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PRODUCT OF PROPER SHAPE EQUIVALENCES OVER FINITE COVERINGS IS AN EQUIVALENCE

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Abstract. In this paper we prove that in the category of proper shape over finite coverings from $Sh_{pF}(X) = Sh_{pF}(X')$ and $Sh_{pF}(Y) = Sh_{pF}(Y')$ it follows that $Sh_{pF}(X \times Y) = Sh_{pF}(X' \times Y')$. Also, we give an example in which the product of two morphisms is not morphism in the category of proper shape.

1. Introduction. The theory of shape provides a good tool for classification of compact metric spaces by considering their global properties. To get a theory that describes global properties of locally-compact metric spaces that are not compact, Ball and Sher at [2] define the proper shape. In [1] and [8] an intrinsic approach to proper shape is given. In [7] is explained their equivalence and the theory is explained in more details. In [6] it is shown that the approach in [2] and the intrinsic approach to proper shape are equivalent. In this paper we will follow the definition from [6] and [10].

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In [9], by using the intrinsic approach, authors obtain new category of proper shape for spaces with compact spaces of quasicomponents. This category is denoted by Sh_{pF}-proper shape over finite coverings. The advantage of using the category Sh_{pF} lies in the fact that it allows us to work only with finite coverings.

In this paper we investigate the proper shape of product space in the category of proper shape over finite coverings.

In [4], Kodama showed the following results about product of shapes:

– For X compact and X', Y, Y' locally-compact:

If $Sh_p(X) = Sh_p(X')$ and $Sh_p(Y) = Sh_p(Y')$, then

$$Sh_p(X \times Y) = Sh_p(X' \times Y').$$

In Borsuks book [3], two types of shape for non-compact spaces are considered: weak shape-Sh_W and (strong) shape-Sh_S. We have the following results from [3] related to these categories:

– If $Sh_W(X) = Sh_W(X')$ and $Sh_W(Y) = Sh_W(Y')$, then

$$Sh_W(X \times Y) = Sh_W(X' \times Y').$$

– An example is given where:

$$Sh_S(X) = Sh_S(X'), \text{ but } Sh_S(X \times Y) \neq Sh_S(X' \times Y').$$

In the Section 3, we prove the main result:

If $Sh_{pF}(X) = Sh_{pF}(X')$ and $Sh_{pF}(Y) = Sh_{pF}(Y')$, then

$$Sh_{pF}(X \times Y) = Sh_{pF}(X' \times Y'),$$

where spaces are assumed to be locally-compact separable metrizable and with compact spaces of quasicomponents.

The definition of proper shape over finite coverings is based on sequence of (not necessarily continuous) functions (f_n), each $f_n : X \to Y$ continuous over a covering $\mathcal{V}_n$ and coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \cdots$ are cofinal among all coverings of open sets with compact boundary.

First, we prove theorems about product of functions continuous over a covering and product of proper functions. We also show that product of two cofinal sequences over finite coverings from X, Y gives a cofinal sequence in $X \times Y$.

In the Section 4, we give an example to show that we could not always form the product morphism in the category of proper shape.

2. Product of functions continuous over a covering. Product of proper functions. First, we will prove theorems about functions continuous over a covering.

Definition 2.1. *Suppose $\mathcal{V}$ is a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$-continuous at point x, if there exists a neighborhood U_x of x, and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$*

A function $f : X \to Y$ is $\mathcal{V}$-continuous, if it is $\mathcal{V}$-continuous at every point $x \in X$

Lemma 2.2. *Let X, Y, Z be arbitrary topological spaces. If functions $f : Z \to X$, $g : Z \to Y$ are $\mathcal{U}$, $\mathcal{V}$-continuous, respectively, then the function $h : Z \to X \times Y$ defined by $h(z) = (f(z), g(z))$ is $\mathcal{U} \times \mathcal{V}$-continuous.*

Proof. Let $z \in Z$ and let D_z, G_z be neighborhoods of z in Z such that $f(D_z) \subseteq U$ and $g(G_z) \subseteq V$ for some $U \in \mathcal{U}$, $V \in \mathcal{V}$. The set $A_z = D_z \cap G_z$, is a neighborhood of z and also:

$$h(A_z) = (f, g)(A_z) \subseteq f(A_z) \times g(A_z) \subseteq$$
$$\subseteq f(D_z) \times g(G_z) \subseteq U \times V.$$

From $U \times V \in \mathcal{U} \times \mathcal{V}$, it implies that h is $\mathcal{U} \times \mathcal{V}$-continuous. $\square$

Lemma 2.3. *Let X, Y, X', Y' be arbitrary topological spaces. If functions $f : X \to X'$, $g : Y \to Y'$ are $\mathcal{U}', \mathcal{V}'$-continuous, respectively, then the function $h : X \times Y \to X' \times Y'$ defined by $h(x, y) = (f(x), g(y))$ is $\mathcal{U}' \times \mathcal{V}'$-continuous.*

Proof. Let $(x, y) \in X \times Y$ and let U_x, V_y be neighborhoods of x, y in X, Y, respectively, such that $f(U_x) \subseteq U'$ and $g(V_y) \subseteq V'$ for some $U' \in \mathcal{U}', V' \in \mathcal{V}'$. The set $U_x \times V_y$ is a neighborhood of (x, y) in $X \times Y$ and:

$$h(U_x \times V_y) = \{h(a, b) \,|a \in U_x, b \in V_y\} =$$

$$= \{(f(a), g(b)) \,|a \in U_x, b \in V_y\} = f(U_x) \times g(V_y) \subseteq U' \times V'.$$

From $U' \times V' \in \mathcal{U}' \times \mathcal{V}'$, it follows that h is $\mathcal{U}' \times \mathcal{V}'$-continuous. $\square$

