = \Gamma_{x,X_0}$.*

*Replacing in Definitions* 4.2, 4.3 *and* 4.4 *the word "extensional" with the word "N-regular" (resp., "regular", "normal"), we introduce the notions of* canonical N-regular (resp., regular; normal) contact algebra of an N-regular (resp., regular; normal) 3-contact space $\underline{X}$, *as well as the notions of* canonical N-regular (resp., regular; normal) 3-contact space of an N-regular (resp., regular; normal) contact algebra $(B, C)$, *and the notions of* isomorphism *between such spaces. An N-regular (resp., regular; normal) 3-contact space $(X, X_0, X_1)$ is called* connected *(resp.,* extremally disconnected*) if the space $X_1$ is connected (resp., the space $X_0$ is extremally disconnected).*

**Lemma 6.2.** *For every N-regular (respectively, regular; normal) 3-contact space $(X, X_0, X_1)$ there exists a unique reflexive and symmetric binary relation $R$ on $X_0$ such that $(X, X_0, X_1, R)$ is an N-regular (resp., regular; normal) 3-precontact space.*

Proof. It is similar to the proof of Lemma 4.5. □

As in the case of extensional contact algebras, the above lemma implies easily the following corollaries:

**Corollary 6.3.** *Let $(X, X_0, X_1)$ be a normal 3-contact space. Then, for every $F, G \in CO(X_0)$, we have that* (19) *holds.*

**Corollary 6.4.** *Let $(X, X_0, X_1)$ be an N-regular (respectively, regular; normal) 3-contact space. Then the canonical N-regular (respectively, regular; normal) contact algebra of the N-regular (respectively, regular; normal) 3-contact space $(X, X_0, X_1)$ is indeed an N-regular (respectively, regular; normal) contact algebra. Also, the Boolean algebra $RC(X, X_0, X_1)$ is a dense subset of the Boolean algebra $RC(X_1)$.*

**Proposition 6.5.** *Let $\underline{B} = (B, C)$ be an N-regular (respectively, regular; normal) CA. Then the canonical N-regular (respectively, regular; normal) 3-contact space $(X, X_0, X_1)$ of $\underline{B}$ is indeed an N-regular (respectively, regular; normal) 3-contact space.*

Proof. It is similar to the proof of Proposition 4.7. □

**Theorem 6.6** (New representation theorem for N-regular and regular contact algebras)**.**

(a) *Let $(B, C)$ be an N-regular (resp., regular) contact algebra and let $(X, X_0, X_1)$ be the canonical N-regular (resp., regular) 3-contact space of $(B, C)$. Then the function $g_1^B$, defined in (21), is a CA-isomorphism between $(B, C)$ and the canonical N-regular (resp., regular) contact algebra $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$ of the N-regular (resp., regular) 3-contact space $(X, X_0, X_1)$. The sets $RC(X_1)$ and $RC(X, X_0, X_1)$ coincide iff the algebra $(B, C)$ is complete iff the space $X_0$ is extremally disconnected. If the N-regular (resp., regular) contact algebra $(B, C)$ is complete, then it is connected iff its canonical N-regular (resp., regular) 3-contact space $(X, X_0, X_1)$ is connected.*

(b) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, N-regular (resp., regular) contact algebras and the class of all, up to respective isomorphism, N-regular (resp., regular) 3-contact spaces.*

(c) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) complete N-regular (resp., regular) contact algebras and the class of all, up to respective isomorphism, (connected) extremally disconnected N-regular (resp., regular) 3-contact spaces.*

Proof. It follows from Lemma 6.2 and Theorem 5.17 (and is analogous to the proof of Theorem 2.31=[9, Theorem 7.9]). □

**Theorem 6.7** (New representation theorem for normal contact algebras)**.**

(a) *Let $(B, C)$ be a normal contact algebra and let $(X, X_0, X_1)$ be the canonical normal 3-contact space of $(B, C)$. Then the function $g_1^B$, defined in (21), is a CA-isomorphism between $(B, C)$ and the canonical normal contact algebra $(RC(X, X_0, X_1), C_{(X,X_0,X_1)})$ of the normal 3-contact space $(X, X_0, X_1)$. The sets $RC(X, X_0, X_1)$ and $RC(X_1)$ coincide iff the algebra $(B, C)$ is complete iff the space $X_0$ is extremally disconnected. The normal contact algebra $(B, C)$ is connected iff its canonical normal 3-contact space $(X, X_0, X_1)$ is connected.*

(b) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) normal contact algebras and the class of all, up to isomorphism, (connected) normal 3-contact spaces.*

(c) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) complete normal contact algebras and the class of all, up to isomorphism, (connected) extremally disconnected normal 3-contact spaces.*

Proof. It follows from Lemma 6.2 and Theorem 5.18 (and is analogous to the proof of Theorem 2.31=[9, Theorem 7.9]). □

**Definition 6.8** ([6])**.** *An N-regular space $X$ is said to be* CN-regular *if every cluster in $(RC(X), C_X)$ is fixed. A regular space $(X, \tau)$ is called* C-regular *if every co-end in $(RC(X), C_X)$ is fixed.*

**Proposition 6.9.** *If $(X, X_0, X_1)$ is an extremally disconnected N-regular (resp., regular) 3-contact space, then the space $X_1$ is CN-regular (resp., C-regular).*

Proof. It is similar to the proof of Proposition 4.11. □

**Proposition 6.10.** *An N-regular $T_1$-space (resp., regular $T_2$-space) $X$ is CN-regular (resp., C-regular) iff $X$ is homeomorphic to the space* (Clust($RC(X)$, $C_X), \tau$) *(resp.,* (Coend($RC(X), C_X), \tau$)*), where the topology $\tau$ has as a closed base the family*

$$\{g_1(F) = \{\Gamma \in Clust(RC(X), C_X) \mid F \in \Gamma\} \mid F \in RC(X)\}$$