Lemma 2.4. *Let X, Y, X', Y' be arbitrary topological spaces and I the unit interval. If the functions $f : X \times I \to X'$, $g : Y \times I \to Y'$ are $\mathcal{U}', \mathcal{V}'$-continuous, respectively, then the function $h : (X \times Y) \times I \to X' \times Y'$ defined by $h((x, y), s) = (f(x, s), g(y, s))$ is $\mathcal{U}' \times \mathcal{V}'$-continuous.*

Proof. Let $((x, y), s) \in (X \times Y) \times I$ and let $U_x \times J_x$, $V_y \times J_y$ be neighborhoods of $(x, s), (y, s)$ in $X \times I$, $Y \times I$, respectively, such that $f(U_x \times J_x) \subseteq U'$ and $g(V_y \times J_y) \subseteq V'$ for some $U' \in \mathcal{U}', V' \in \mathcal{V}'$. The set $(U_x \times V_y) \times (J_x \cap J_y)$ is neighborhood of $((x,y),s)$ in $(X \times Y) \times I$ and the following holds:

$$h((U_x \times V_y) \times (J_x \cap J_y)) =$$
$$= \{h((a,b),s)\,|a \in U_x, b \in V_y, s \in J_x \cap J_y\} =$$
$$= \{(f(a,s), g(b,s))\,|a \in U_x, b \in V_y, s \in (J_x \cap J_y)\} \subseteq$$
$$\subseteq f(U_x \times (J_x \cap J_y)) \times g(V_y \times (J_x \cap J_y)) \subseteq U' \times V'.$$

Since $U' \times V' \in \mathcal{U}' \times \mathcal{V}'$, it follows that h is $\mathcal{U}' \times \mathcal{V}'$-continuous. □

Now, we define a proper (noncontinuous) function.

Definition 2.5. *A function $f : X \to Y$ is proper if for every compact D in Y there exists compact set C in X such that $f(X \backslash C) \subseteq Y \backslash D$.*

Note that if the function is continuous then this notion of proper function, coincides with the standard one.

Lemma 2.6. *Let X, Y, Z be arbitrary topological spaces. If $f : Z \to X$ and $g : Z \to Y$ are proper functions, then the function $H : Z \to X \times Y$ defined by $H(z) = (f(z), g(z))$ is proper.*

Proof. Let $\underline{K} \subseteq X \times Y$ be compact. Taking $P = p_X(\underline{K})$ and $Q = p_Y(\underline{K})$, yields $\underline{K} \subseteq P \times Q$. The set P is compact in X, so there exists a compact $M \subseteq Z$ such that $f(Z \backslash M) \subseteq X \backslash P$.
Similarly, from compactness of Q in Y, there exists a compact set $N \subseteq Z$ such that:

$$g(Z \backslash N) \subseteq Y \backslash Q.$$

The set $A = M \cup N$ is compact in Z. Now, by the fact that:

$$H(Z \backslash A) \subseteq f(Z \backslash A) \times g(Z \backslash A) \subseteq$$
$$\subseteq f(Z \backslash M) \times g(Z \backslash N) \subseteq (X \backslash P) \times (Y \backslash Q).$$

and

$$(X \backslash P) \times (Y \backslash Q) \subseteq (X \times Y) \backslash (P \times Q) \subseteq (X \times Y) \backslash \underline{K}.$$

we conclude that $H(Z \backslash A) \subseteq (X \times Y) \backslash \underline{K}$. So, H is proper function. □

Lemma 2.7. *Let X, Y, Z, V be arbitrary topological spaces. If $f : X \to Z$ and $g : Y \to V$ are proper functions, then the function $H : X \times Y \to Z \times V$ defined by $H(x,y) = (f(x), g(y))$ is proper.*

Proof. Let $\underline{K} \subseteq Z \times V$ be compact. Take $P = p_Z(\underline{K})$ and $Q = p_V(\underline{K})$, then $\underline{K} \subseteq P \times Q$. From compactness of P in Z, there exists a compact set $C_X \subseteq X$ such that

$$f(X \backslash C_X) \subseteq Z \backslash P.$$

Similarly, from compactness of Q in V, there exists a compact set $C_Y \subseteq Y$ such that:

$$g\left(Y \backslash C_Y\right) \subseteq V \backslash Q.$$

Then:

$$\begin{aligned}
&H\left((X \times Y) \backslash (C_X \times C_Y)\right) = \\
&= H\left(X \times (Y \backslash C_Y)\right) \cup H\left((X \backslash C_X) \times Y\right) = \\
&= \left(f\left(X\right) \times g\left(Y \backslash C_Y\right)\right) \cup \left(f\left(X \backslash C_X\right) \times g\left(Y\right)\right) \subseteq \\
&\subseteq \left(Z \times (V \backslash Q)\right) \cup \left((Z \backslash P) \times V\right) = (Z \times V) \backslash (P \times Q)
\end{aligned}$$

from where we conclude that H is proper function. □

Lemma 2.8. *Let X, Y, Z, V be arbitrary topological spaces and I the unit interval. If $f : X \times I \to Z$ and $g : Y \times I \to V$ are proper functions, then the function $H : (X \times Y) \times I \to Z \times V$ defined by $H((x, y), s) = (f(x, s), g(y, s))$ is proper.*

Proof. Let $\underline{K} \subseteq Z \times V$ be compact. Take $P = p_Z\left(\underline{K}\right)$ and $Q = p_V\left(\underline{K}\right)$, then $\underline{K} \subseteq P \times Q$. From compactness of P in Z, there exists compact set $C_X \times I \subseteq X \times I$ such that

$$f\left((X \times I) \setminus (C_X \times I)\right) \subseteq Z \backslash P.$$

Similarly, from compactness of Q in V, there exists compact set $C_Y \times I \subseteq Y \times I$ such that:

$$g\left((Y \times I) \setminus (C_Y \times I)\right) \subseteq V \backslash Q.$$