*(resp.,* $\{g_2(F) = \{\Gamma \in \mathrm{Coend}(RC(X), C_X) \mid F \in \Gamma\} \mid F \in RC(X)\}$*).*

Proof. See [6, Proposition 4.7 and the proof of I(ii) on page 241] for the N-regular case. For the proof of the case of regular spaces see [6, Proposition 4.10, Theorem 4.2 and the proof of II(ii) on page 241]. □

**Proposition 6.11.** *If $(X, X_0, X_1)$ and $(X', X'_0, X'_1)$ are two N-regular (resp., regular; normal) extremally disconnected 3-contact spaces such that the spaces $X_1$ and $X'_1$ are homeomorphic, then $(X, X_0, X_1)$ and $(X', X'_0, X'_1)$ are isomorphic.*

Proof. It is similar to the proof of Proposition 4.13. The only difference is that we now use Theorems 6.6(c) and 6.7(c) instead of Theorem 4.8(c). □

**Corollary 6.12** ([6])**.** *There exists a bijective correspondence between the class of all, up to CA-isomorphism, N-regular (resp., regular; normal) (connected)* *complete contact algebras and the class of all, up to homeomorphism, (connected) CN-regular $T_1$-spaces (resp., C-regular $T_2$-spaces; compact $T_2$-spaces).*

Proof. Using Propositions 6.10 and 6.11, we can argue as in the proof of Corollary 4.14. □

## 7. Some classes of compact $T_0$-extensions.

**Definition 7.1.** *An* extension *of a space $X$ is a pair $(Y, f)$, where $Y$ is a space and $f : X \longrightarrow Y$ is a dense embedding of $X$ into $Y$.*

*Two extensions $(Y_i, f_i)$, $i = 1, 2$, of $X$ are called* isomorphic *(or* equivalent*) if there exists a homeomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ f_1 = f_2$. Clearly, the relation of isomorphism is an equivalence in the class of all extensions of $X$; the equivalence class of an extension $(Y, f)$ of $X$ will be denoted by $[(Y, f)]$.*

*We write*

$$(Y_1, f_1) \leq (Y_2, f_2)$$

*and say that the extension $(Y_2, f_2)$ is* projectively larger *than the extension $(Y_1, f_1)$ if there exists a continuous mapping $f : Y_2 \longrightarrow Y_1$ such that $f \circ f_2 = f_1$. This relation is a* preorder *(i.e., it is reflexive and transitive). Setting for every two extensions $(Y_i, f_i)$, $i = 1, 2$, of a space $X$, $[(Y_1, f_1)] \leq [(Y_2, f_2)]$ iff $(Y_1, f_1) \leq (Y_2, f_2)$, we obtain a well-defined relation on the class of all, up to equivalence, extensions of $X$; obviously, it is also a preorder (see, e.g., [1]).*

*We write*

$$(Y_1, f_1) \leq_{in} (Y_2, f_2)$$

*and say that the extension $(Y_2, f_2)$ is* injectively larger *than the extension $(Y_1, f_1)$ if there exists a continuous mapping $f : Y_1 \longrightarrow Y_2$ such that $f \circ f_1 = f_2$ and $f$ is a homeomorphism from $Y_1$ to the subspace $f(Y_1)$ of $Y_2$. This relation is a preorder. Setting for every two extensions $(Y_i, f_i)$, $i = 1, 2$, of a space $X$, $[(Y_1, f_1)] \leq_{in} [(Y_2, f_2)]$ iff $(Y_1, f_1) \leq_{in} (Y_2, f_2)$, we obtain a well-defined relation on the class of all, up to equivalence, extensions of $X$; obviously, it is also a preorder (see, e.g., [1]).*

**Notation 7.2.** *Let $Y$ be a space. We will denote by* C-semireg$(Y)$ *(resp., by* ConC-semireg$(Y)$*) the class of all, up to equivalence, (connected) C-semiregular extensions of $Y$.*

*If $B$ is a Boolean algebra, then we denote by CRel$(B)$ (resp., CCRel$(B)$) the set of all (connected) contact relations on $B$. We define a relation "$\leq$" on the set CRel$(B)$ setting, for any $C_1, C_2 \in$ CRel$(B)$, $C_1 \leq C_2 \iff C_1 \supseteq C_2$. We will denote again by "$\leq$" the restriction of the relation "$\leq$" to the set CCRel$(B)$.*

The next theorem was proved in [9]. We will give here a sketch of its proof because it and the functions defined in it will be used later on.

**Theorem 7.3** ([9]). *Let $Y$ be an extremally disconnected compact Hausdorff space and $B = RC(Y)$. Then the ordered sets $(CRel(B), \leq)$ and $(C\text{-semireg}(Y), \leq)$, as well as the ordered sets $(CRel(B), \subseteq)$ and $(C\text{-semireg}(Y), \leq_{in})$, are isomorphic (see Definition 7.1 for the relations "$\leq$" and "$\leq_{in}$" on $C\text{-semireg}(Y)$). Also, the ordered sets $(CCRel(B), \leq)$ and $(ConC\text{-semireg}(Y), \leq)$, as well as the ordered sets $(CCRel(B), \subseteq)$ and $(ConC\text{-semireg}(Y), \leq_{in})$, are isomorphic.*