Then:

$$\begin{aligned}
&H(((X \times Y) \times I) \backslash ((C_X \times C_Y) \times I)) = \\
&= H\left(((X \backslash C_X \times Y) \cup (X \times Y \backslash C_Y)) \times I\right) \subseteq \\
&\subseteq (f(X \backslash C_X \times I) \times g(Y \times I)) \cup (f(X \times I) \times g(Y \backslash C_Y \times I)) \subseteq \\
&\subseteq (Z \times V) \backslash (P \times Q)
\end{aligned}$$

from where we have that H is proper function. □

Lemma 2.9. *Let $f : X_1 \to Y$, $g : X_2 \to Y$ be proper functions, where X_1, X_2 are closed in X. If $f(x) = g(x)$ for $x \in X_1 \cap X_2$, then the combined function $h : X_1 \cup X_2 \to Y$ is proper.*

Proof. Let $K \subseteq Y$ be compact subset of Y. From the supposition there exists compact subsets $D_1 \subseteq X_1$, $D_2 \subseteq X_2$ such that:

$$f\left(X_1 \backslash D_1\right) \subseteq Y \backslash K \quad \text{and} \quad g\left(X_2 \backslash D_2\right) \subseteq Y \backslash K.$$

The set $D = D_1 \cup D_2$ is compact in X and also we have:

$$h(X \backslash D) = h((X_1 \cup X_2) \backslash D) = h(X_1 \backslash D) \cup h(X_2 \backslash D) \subseteq$$
$$\subseteq h(X_1 \backslash D_1) \cup h(X_2 \backslash D_2) \subseteq Y \backslash K.$$

Thus, we have proved that h is proper. $\square$

Proposition 2.10. *Let $W = ((X \backslash C) \times Y) \cup (X \times (Y \backslash K))$, where C, K are compact subsets of the T_2 spaces X, Y, respectively. Then the boundary ∂W is compact.*

Proof. From

$$((X \backslash C) \times Y) \cup (X \times (Y \backslash K)) = (X \times Y) \backslash (C \times K)$$

we have $(X \times Y) \backslash W = C \times K$.

On the other hand, from $\partial W = \partial((X \times Y) \backslash W)$, it implies that $\partial W = \partial(C \times K)$. The set $C \times K$ is compact, so it is a closed subset of $X \times Y$. From $\partial(C \times K) \subseteq C \times K$, the set $\partial(C \times K)$ is compact as closed subset of compact set. $\square$

3. Product of equivalences of proper shape over finite coverings. About the notion of space of quasicomponents, we refer to [9].

Now we will give a short description of the category of proper shape over finite coverings from [9].

For arbitrary space W we denote by $\mathrm{Cov}_F(W)$ the set of all finite open coverings of W consisting of sets with compact boundary.

Definition 3.1. *Proper functions $f, g : X \to Y$ are $\mathcal{V}$-properly homotopic, if there exists a proper function $F : X \times I \to Y$ such that:*

1) *$F : X \times I \to Y$ is st$\mathcal{V}$-continuous.*
2) *$F : X \times I \to Y$ is $\mathcal{V}$-continuous at all points of $X \times \partial I$.*
3) *$F(x, 0) = f(x)$, $F(x, 1) = g(x)$. We denote this by $f \sim_{p\mathcal{V}} g$.*

Definition 3.2. *A sequence (f_n) of proper functions $f_n : X \to Y$ is proper proximate sequence over finite coverings from X to Y, if there exists cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \cdots$ from $Cov_F(Y)$ such that for every $m \geq n$, f_n and f_m are $\mathcal{V}_n$-properly homotopic. We say that (f_n) is proper proximate sequence over $(\mathcal{V}_n)$.*

Definition 3.3. *Two proximate sequences $(f_n), (g_n)$ are homotopic if there exists a cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \cdots$ in $Cov_F(Y)$, such that $(f_n), (g_n)$ are proximate sequences over $(\mathcal{V}_n)$ and for all $n \in \mathbb{N}$, f_n and g_n are properly $\mathcal{V}_n$-homotopic. We denote this by $(f_n) \sim_{pF} (g_n)$.*

In the paper [9], it is proven that locally-compact separable metric spaces with compact spaces of quasicomponents and homotopy classes of proximate sequences over finite coverings form a category of proper shape over finite coverings.

Two spaces X, Y have same proper shape over finite coverings if there exists a proper sequence over finite coverings $(f_n) : X \to Y$ and a proper proximate sequence $({f_n}^*) : Y \to X$ such that:

$$(f_n)({f_n}^*) \sim_{pF} 1_Y, \quad ({f_n}^*)(f_n) \sim_{pF} 1_X.$$

We denote this by $Sh_{pF}(X) = Sh_{pF}(Y)$.

The following definitions are from [11].

Definition 3.4. *A topological space X is clp-compact if every clopen covering of X has a finite subcovering.*

Definition 3.5. *A product space $X \times Y$ is clp-rectangular if every clopen set W in $X \times Y$ is the union of clopen rectangles.*

We need the following theorem to prove the main result.

Theorem 3.6. *Let X and Y be locally compact separable metric spaces with compact spaces of quasicomponents. If $\mathcal{W} \in \mathrm{Cov}_F(X \times Y)$, then there exists $\mathcal{U} \in \mathrm{Cov}_F(X)$ and $\mathcal{V} \in \mathrm{Cov}_F(Y)$ such that $\mathcal{U} \times \mathcal{V} \prec \mathcal{W}$.*

P r o o f. Let $\mathcal{W} = \{W_1, W_2, \dots, W_n, W_1^\infty, W_2^\infty, \dots, W_k^\infty\} = \mathcal{W}^0 \cup \mathcal{W}^\infty$ be a covering form $Cov_F(X \times Y)$, where $\mathcal{W}^0 = \{W_1, W_2, \dots, W_n\}$ consists of sets with compact closure and $\mathcal{W}^\infty = \{W_1^\infty, W_2^\infty, \dots, W_k^\infty\}$ consists of sets with compact boundary. From [9] we could assume that there exists compact sets $C \subseteq X$, $D \subseteq Y$ such that

$$(X \times Y) \setminus (C \times D) = \bigcup_{i=1}^{k} W_i^\infty, \quad W_i^\infty \cap W_j^\infty = \emptyset \ \text{ for } \ i \neq j$$

and W_i^∞ is clopen in $(X \times Y) \setminus (C \times D)$ for every $i \in \{1, 2, \dots, k\}$.