Sketch of the proof. Let $(X, f)$ be a C-semiregular extensions of $Y$. Set $X' = f(Y)$. Then, clearly, the map

$$e : (RC(X'), \delta_{(X,X')}) \longrightarrow (RC(X), C_X), \quad F \mapsto \mathrm{cl}_X(F),$$

is a CA-isomorphism (note that $RC(X') = CO(X')$). For every $F, G \in B$, set

$$FC_{(X,f)}G \iff \mathrm{cl}_X(f(F)) \cap \mathrm{cl}_X(f(G)) \neq \emptyset, \tag{26}$$

i.e., $FC_{(X,f)}G \iff f(F)\delta_{(X,X')}f(G)$. Then, obviously, $(B, C_{(X,f)})$ is a contact algebra. Set

$$\varphi(X, f) = (B, C_{(X,f)}).$$

Clearly, two equivalent extension of $Y$ define two coinciding contact relations on the Boolean algebra $B$. Thus we have that $\varphi([(X, f)]) = (B, C_{(X,f)})$ and, for simplicity, we will denote by the same letter $\varphi$ the induced map on the set of equivalence classes of the C-semiregular extensions of $Y$.

Conversely, let $C$ be a contact relation on the Boolean algebra $B$ and let $(\widehat{X}, \widehat{X}_0)$ be the canonical 2-contact space of the complete contact algebra $(B, C)$ (see Definition 2.28). Then, by the definition of the space $\widehat{X}_0$ and the Stone Representation Theorem, we have that the map

$$\widehat{f} : Y \longrightarrow \widehat{X}, \quad y \mapsto u_y,$$

(see (5) for the notation $u_y$) is a homeomorphic embedding and $\widehat{f}(Y) = \widehat{X}_0$. Hence, $(\widehat{X}, \widehat{f})$ is an extension of the space $Y$. Using Lemma 2.34, we obtain that $\widehat{X}$ is a C-semiregular space. So, $(\widehat{X}, \widehat{f})$ is a C-semiregular extension of the space $Y$. Set

$$\psi(B, C) = (\widehat{X}, \widehat{f}).$$

Then

$$\psi = \varphi^{-1}, \tag{27}$$

$\varphi$ is an isomorphism between the ordered sets ($C$-*semireg*$(Y), \leq$) and (*CRel*$(B), \leq$), and also between the ordered sets ($C$-*semireg*$(Y), \leq_{in}$) and (*CRel*$(B), \subseteq$). Clearly, this implies that $C$-*semireg*$(Y)$ is a set and the preorders "$\leq$" and "$\leq_{in}$" on $C$-*semireg*$(Y)$, defined in Definition 7.1, are, in fact, orders. Also,

Now, the assertions about connected contact relations on $B$ follow immediately. $\square$

Recall that a subspace $A$ of a topological space $X$ is said to be *2-combinatorially embedded in* $X$ ([3]) if the closures in $X$ of any two disjoint closed in $A$ subsets of $A$ are disjoint; also, a subspace $A$ of a topological space $X$ is said to be *open combinatorially embedded in* $X$ ([17]) if, for any two open in $A$ subsets of $A$, the disjointness of their closures in $A$ implies the disjointness of their closures in $X$.

Clearly, if $A$ is 2-combinatorially embedded in $X$ then $A$ is open combinatorially embedded in $X$; also, if $A$ is a normal subspace of $X$, then $A$ is open combinatorially embedded in $X$ iff $A$ is 2-combinatorially embedded in $X$. Using Lemma 3.1, we obtain immediately the following assertion as well:

**Fact 7.4.** *Let $A$ be a dense subspace of a topological space $X$. Then $A$ is open combinatorially embedded in $X$ if and only if the contact algebras $(RC(A), C_A)$ and $(RC(X), C_X)$ are CA-isomorphic.*

Recall that if $(X, \mathfrak{T})$ is a topological space and $x \in X$, then the point $x$ is said to be an *u-point* ([9]) if for every $U, V \in \mathfrak{T}$, $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$ implies that $x \in \mathrm{cl}(U \cap V)$. In [9] we proved that the set of all u-points of a C-semiregular space has very interesting and unexpected properties. In the next theorem, we recall (without proof) our result from [9] and show that this set is connected not only with our notion of 2-contact space but also with all kinds of 3-contact spaces introduced here. Theorem 7.5 will be used later on in the proofs of our results about extensions.

**Theorem 7.5.** *Let $(X, \mathfrak{T})$ be a C-semiregular space. Then:*

$(a)$ ([9, Corollary 7.21]) *The set*

$$u(X) = \{x \in X \mid x \textit{ is an u-point of } X\}$$

*endowed with its subspace topology is a dense extremally disconnected compact Hausdorff subspace of $(X, \mathfrak{T})$; the pair $(X, u(X))$ is a 2-contact space; if $X_0$ is a dense extremally disconnected compact Hausdorff subspace of $(X, \mathfrak{T})$, then $X_0 \equiv u(X)$;*

*(b) if $X_1$ is a dense C-weakly regular (resp., dense CN-regular; dense C-regular; dense compact Hausdorff) subspace of $X$ which is open combinatorially embedded in $X$, then the triple $(X, u(X), X_1)$ is an extensional (resp., N-regular; regular; normal) 3-contact space;*

*(c) if $X_1$ and $X_1'$ are two dense C-weakly regular (resp., dense CN-regular; dense C-regular; dense compact Hausdorff) subspaces of $X$ which are open combinatorially embedded in $X$, then $X_1 \equiv X_1'$.*

Proof. (b) Let $X_1$ be a dense C-weakly regular subspace of $X$ which is open combinatorially embedded in $X$. Set $X_0 = u(X)$. Then, by (a), the pair $(X, X_0)$ is a 2-contact space. Obviously, the axiom (3ECS2) is fulfilled. So, it remains only to prove that the axiom (3ECS3) is satisfied. First of all, since $X_1$ is open combinatorially embedded in $X$, we obtain, using Fact 7.4, Lemma 3.1 and the definition of the relation $\delta_{(X,X_0)}$, that the function

$$i_r : (RC(X_0), \delta_{(X,X_0)}) \longrightarrow (RC(X_1), C_{X_1}),\ F \mapsto X_1 \cap \mathrm{cl}_X(F), \tag{28}$$

is a CA-isomorphism.