Take the coverings $\mathcal{U} = \{U_1\} \cup \{X \backslash C\}$, $\mathcal{V} = \{V_1\} \cup \{Y \backslash D\}$ of X, Y, respectively, such that:

$$U_1 \text{ is neighborhood of } C \text{ , } V_1 \text{ is neighborhood of } D.$$

The sets U_1, V_1 are open and with compact closure in spaces X, Y, respectively.

Since X and Y have compact spaces of quasicomponents it follows that $Q(X \backslash U_1)$, $Q(Y \backslash V_1)$ are compact.

Now, taking into consideration the paper [5] the spaces

$$Q\left((X \backslash U_1) \times Y\right), \quad Q\left(X \times (Y \backslash V_1)\right)$$

are compact, thus $(X \backslash U_1) \times Y$, $X \times (Y \backslash V_1)$ are clp-compact [11], hence from [11] (Proposition 2.5) they are also clp-rectangular.

Take compact subsets $C_1 = \overline{U_1} \subseteq X$, $D_1 = \overline{V_1} \subseteq Y$.

Fix a point $y \in Y \backslash V_1$. If $x \in X$, then there exists a clopen neighborhood U_x^y of x in X and a clopen neighborhood V_x^y of y in $Y \backslash V_1$ such that $(x, y) \in U_x^y \times V_x^y \subseteq W_i^\infty$ for some $W_i^\infty \in \mathcal{W}^\infty$. From the compactness of $Q(X)$, there exists a finite subcovering $\mathcal{U}_y^0$ of X from the covering $\{U_x^y \,|x \in X\}$. The set $V_y = \bigcap\limits_{U_x^y \in \mathcal{U}_y^0} V_x^y$ is clopen in $Y \backslash V_1$. The family of sets $\{V_y \,|y \in Y \backslash V_1\}$ is a covering for $Y \backslash V_1$ consisting of clopen subsets of $Y \backslash V_1$, and hence it has a finite subcovering $\{V_{y_1}, V_{y_2}, \ldots, V_{y_p}\}$. Take $V'_{y_j} = V_{y_j} \cap Y \backslash D_1$ for $j \in \{1, 2, 3, \ldots, p\}$.

We obtained finite number of coverings $\mathcal{U}_{y_1}^0, \mathcal{U}_{y_2}^0, \ldots, \mathcal{U}_{y_p}^0$ such that for every $U \in \mathcal{U}_{y_i}^0$ and $V \in \{V_{y_1}, V_{y_2}, \ldots, V_{y_p}\}$ it follows that $U \times V \subseteq W_j^\infty$ for some $W_j^\infty \in \mathcal{W}^\infty$.

Similarly, we could find a finite covering $\{U_{x_1}, U_{x_2}, \ldots, U_{x_s}\}$ of $X \backslash U_1$ consisting of clopen subsets of $X \backslash U_1$ and a finite coverings $\mathcal{V}_{x_1}^0, \mathcal{V}_{x_2}^0, \ldots, \mathcal{V}_{x_s}^0$ of Y consisting of clopen subsets of Y.

Take $U'_{x_i} = U_{x_i} \cap X \backslash C_1$ for $i \in \{1, 2, 3, \ldots, s\}$.

For the compact set $C_1 \times D_1$ there exists a covering $\mathcal{U}^1$ of C_1 in X and a covering $\mathcal{V}^1$ of D_1 in Y such that $\mathcal{U}^1 \times \mathcal{V}^1 \prec \mathcal{W}$ and the sets of $\mathcal{U}^1$, $\mathcal{V}^1$ have compact closure.

We can choose a covering $\mathcal{U}$ of X from $Cov_F(X)$ and a covering $\mathcal{V}$ of Y from $Cov_F(Y)$ such that:

$$\mathcal{U} \prec \mathcal{U}_{y_1}^0, \ldots, \mathcal{U} \prec \mathcal{U}_{y_p}^0, \mathcal{U} \prec \{U_{x_1}{}', U_{x_2}{}', \ldots, U_{x_s}{}'\} \cup \mathcal{U}^1,$$
$$\mathcal{V} \prec \mathcal{V}_{y_1}^0, \ldots, \mathcal{V} \prec \mathcal{V}_{x_s}^0, \mathcal{V} \prec \{V_{y_1}{}', V_{y_2}{}', \ldots, V_{y_p}{}'\} \cup \mathcal{V}^1.$$

In this way we have constructed a covering $\mathcal{U}$ for X and $\mathcal{V}$ for Y consisting of sets with compact boundary with the property $\mathcal{U} \times \mathcal{V} \prec \mathcal{W}$. $\square$

Theorem 3.7. *Let X, Y be separable locally-compact (noncompact) metric spaces with compact spaces of quasicomponents. If $(f_n : Z \to X)$ and $(g_n : Z \to Y)$ are two proper proximate sequences over finite coverings, then:*

1) *The sequence $(h_n = (f_n, g_n) : Z \to X \times Y)$ is proper proximate sequence over finite coverings and:*

2) *If $(f_n) \sim_{pF} (f_n{}')$ and $(g_n) \sim_{pF} (g_n{}')$, then $(f_n, g_n) \sim_{pF} (f_n, g_n)$.*
(We say that h_n is diagonal product of f_n and g_n).

P r o o f. Let the assumptions of the theorem be fulfilled.

From Theorem 3.6 we have that for the covering $\mathcal{W} \in Cov_F(X \times Y)$, there exists $\mathcal{U} \in Cov_F(X)$ and $\mathcal{V} \in Cov_F(Y)$ such that $\mathcal{U} \times \mathcal{V} \prec \mathcal{W}$. Let $(f_n : Z \to X)$, $(g_n : Z \to Y)$ be proper proximate sequences over $(\mathcal{U}_n)$, $(\mathcal{V}_n)$, respectively, and take a cofinal sequence $\mathcal{W}_1 \succ \mathcal{W}_2 \succ \mathcal{W}_3 \succ \cdots$ in $Cov_F(X \times Y)$. We could find a subsequence $(\mathcal{U}_{n_k})$ from $(\mathcal{U}_n)$ and a subsequence $(\mathcal{V}_{n_k})$ from $(\mathcal{V}_n)$ with property $\mathcal{U}_{n_1} \times \mathcal{V}_{n_1} \succ \mathcal{U}_{n_2} \times \mathcal{V}_{n_2} \succ \mathcal{U}_{n_3} \times \mathcal{V}_{n_3} \succ \cdots$ and $\mathcal{U}_{n_k} \times \mathcal{V}_{n_k} \prec \mathcal{W}_k$ for $k \in \mathbb{N}$.

Now, define a sequence of coverings $(\mathcal{W}'_n)$ by:

$\mathcal{W}'_i = \{X \times Y\}$, for $1 \leq i < n_1$,

$\mathcal{W}'_i = \mathcal{W}_1$, for $n_1 \leq i < n_2$,

$\mathcal{W}'_i = \mathcal{W}_2$, for $n_2 \leq i < n_3$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\mathcal{W}'_i = \mathcal{W}_k$, for $n_k \leq i < n_{k+1}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The sequence $(\mathcal{W}'_n)$ is cofinal in $Cov_F(X \times Y)$ and $\mathcal{U}_n \times \mathcal{V}_n \prec \mathcal{W}'_n$ for every $n \in \mathbb{N}$, so we could investigate the sequence of functions $((f_n, g_n) : Z \to X \times Y)$ as a proper proximate sequence over $(\mathcal{W}'_n)$. We will prove that $(h_n : Z \to X \times Y)$ is proper proximate sequence over $(\mathcal{W}'_n)$.

1) h_n is proper function for every $n \in \mathbb{N}$:

Proof of 1). From Lemma 2.6 h_n is proper function for every $n \in \mathbb{N}$.

2) h_n is $\mathcal{W}'_n$-continuous for every $n \in \mathbb{N}$:

Proof of 2). From Lemma 2.2 product of $\mathcal{U}_n$-continuous function with $\mathcal{V}_n$-continuous function is $\mathcal{U}_n \times \mathcal{V}_n$-continuous. From the construction of $\mathcal{W}'_n$, it implies that h_n is $\mathcal{W}'_n$-continuous for every $n \in \mathbb{N}$.

3) If $m \geq n$, then h_n and h_m are $(\mathcal{W}'_n)$-properly homotopic.

Proof of 3). We define function $H_{n,m} : Z \times I \to X \times Y$ by:

$$H_{n,m}(z,s) = (F_{n,m}(z,s), G_{n,m}(z,s))$$

where $F_{n,m}(z,s)$ is a proper homotopy over $\mathcal{U}_n$ which connects the functions f_n and f_m (from $m \geq n$) and $G_{n,m}(z,s)$ is a proper homotopy over $\mathcal{V}_n$ connecting the functions g_n and g_m (from $m \geq n$).

The function $H_{n,m}$ is proper homotopy over $\mathcal{U}_n \times \mathcal{V}_n$. From $\mathcal{U}_n \times \mathcal{V}_n \prec \mathcal{W}'{}_n$ we have that $H_{n,m}$ is $\mathcal{W}'{}_n$-homotopy connecting $h_n = (f_n, g_n)$ and $h_m = (f_m, g_m)$.

4) Let $\left(h_n{}' = (f_n{}', g_n{}') : Z \to X \times Y\right)_n$ be proper proximate sequence over $(\mathcal{W}''_n)$, where $\mathcal{W}''_1 \succ \mathcal{W}''_2 \succ \mathcal{W}''_3 \succ \cdots$ is cofinal in $Cov_F(X \times Y)$ and also:

$$\left[(f_n{}')\right]_{pF} = \left[(f_n)\right]_{pF}, \quad \left[(g_n{}')\right]_{pF} = \left[(g_n)\right]_{pF}. \tag{1}$$

Then $\left[(h_n{}')\right]_{pF} = \left[(h_n)\right]_{pF}$.

Proof of 4). We have to show that there exists a cofinal sequence $\mathcal{M}_1 \succ \mathcal{M}_2 \succ \mathcal{M}_3 \succ \cdots$ of coverings from $\mathrm{Cov}_F(X \times Y)$ such that (h_n) and $\left(h_n{}'\right)$ are proximate sequences over $(\mathcal{M}_n)$ and for every $n \in \mathbb{N}$ it holds $h_n \sim_{\mathcal{M}_n} h_n{}'$. (they are $\mathcal{M}_n$-properly homotopic.)

From (1) there exists a cofinal sequence $\mathcal{F}_1 \succ \mathcal{F}_2 \succ \mathcal{F}_3 \succ \cdots$ in $\mathrm{Cov}_F(X)$ and $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \mathcal{G}_3 \succ \cdots$ from $\mathrm{Cov}_F(Y)$ such that:

$$f_n \sim_{\mathcal{F}_n} f_n{}' \text{ and } g_n \sim_{\mathcal{G}_n} g_n{}' \text{ for every } n \in \mathbb{N}.$$

From Theorem 3.6 there exists a cofinal sequence $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \mathcal{B}_3 \succ \cdots$ in $\mathrm{Cov}_F(X \times Y)$ such that $\mathcal{F}_n \times \mathcal{G}_n \prec \mathcal{B}_n$ for all $n \in \mathbb{N}$.

We will prove that (h_n) and $\left(h_n{}'\right)$ are properly homotopic proximate sequences over $(\mathcal{M}_n)$ where:

$$\mathcal{M}_n = \mathcal{B}_n \cup \mathcal{W}_n' \cup \mathcal{W}_n''.$$

The sequence $(\mathcal{M}_n)$ is cofinal in $Cov_F(X \times Y)$.