Let now $\Gamma$ be a maximal clan in $(RC(X_0), \delta_{(X,X_0)})$. Then $\Gamma' = i_r(\Gamma)$ is a maximal clan in $(RC(X_1), C_{X_1})$. Thus there exists $x \in X_1$ such that $\Gamma' = \sigma_x^{X_1}$. Then we have that, for any $F \in RC(X_0)$, $F \in \Gamma \iff x \in i_r(F) \iff x \in X_1 \cap \mathrm{cl}_X(F) \iff x \in \mathrm{cl}_X(F)$. Hence $\Gamma \equiv \Gamma_{x,X_0}$ and $x \in X_1$.

Conversely, let $x \in X_1$. Then $i_r(\Gamma_{x,X_0}) = \{i_r(F) \mid F \in RC(X_0), x \in \mathrm{cl}_X(F)\} = \{i_r(F) \mid F \in RC(X_0), x \in X_1 \cap \mathrm{cl}_X(F)\} = \{i_r(F) \mid F \in RC(X_0), x \in i_r(F)\} = \sigma_x^{X_1}$. By [6, Proposition 4.4(ii) and Definition 4.4], $\sigma_x^{X_1}$ is a maximal clan in $(RC(X_1), C_{X_1})$. Therefore $\Gamma_{x,X_0}$ is a maximal clan in $(RC(X_0), \delta_{(X,X_0)})$.

All this shows that the triple $(X, u(X), X_1)$ is an extensional 3-contact space.

The proof for the remaining three cases is analogous. The only difference is that instead of [6, Proposition 4.4(ii) and Definition 4.4], we have to use [6, Lemma 4.1] (respectively, [6, Lemma 4.2]; Proposition 5.10(a) and [6, Lemma 4.1]).

(c) Let $X_1$ and $X_1'$ be two dense C-weakly regular subspaces of $X$ which are open combinatorially embedded in $X$. Set $X_0 = u(X)$. In the proof of (b), we have shown, in particular, that the function

$$i_b : X_1 \longrightarrow \mathrm{MClans}(RC(X_0), \delta_{(X,X_0)}),\ x \mapsto \Gamma_{x,X_0},$$

is a bijection. Hence, the function

$$i_b' : X_1' \longrightarrow \mathrm{MClans}(RC(X_0), \delta_{(X,X_0)}),\ x \mapsto \Gamma_{x,X_0},$$

is also a bijection. Thus, for proving that $X_1 \equiv X_1'$, it remains to show that if $x, y \in X$, then $\Gamma_{x,X_0} \equiv \Gamma_{y,X_0}$ iff $x = y$.

Let $x, y \in X$ and $\Gamma_{x,X_0} \equiv \Gamma_{y,X_0}$. By Lemma 3.1, the map

$$e = e_{X_0,X} : RC(X_0) \longrightarrow RC(X),\ F \mapsto \mathrm{cl}_X(F),$$

is a Boolean isomorphism. Thus $e(\Gamma_{x,X_0}) = \sigma_x^X$ and $e(\Gamma_{y,X_0}) = \sigma_y^X$. Hence $\sigma_x^X \equiv \sigma_y^X$. Now, [6, Proposition 4.2(i)] implies that $x = y$.

The converse implication is clear. So, $X_1 \equiv X_1'$.

The proof for the remaining three cases is analogous. □

**Notation 7.6.** *Let $B$ be a Boolean algebra. We denote by ECRel($B$) (resp., by ECCRel($B$)) the set of all (connected) extensional contact relations on $B$. Clearly, the set ECRel($B$) (resp., ECCRel($B$)) is a subset of the set CRel($B$) (resp., CCRel($B$)). The restriction of the relation "$\leq$" (defined in* 7.2 *on the set CRel($B$) (resp., CCRel($B$))) on the set ECRel($B$) (resp., ECCRel($B$)) will be denoted again by "$\leq$".*

*Let $Y$ be a compact Hausdorff extremally disconnected space. The class of all, up to equivalence, C-semiregular extensions $[(X, f)]$ of $Y$ such that $X$ contains a dense (connected) C-weakly regular subspace $X_1$ having the following property:*

$$(29)\quad \forall F, G \in CO(Y), \mathrm{cl}_X(f(F)) \cap \mathrm{cl}_X(f(G)) \neq \emptyset \quad \textit{iff}\ \mathrm{cl}_X(f(F)) \cap \mathrm{cl}_X(f(G)) \cap X_1 \neq \emptyset,$$

*will be denoted by EC-semireg($Y$) (resp., by EConC-semireg($Y$)).*

*Note that the condition "$X_1$ satisfies* (29)*" is equivalent to the following one: "$X_1$ is open combinatorially embedded in $X$". Hence, using Theorem* 7.5*, we obtain that EC-semireg($Y$) $\subseteq$ C-semireg($Y$) and EConC-semireg($Y$) $\subseteq$ ConC-semireg($Y$) (see* 7.2 *for C-semireg($Y$) and ConC-semireg($Y$)).*