For every $n \in \mathbb{N}$, we define function $H_n : Z \times I \to X \times Y$ by:

$$H_n(z,s) = (F_n(z,s), G_n(z,s)),$$

where $F_n(z,s)$ is proper homotopy over $\mathcal{F}_n$ connecting f_n and $f_n{}'$ and $G_n(z,s)$ is proper homotopy over $\mathcal{G}_n$ between g_n and $g_n{}'$. Clearly, from Lemma 2.6 H_n is proper function. From $\mathcal{F}_n \times \mathcal{G}_n \prec \mathcal{M}_n$ and considering the Lemma 2.2, we have that $H_n(z,s)$ is $st\mathcal{M}_n$-continuous in $X \times I$ and $\mathcal{M}_n$-continuous in $X \times \partial I$. Since

$$H_n(z,0) = (F_n(z,0), G_n(z,0)) = (f_n(z), g_n(z)),$$
$$H_n(z,1) = (F_n(z,1), G_n(z,1)) = \left(f_n{}'(z), g_n{}'(z)\right),$$

it follows, the function $H_n(z,s)$ is proper $\mathcal{M}_n$-homotopy between $(f_n(z), g_n(z))$ and $\left(f_n{}'(z), g_n{}'(z)\right)$.

We proved that $[((f_n, g_n))_n]_{pF} = \left[\left(\left(f_n{}', g_n{}'\right)\right)_n\right]_{pF}$ so

$$[(h_n)_n]_{pF} = \left[\left(h_n{}'\right)_n\right]_{pF}. \qquad \square$$

Similarly as in Theorem 3.7, by taking in consideration Lemmas 2.3, 2.4, 2.7 and 2.8, we obtain the following theorem.

Theorem 3.8. *Let X, Y, X', Y' be separable locally-compact (noncompact) metric spaces with compact spaces of quasicomponents. If $\left(f_n : X \to X'\right)$ and $\left(g_n : Y \to Y'\right)$ are two proper proximate sequences over finite coverings, then:*

1) The sequence $(h_n = f_n \times g_n : X \times Y \to X' \times Y')$ is proper proximate sequence over finite coverings and:

2) If $(f_n) \sim_{pF} \left(f_n{}'\right)$ and $(g_n) \sim_{pF} \left(g_n{}'\right)$, then $(f_n \times g_n) \sim_{pF} (f_n{}' \times g_n{}')$. (We say that h_n is product of f_n and g_n).

P r o o f. Let (f_n), (g_n) be proper sequences over $(\mathcal{U'}_n)$, $(\mathcal{V'}_n)$, respectively. From Theorem 3.6 there exists a cofinal sequence of coverings $\mathcal{W'}_1 \succ \mathcal{W'}_2 \succ \mathcal{W'}_3 \succ \cdots$ of the space $X' \times Y'$ such that $\mathcal{U'}_n \times \mathcal{V'}_n \prec \mathcal{W'}_n$ for all $n \in \mathbb{N}$. From Lemma 2.3 and Lemma 2.7 h_n is a $\mathcal{W'}_n$-continuous proper function for every $n \in \mathbb{N}$. It suffices to show that h_n is proper sequence over $(\mathcal{W'}_n)$.

Let $m \geq n$. We will prove that the functions h_n, h_m are $\mathcal{W'}_n$-properly homotopic.

We define a function $\underline{H}_{n,m} : (X \times Y) \times I \to X' \times Y'$ by:

$$\underline{H}_{n,m}((x,y),s) = (F_{n,m}(x,s), G_{n,m}(y,s)),$$

where $F_{n,m}(x,s)$ is proper homotopy over $\mathcal{U'}_n$ connecting the functions f_n, f_m (from $m \geq n$) and $G_{n,m}(y,s)$ is proper homotopy over $\mathcal{V'}_n$ connecting the functions g_n, g_m (from $m \geq n$). The function $\underline{H}_{n,m}$ is $\mathcal{U'}_n \times \mathcal{V'}_n$-proper homotopy. From $\mathcal{U'}_n \times \mathcal{V'}_n \prec \mathcal{W'}_n$, it implies that $\underline{H}_{n,m}$ is $\mathcal{W'}_n$-proper homotopy connecting the functions $h_n = f_n \times g_n$ and $h_m = f_m \times g_m$. Similarly as in the previous theorem, by using Lemma 2.4 and Lemma 2.8, we could prove the uniqueness of (h_n). □

Theorem 3.9. *Let X, Y, X', Y' be locally-compact (noncompact) separable metric spaces with compact spaces of quasicomponents and $Sh_{pF}(X) = Sh_{pF}(X')$, $Sh_{pF}(Y) = Sh_{pF}(Y')$. Then $Sh_{pF}(X \times Y) = Sh_{pF}(X' \times Y')$ for the category of proper shape over finite coverings.*

P r o o f. From $Sh_{pF}(X) = Sh_{pF}(X')$: there exists a morphism $\big(f_n : X \to X'\big)$ over the sequence of coverings $\mathcal{U'}_1 \succ \mathcal{U'}_2 \succ \mathcal{U'}_3 \succ \cdots$ and a proper proximate sequence $\big({f_n}^* : X' \to X\big)$ over $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \mathcal{U}_3 \succ \cdots$ such that

$$\begin{aligned} &\big({f_n}^* : X' \to X\big) \circ \big(f_n : X \to X'\big) \sim_{pF} 1_X \text{ by a proper homotopy } F_n \\ (2) \quad &\text{and} \\ &\big(f_n : X \to X'\big) \circ \big({f_n}^* : X' \to X\big) \sim_{pF} 1_{X'} \text{ by a proper homotopy } {F_n}^*. \end{aligned}$$

On the other hand, from $Sh_{pF}(Y) = Sh_{pF}(Y')$, there exists proper proximate sequence $\big(g_n : Y \to Y'\big)$ over $\mathcal{V'}_1 \succ \mathcal{V'}_2 \succ \mathcal{V'}_3 \succ \cdots$ and a proper proximate sequence $\big({g_n}^* : Y' \to Y\big)$ over $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \cdots$ such that:

$$\begin{aligned} &\big({g_n}^* : Y' \to Y\big) \circ \big(g_n : Y \to Y'\big) \sim_{pF} 1_Y \text{ by a proper homotopy } G_n \\ (3) \quad &\text{and} \\ &\big(g_n : Y \to Y'\big) \circ \big({g_n}^* : Y' \to Y\big) \sim_{pF} 1_{Y'} \text{ by a proper homotopy } {G_n}^* \end{aligned}$$

From Theorem 3.6 there exists a cofinal sequence of coverings $\mathcal{W'}_1 \succ \mathcal{W'}_2 \succ \mathcal{W'}_3 \succ \cdots$ of the space $X' \times Y'$ such that $\mathcal{U'}_n \times \mathcal{V'}_n \prec \mathcal{W'}_n$ for all $n \in \mathbb{N}$.

Similarly, there exists a cofinal sequence $\mathcal{W}_1 \succ \mathcal{W}_2 \succ \mathcal{W}_3 \succ \cdots$ of $X \times Y$ such that $\mathcal{U}_n \times \mathcal{V}_n \prec \mathcal{W}_n$ for all $n \in \mathbb{N}$.

From [5], the space $Q(X \times Y)$ is compact, also $X \times Y$ is locally compact separable metric space. Thus, we could define proper shape over finite coverings for $X \times Y$.

From Theorem 3.8 we could define:

1) proper proximate sequence $\big(h_n : X \times Y \to X' \times Y'\big)$ over $\mathcal{W}'_1 \succ \mathcal{W}'_2 \succ \mathcal{W}'_3 \succ \cdots$ by

$$h_n(x,y) = (f_n(x), g_n(y)) \quad \text{for every} \quad n \in \mathbb{N}.$$

2) proper proximate sequence $\big({h_n}^* : X' \times Y' \to X \times Y\big)$ over $\mathcal{W}_1 \succ \mathcal{W}_2 \succ \mathcal{W}_3 \succ \cdots$ by

$${h_n}^*(x',y') = \big({f_n}^*(x'), {g_n}^*(y')\big) \quad \text{for every} \quad n \in \mathbb{N}.$$

Hence:

$$\begin{aligned} {h_n}^* \circ h_n(x,y) &= ({f_n}^* \times {g_n}^*) \circ (f_n \times g_n)\,(x,y) = \\ &= \left({f_n}^*\left(f_n(x)\right), {g_n}^*\left(g_n(y)\right)\right) \end{aligned}$$

It suffices to prove the following

$$\left({f_n}^*\left(f_n(x)\right), {g_n}^*\left(g_n(y)\right)\right) \sim_{pF} 1_{X \times Y}(x,y).$$

Using Lemma 2.4 and Lemma 2.8, we define the desired proper homotopy by:

$$\underline{K}_n((x,y),s) = (F_n(x,s), G_n(y,s))\,.$$

In a similar way, we could show that

$$\left(f_n\left({f_n}^*(x')\right), g_n\left({g_n}^*(y')\right)\right) \sim_{pF} 1_{X' \times Y'}(x',y')$$

by the proper homotopy ${\underline{K}_n}^*((x',y'),s) = \big({F_n}^*(x',s), {G_n}^*(y',s)\big)$.

Functions F_n, G_n, ${F_n}^*$, ${G_n}^*$ are taken from (2) and (3). $\square$

4. Example in proper shape. At the end we will show that is not possible to prove a result in proper shape theory, corresponding to the main result in proper shape over finite coverings (proven above).

For arbitrary space W we denote by $\operatorname{Cov}(W)$ the set of all open coverings of W consisting of sets with compact closure.

Definition 4.1. *A proper proximate net $(f_{\mathcal{V}} : X \to Y)$ is a net of functions $f_{\mathcal{V}} : X \to Y$, $f_{\mathcal{V}}$ is proper $\mathcal{V}$-continuous function, indexed by all coverings from* $\operatorname{Cov}(Y)$*, such that if $\mathcal{V} \prec \mathcal{W}$ then $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{W}$-properly homotopic.*

We say $(f_{\mathcal{V}})$ is a proper proximate net.

Definition 4.2. *Two proper proximate nets $(f_{\mathcal{U}})$, $(g_{\mathcal{V}}) : X \to Y$ are homotopic if for every $\mathcal{U} \in \operatorname{Cov}(Y)$, $f_{\mathcal{U}}$ and $g_{\mathcal{U}}$ are $\mathcal{U}$-properly homotopic.*

We denote this by $(f_{\mathcal{U}}) \sim_p (g_{\mathcal{V}})$.

Example. We will show that in the category of proper shape, product of morphisms is not always a morphism.

Proof. Let $X' = X = [0,1]$, $Y' = Y = [0,1)$. We have that X', Y', X, Y are separable, locally-compact metric spaces.

The set of all minimal finite coverings of X consisting of intervals is cofinal in $\mathrm{Cov}(X)$. We denote this set by $\mathrm{Cov}_M(X)$.

We consider the proper proximate nets:

$$\underline{f} = \{\ f_{\mathcal{U}} |\quad f_{\mathcal{U}} \text{ is } \mathcal{U}\text{-continuous and } \mathcal{U} \in \mathrm{Cov}_M(X)\}\colon X' \to X,$$
$$\underline{g} = \{\ g_{\mathcal{V}} |\quad g_{\mathcal{V}} \text{ is } \mathcal{V}\text{-continuous and } \mathcal{V} \in \mathrm{Cov}(Y)\}\colon Y' \to Y.$$

defined as follows:

Fix $\mathcal{U} \in Cov_M(X)$. There exists an integer $n_{\mathcal{U}}$ with the property: if $1/2 \in U \in \mathcal{U}$, then $\frac{1}{2} + \frac{1}{n_{\mathcal{U}}} \in U$.