**Theorem 7.7.** *Let $Y$ be an extremally disconnected compact Hausdorff space and $B = CO(Y)$. Then the ordered sets (ECRel($B$), $\leq$) and (EC-semireg($Y$), $\leq$), as well as the ordered sets (ECRel($B$), $\subseteq$) and (EC-semireg($Y$), $\leq_{in}$), are isomorphic. Further, the ordered sets (ECCRel($B$), $\leq$) and (EConC-semireg($Y$), $\leq$), as well as the ordered sets (ECCRel($B$), $\subseteq$) and (EConC-semireg($Y$), $\leq_{in}$), are isomorphic. (See Definition* 7.1 *for the notation "$\leq$" and "$\leq_{in}$" on the respective extensions.)*

Proof. Clearly, $B$ is a complete Boolean algebra and $B = RC(Y)$. By Theorem 7.3, we have that the ordered sets (*CRel*($B$), $\leq$) and (*C-semireg*($Y$), $\leq$), as well as the ordered sets $(\mathit{CRel}(B), \subseteq)$ and $(\mathit{C\text{-}semireg}(Y), \leq_{in})$, are isomorphic. We will show that the restrictions of these isomorphisms to, respectively, $(\mathit{ECRel}(B), \leq)$ and $(\mathit{ECRel}(B), \subseteq)$ are the desired isomorphisms.

Let $C$ be an extensional contact relation on $B$ and $(X, X_0, X_1)$ be the canonical extensional 3-contact space of the complete contact algebra $(B, C)$ (see Definition 4.3). Then, by Definition 4.3, $(X, X_0)$ is the canonical 2-contact space of the contact algebra $(B, C)$. Hence $\psi(B, C) = (X, f)$ (see the sketch of the proof of Theorem 7.3 for $\psi$ and $f$), $(X, f)$ is a C-semiregular extension of $Y$ and $X_0 = f(Y)$. Now Proposition 4.11, Lemma 4.5 and (19) show that $[(X, f)] \in \mathit{EC\text{-}semireg}(Y)$. Therefore, $\psi(\mathit{ECRel}(B)) \subseteq \mathit{EC\text{-}semireg}(Y)$. We will show that $\psi(\mathit{ECRel}(B)) = \mathit{EC\text{-}semireg}(Y)$.

Let $[(X, f)] \in \mathit{EC\text{-}semireg}(Y)$ and $X_1$ be a dense C-weakly regular subset of $X$ such that (29) is satisfied. We have that $\varphi([(X, f)]) = (B, C_{(X,f)})$ (see the sketch of the proof of Theorem 7.3 for $\varphi$ and $C_{(X,f)}$). Using twice Lemma 3.1, we obtain that the correspondence $F \mapsto X_1 \cap \mathrm{cl}_X(f(F))$ is a Boolean isomorphism between the Boolean algebras $B$ and $RC(X_1)$. Further, (29) implies that this isomorphism is in fact a CA-isomorphism between the contact algebras $(B, C_{(X,f)})$ and $(RC(X_1), C_{X_1})$. Since $X_1$ is a weakly regular space, we obtain, by Proposition 3.5, that $(B, C_{(X,f)})$ is an extensional contact algebra. Now, by (27), $\psi(B, C_{(X,f)})$ is equivalent to $(X, f)$. Hence, $\psi(\mathit{ECRel}(B)) = \mathit{EC\text{-}semireg}(Y)$. Thus, using the sketch of the proof of Theorem 7.3, we obtain that $\psi$ is an isomorphism between the ordered sets $(\mathit{ECRel}(B), \leq)$ and $(\mathit{EC\text{-}semireg}(Y), \leq)$, as well as between the ordered sets $(\mathit{ECRel}(B), \subseteq)$ and $(\mathit{EC\text{-}semireg}(Y), \leq_{in})$.

Now, the last two assertions of our theorem follow easily. □

**Notation 7.8.** *Let $B$ be a Boolean algebra. We denote by NECRel$(B)$ (resp., by NECCRel$(B)$) the set of all (connected) extensional contact relations on $B$ satisfying N-regularity axiom. Clearly, the set NECRel$(B)$ (resp., NECCRel$(B)$) is a subset of the set CRel$(B)$ (resp., CCRel$(B)$). The restriction of the relation "$\leq$" (defined in* 7.2 *on the set CRel$(B)$ (resp., CCRel$(B)$)) on the set NECRel$(B)$ (resp., NECCRel$(B)$) will be denoted again by "$\leq$".*

*Let $Y$ be a compact Hausdorff extremally disconnected space. The class of all, up to equivalence, C-semiregular extensions $[(X, f)]$ of $Y$ such that $X$ contains a dense (connected) CN-regular $T_1$-subspace $X_1$ having the property* (29)*, will be denoted by NC-semireg$(Y)$ (resp., by NConC-semireg$(Y)$).*

*Note that the condition "$X_1$ satisfies* (29)*" is equivalent to the following one: "$X_1$ is open combinatorially embedded in $X$". Hence, using Theorem* 7.5*, we obtain that NC-semireg$(Y) \subseteq$ C-semireg$(Y)$ and NConC-semireg$(Y) \subseteq$ ConC-semireg$(Y)$ (see* 7.2 *for the notation C-semireg$(Y)$ and ConC-semireg$(Y)$).*

**Theorem 7.9.** *Let $Y$ be an extremally disconnected compact Hausdorff space and $B = CO(Y)$. Then the ordered sets $(\mathit{NECRel}(B), \leq)$ and $(\mathit{NC}\text{-semireg}(Y), \leq)$, as well as the ordered sets $(\mathit{NECRel}(B), \subseteq)$ and $(\mathit{NC}\text{-semireg}(Y), \leq_{in})$ are isomorphic. Also, the ordered sets $(\mathit{NECCRel}(B), \leq)$ and $(\mathit{NConC}\text{-semireg}(Y), \leq)$, as well as the ordered sets $(\mathit{NECCRel}(B), \subseteq)$ and $(\mathit{NConC}\text{-semireg}(Y), \leq_{in})$, are isomorphic. (See Definition* 7.1 *for the notation "$\leq$" and "$\leq_{in}$" on the respective extensions.)*

Proof. It is similar to the proof of Theorem 7.7. □

**Notation 7.10.** *Let $B$ be a Boolean algebra. We denote by $\mathit{RECRel}(B)$ (resp., by $\mathit{RECCRel}(B)$) the set of all (connected) extensional contact relations on $B$ satisfying Regularity axiom. Clearly, the set $\mathit{RECRel}(B)$ (resp., $\mathit{RECCRel}(B)$) is a subset of the set $\mathit{CRel}(B)$ (resp., $\mathit{CCRel}(B)$). The restriction of the relation "$\leq$" (defined in* 7.2 *on the set $\mathit{CRel}(B)$ (resp., $\mathit{CCRel}(B)$)) on the set $\mathit{RECRel}(B)$ (resp., $\mathit{RECCRel}(B)$) will be denoted again by "$\leq$".*

*Let $Y$ be a compact Hausdorff extremally disconnected space. The class of all, up to equivalence, $C$-semiregular extensions $[(X, f)]$ of $Y$ such that $X$ contains a dense (connected) $C$-regular $T_2$-subspace $X_1$ having the property* (29), *will be denoted by $\mathit{RC}\text{-semireg}(Y)$ (resp., by $\mathit{RConC}\text{-semireg}(Y)$).*

*Note that the condition "$X_1$ satisfies* (29)*" is equivalent to the following one: "$X_1$ is open combinatorially embedded in $X$". Hence, using Theorem* 7.5*, we obtain that $\mathit{RC}\text{-semireg}(Y) \subseteq C\text{-semireg}(Y)$ and $\mathit{RConC}\text{-semireg}(Y) \subseteq \mathit{ConC}\text{-semireg}(Y)$ (see* 7.2 *for the notation $C\text{-semireg}(Y)$ and $\mathit{ConC}\text{-semireg}(Y)$).*

**Theorem 7.11.** *Let $Y$ be an extremally disconnected compact Hausdorff space and $B = CO(Y)$. Then the ordered sets $(\mathit{RECRel}(B), \leq)$ and $(\mathit{RC}\text{-semireg}(Y), \leq)$, as well as the ordered sets $(\mathit{RECRel}(B), \subseteq)$ and $(\mathit{RC}\text{-semireg}(Y), \leq_{in})$ are isomorphic. Also, the ordered sets $(\mathit{RECCRel}(B), \leq)$ and $(\mathit{RConC}\text{-semireg}(Y), \leq)$, as well as the ordered sets $(\mathit{RECCRel}(B), \subseteq)$ and $(\mathit{RConC}\text{-semireg}(Y), \leq_{in})$, are isomorphic. (See Definition* 7.1 *for the notation "$\leq$" and "$\leq_{in}$" on the respective extensions.)*

Proof. It is similar to the proof of Theorem 7.7. □

**Notation 7.12.** *Let $B$ be a Boolean algebra. We denote by $\mathit{IECRel}(B)$ (resp., by $\mathit{IECCRel}(B)$) the set of all (connected) extensional contact relations on $B$ satisfying the Interpolation axiom ((Ctr#)). Clearly, the set $\mathit{IECRel}(B)$ (resp., $\mathit{IECCRel}(B)$) is a subset of the set $\mathit{CRel}(B)$ (resp., $\mathit{CCRel}(B)$). The restriction of the relation "$\leq$" (defined in* (7.2) *on the set $\mathit{CRel}(B)$ (resp., $\mathit{CCRel}(B)$)) on the set $\mathit{IECRel}(B)$ (resp., $\mathit{IECCRel}(B)$) will be denoted again by "$\leq$".*

*Let $Y$ be a compact Hausdorff extremally disconnected space. The class of all, up to equivalence, C-semiregular extensions $[(X,f)]$ of $Y$ such that $X$ contains a dense (connected) compact $T_2$-subspace $X_1$ having the property* (29), *will be denoted by IC-semireg$(Y)$ (resp., by IConC-semireg$(Y)$).*

*Note that the condition "$X_1$ satisfies* (29)*" is equivalent to the following one: "$X_1$ is open combinatorially embedded in $X$". Hence, using Theorem* 7.5, *we obtain that IC-semireg$(Y) \subseteq$ C-semireg$(Y)$ and IConC-semireg$(Y) \subseteq$ ConC-semireg$(Y)$ (see* 7.2 *for the notation C-semireg$(Y)$ and ConC-semireg$(Y)$).*

**Theorem 7.13.** *Let $Y$ be an extremally disconnected compact Hausdorff space and $B = CO(Y)$. Then the ordered sets* (IECRel$(B), \leq$) *and* (IC-semireg$(Y)$, $\leq$)*, as well as the ordered sets* (IECRel$(B), \subseteq$) *and* (IC-semireg$(Y), \leq_{in}$) *are isomorphic. Also, the ordered sets* (IECCRel$(B), \leq$) *and* (IConC-semireg$(Y), \leq$)*, as well as the ordered sets* (IECCRel$(B), \subseteq$) *and* (IConC-semireg$(Y), \leq_{in}$)*, are isomorphic. (See Definition* 7.1 *for the notation "$\leq$" and "$\leq_{in}$" on the respective extensions.)*

P r o o f. It is similar to the proof of Theorem 7.7. □

Recall that (see, e.g., [18]) if $X$ is a regular space then a space $EX$ is called an *absolute of* $X$ if there exists a perfect irreducible map $\pi_X : EX \longrightarrow X$ and every perfect irreducible preimage of $EX$ is homeomorphic to $EX$; it is well-known that: a) the absolute is unique up to homeomorphism; b) a space $Y$ is an absolute of a regular space $X$ iff $Y$ is an extremally disconnected Tychonoff space for which there exists a perfect irreducible map $\pi_X : Y \longrightarrow X$; c) if $X$ is a compact Hausdorff space then $EX = S(RC(X))$, where $S$ is the Stone contravariant functor (see (3)).

**Proposition 7.14.** *Let $Y$ be a compact Hausdorff extremally disconnected space and $B = RC(Y)(= CO(Y))$. Then there exists a bijective correspondence between the set of all normal contact relations on $B$ and the set of all, up to homeomorphism, compact Hausdorff spaces $X$ such that $Y$ is homeomorphic to $EX$.*

P r o o f. It follows from Corollary 6.12, the Stone Duality Theorem and (c) from the preceding paragraph. □

**Theorem 7.15.** *Let $Z$ be a C-weakly regular (resp., CN-regular; C-regular; compact Hausdorff) space and $(X, X_0, X_1)$ be the canonical extensional (resp., N-regular; regular; normal) 3-contact space of the contact algebra $(RC(Z), C_Z)$. Then the function*

$$\varkappa : Z \longrightarrow X,\ z \mapsto \sigma_z^Z, \tag{30}$$

*is a dense homeomorphic embedding of $Z$ in $X$. Set $\varkappa Z = X$. Then $(\varkappa Z, \varkappa)$ is the unique, up to equivalence, C-semiregular extension of $Z$ such that $\varkappa(Z)$ is open combinatorially embedded in $\varkappa Z$.*

P r o o f. Let $Z$ be a C-weakly regular space and $(X, X_0, X_1)$ be the canonical extensional 3-contact space of the complete contact algebra $(RC(Z), C_Z)$. Then, by [6, Proposition 4.4], we have that MClans$(RC(Z), C_Z) = \{\sigma_z^Z \mid z \in Z\}$. Now, Definition 4.3 and Proposition 4.12 (together with its proof) imply that the spaces $Z$ and $\varkappa(Z) \equiv X_1$ are homeomorphic. Since $(X, X_0, X_1)$ is an extensional 3-contact space, we obtain that $\varkappa(Z)$ is a dense subspace of $X$. Using (19) and Fact 7.4, we obtain that $\varkappa(Z)$ is open combinatorially embedded in $X$ and, thus, in $\varkappa Z$. By Lemma 2.34, the space $X$ is C-semiregular. Let now $(cZ, c)$ be a C-semiregular extension of $Z$ such that $c(Z)$ is open combinatorially embedded in $cZ$. Then, by Theorem 7.5(b), $(cZ, u(cZ), c(Z))$ is an extensional 3-contact space. Set, for short, $X' = cZ$, $X'_0 = u(X')$, $X'_1 = c(Z)$, $\underline{B} = (RC(Z), C_Z)$, $\underline{B}_1 = (RC(X'_1), C_{X'_1})$ and $\underline{B}' = (RC(X'_0), \delta_{(X', X'_0)})$. Clearly, the map

$$\varphi_c : \underline{B} \longrightarrow \underline{B}_1,\ F \mapsto c(F),$$

is a CA-isomorphism. Since $X'_1$ is open combinatorially embedded in $X'$, we obtain, using Fact 7.4, that the map

$$\varphi_1 : \underline{B}_1 \longrightarrow \underline{B}',\ F \mapsto X'_0 \cap \mathrm{cl}_{X'}(F),$$

is a CA-isomorphism. Let $\varphi = \varphi_1 \circ \varphi_c$. Then

$$\varphi : \underline{B} \longrightarrow \underline{B}'$$

is a CA-isomorphism. Let $(\widehat{X'}, \widehat{X'_0}, \widehat{X'_1})$ be the canonical extensional 3-contact space of the complete contact algebra $\underline{B}'$. Set

$$f_1 : X' \longrightarrow \widehat{X'},\ x \mapsto \Gamma_{x, X'_0}.$$

Then, by Theorem 4.8(c) (and its proof), we have that $f_1$ is a homeomorphism. Let

$$f_0 : X \longrightarrow \widehat{X'},\ \Gamma \mapsto \varphi(\Gamma).$$

Then the fact that $\varphi$ is a CA-isomorphism implies easily that $f_0$ is a homeomorphism. Set $f = (f_1)^{-1} \circ f_0$. Then

$$f : X \longrightarrow X'$$

is a homeomorphism. Using the proof of Theorem 7.5, we obtain that, for every $x \in X_1'$, $\varphi_1(\sigma_x^{X_1'}) = \Gamma_{x,X_0'}$. Now we obtain that, for every $z \in Z$,

$$f(\varkappa(z)) = f(\sigma_z^Z) = (f_1)^{-1}(f_0(\sigma_z^Z)) = (f_1)^{-1}(\varphi(\sigma_z^Z)) = (f_1)^{-1}(\varphi_1(\varphi_c(\sigma_z^Z)))$$
$$= (f_1)^{-1}(\varphi_1(\sigma_{c(z)}^{X_1'})) = (f_1)^{-1}(\Gamma_{c(z),X_0'}) = c(z).$$

So, we have obtained that there exists a homeomorphism $f : \varkappa Z \longrightarrow cZ$ such that $f \circ \varkappa = c$. Therefore, the extensions $(cZ, c)$ and $(\varkappa Z, \varkappa)$ are equivalent.

The remaining three cases can be proved analogously. □

**Corollary 7.16.** *Let $Z$ be a compact Hausdorff space. Then:*

*(a) $(\varkappa Z, \varkappa)$ is the unique, up to equivalence, C-semiregular extension of $Z$ such that $\varkappa(Z)$ is 2-combinatorially embedded in $\varkappa Z$;*

*(b) the absolute $EZ$ of $Z$ is a dense subspace of $\varkappa Z$ and $EZ \equiv u(\varkappa Z)$;*

*(c) the triple $(\varkappa Z, EZ, \varkappa(Z))$ is the canonical normal 3-contact space of the contact algebra $(RC(Z), C_Z)$.*

Proof. It follows from Theorem 7.15 and the fact that $Z$ is normal (indeed, then $\varkappa(Z)$ is 2-combinatorially embedded in $\varkappa Z$ iff $\varkappa(Z)$ is open combinatorially embedded in $\varkappa Z$). □

**8. Conclusion.** In this paper we supplied with proofs our representation theorems for extensional (and other kinds) (pre)contact algebras announced in the 16-pages-long paper [8]. The writing of the full version of [8] was always postponed because we knew that we had to write a very long paper and we had no time for doing this having many new projects which had to be finished urgently. Now, when we finally have completed the paper, we see that the work on it is still not finished. For example, in [10] we extended to duality theorems our representation theorems proved in [9]. Our next task is to extend to duality theorems the representation theorems which we proved here.

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## REFERENCES

[1] B. Banaschewski. Extensions of topological spaces. *Canad. Math. Bull.* **7** (1964), 1–22.

[2] W. Comfort, S. Negrepontis. Chain Conditions in Topology. Cambridge Tracts in Mathematics, vol. **79**. Cambridge-New York, Cambridge University Press, 1982.

[3] E. Čech, J. Novák. On regular and combinatorial imbedding. *Časopis Pěst. Mat. Fys.* **72** (1947), 7–16.

[4] de Vries, H. Compact Spaces and Compactifications, an Algebraic Approach. Dissertation, University Amsterdam. Assen, Van Gorcum, 1962.

[5] G. Dimov. Some generalizations of the Fedorchuk duality theorem – I. *Topology Appl.* **156**, *4* (2009), 728–746.

[6] G. Dimov, D. Vakarelov. Contact algebras and region-based theory of space: a proximity approach – I. *Fund. Inform.* **74**, *2–3* (2006), 209–249.

[7] G. Dimov, D. Vakarelov. Contact algebras and region-based theory of space: a proximity approach – II. *Fund. Inform.* **74**, *2–3* (2006), 251–282.

[8] G. Dimov, D. Vakarelov. Topological representation of precontact algebras. In: Relation Methods in Computer Science (Eds W. MacCaull, M. Winter, I. Duentsch) Lecture Notes in Computer Science, vol. **3929**. Berlin Heidelberg, Springer-Verlag, 2006, 1–16.

[9] G. Dimov, D. Vakarelov. Topological representation of precontact algebras and a connected version of the Stone Duality Theorem – I. *Topology Appl.* **227** (2017), 64–101.

[10] G. Dimov, E. Ivanova-Dimova, D. Vakarelov. A generalization of the Stone duality theorem. *Topology Appl.* **221** (2017), 237–261.

[11] I. Düntsch, D. Vakarelov. Region-based theory of discrete spaces: a proximity approach. In: Proceedings of Fourth International Conference Journées de l'informatique Messine(Eds M. Nadif, A. Napoli, E. SanJuan, E., A. Sigayret), Metz, France, 2003, 123–129; Journal version in: *Ann. Math. Artif. Intell.* **49**, *1–4* (2007), 5–14.

[12] I. Düntsch, M. Winter. A representation theorem for Boolean contact algebras. *Theoret. Comput. Sci. (B)* **347**, *3* (2005), 498–512.

[13] R. Engelking. General Topology. Warszawa, PWN, 1977.

[14] Fedorchuk, V. V. Boolean $\delta$-algebras and quasi-open mappings. *Sibirsk. Mat. Ž.* **14**, *5* (1973), 1088–1099 (in Russian); English translation in *Siberian Math. J.* **14** (1973), 759–767 (1974).

[15] A. Galton. The mereotopology of discrete spaces. In: Spatial Information Theory, Proceedings of the International Conference COSIT'99 (Eds C. Freksa, D. M. Mark) Lecture Notes in Computer Science, 251–266, Springer-Verlag, 1999.

[16] M. Ganster. On strongly s-regular spaces. *Glasnik Mat. Ser. III*, **25(45)**, *1* (1990), 195–201.

[17] J. R. Porter, Ch. Votaw. $S(\alpha)$ spaces and regular Hausdorff extensions. *Pacific J. Math.* **45** (1973), 327–345.

[18] J. R. Porter, R. G. Woods. Extensions and absolutes of Hausdorff spaces. New York, Springer-Verlag, 1988.

[19] D. Randell, Z. Cui, A. G. Cohn. A spatial logic based on regions and connection. In: Principles of Knowledge Representation and Reasoning: Proc. of the 3rd International Conference (Eds B. Nebel, W. Swartout, C. Rich) Cambridge, Morgan Caufmann, 1992, 165–176.

[20] Yu. M. Smirnov. On proximity spaces. *Mat. Sb. (N.S.)* **31(73)**, *3* (1952), 543–574.

[21] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.* **40**, *1* (1936), 37–111.

[22] D. Vakarelov, G. Dimov, I. Düntsch, B. Bennett. A proximity approach to some region-based theory of space. *J. Appl. Non-classical Logics* **12**, *3-4* (2002), 527–559.

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