For arbitrary $\mathcal{U} \in \mathrm{Cov}_M(X)$ define the function $f_{\mathcal{U}} : [0,1] \to [0,1]$ such that:

$$f_{\mathcal{U}}(x) = x \ \text{ if } \ x \neq \frac{1}{2} \ \text{ and } \ f_{\mathcal{U}}(\frac{1}{2}) = \frac{1}{2} + \frac{1}{n_{\mathcal{U}}}.$$

Fix $\mathcal{V} \in \mathrm{Cov}(Y)$. We define $g_{\mathcal{V}} : [0,1) \to [0,1)$ by $g_{\mathcal{V}}(y) = y$.

Clearly, the function $g_{\mathcal{V}}$ is continuous for every $\mathcal{V} \in \mathrm{Cov}\,[0,1)$ and function $f_{\mathcal{U}}$ is continuous in all points from $[0,1] \setminus \{\frac{1}{2}\}$, where $\mathcal{U} \in \mathrm{Cov}\,[0,1]$. For the point $x = \frac{1}{2}$, if we take a set $U \in \mathcal{U}$ that contains $\frac{1}{2}$, then we have that $f_{\mathcal{U}}(U) \subseteq U$. So, $f_{\mathcal{U}}$ is $\mathcal{U}$-continuous. It is also clear that functions $f_{\mathcal{U}}$, $g_{\mathcal{V}}$ are proper.

Let $\mathcal{U}_1 \prec \mathcal{U}_2$. We will prove that $f_{\mathcal{U}_1}$ and $f_{\mathcal{U}_2}$ are $\mathcal{U}_2$-properly homotopic. From [1] (Lemma 3.2) it is enough to prove that $f_{\mathcal{U}_1}$ and $f_{\mathcal{U}_2}$ are $\mathcal{U}_2$-near.

Let $x \in [0,1]$. If $x \neq 1/2$, then $f_{\mathcal{U}_1}(x) = x = f_{\mathcal{U}_2}(x)$. Now, take $x = 1/2$. Let $U_1 \in \mathcal{U}_1$ such that $1/2 \in U_1$. Hence $f_{\mathcal{U}_1}(\frac{1}{2}) = \frac{1}{2} + \frac{1}{n_{\mathcal{U}_1}}$. We could find $U_2 \in \mathcal{U}_2$ such that $U_1 \subseteq U_2$. From the fact $f_{\mathcal{U}_1}(\frac{1}{2}) \in U_1$, it implies that $f_{\mathcal{U}_1}(\frac{1}{2}) \in U_2$. On the other hand, from $1/2 \in U_2$ we have $f_{\mathcal{U}_2}(\frac{1}{2}) \in U_2$. Consequently, $f_{\mathcal{U}_1}$ and $f_{\mathcal{U}_2}$ are $\mathcal{U}_2$-near.

Similarly, $g_{\mathcal{V}_1}$,$g_{\mathcal{V}_2}$ are $\mathcal{V}_2$-near for $\mathcal{V}_1 \prec \mathcal{V}_2$.

We proved that $\underline{f} = \{\ f_{\mathcal{U}} |\quad \mathcal{U} \in \mathrm{Cov}_M(X)\}$, $\underline{g} = \{\ f_{\mathcal{V}} |\quad \mathcal{V} \in \mathrm{Cov}(Y)\}$ are proper proximate nets. Their product is defined as:

$$\underline{h} = \{f_{\mathcal{U}} \times g_{\mathcal{V}} |\ \mathcal{U} \in \mathrm{Cov}_M(X), \mathcal{V} \in \mathrm{Cov}(Y)\} : X' \times Y' \to X \times Y.$$

We will show that this net is not a proper proximate net.

Take a covering $\mathcal{W} \in \operatorname{Cov}(X \times Y) = \operatorname{Cov}([0,1] \times [0,1))$ defined by:

$\mathcal{W} = \{\ W_n|\quad n \in \mathbb{N}\} \cup \{\ T_n|\quad n \in \mathbb{N}\}$ where:

$W_1 = \left[0, \frac{1}{2} + \frac{1}{3}\right) \times \left[0, \frac{1}{2}\right),$

$W_2 = \left[0, \frac{1}{2} + \frac{1}{4}\right) \times \left(\frac{1}{4}, \frac{1}{2} + \frac{1}{4}\right),$

$W_3 = \left[0, \frac{1}{2} + \frac{1}{5}\right) \times \left(\frac{1}{2} + \frac{1}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right),$

. . . .

$W_n = \left[0, \frac{1}{2} + \frac{1}{n+2}\right) \times \left(\frac{1}{2} + \cdots + \frac{1}{2^{n-2}} + \frac{1}{2^n},\ \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right),$

. . . .

On the other side:

$T_1 = \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right),$

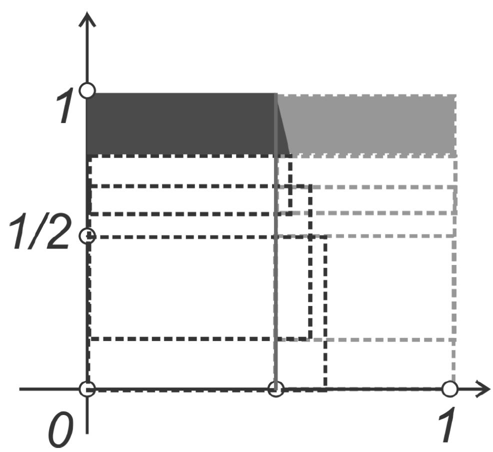
$T_2 = \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{4}, \frac{1}{2} + \frac{1}{4}\right),$

$T_3 = \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2} + \frac{1}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right),$

. . . .

$T_n = \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2} + \cdots + \frac{1}{2^{n-2}} + \frac{1}{2^n},\ \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right),$

. . . .



If we suppose that $\underline{h} = \{f_{\mathcal{U}} \times g_{\mathcal{V}} | \ \mathcal{U} \in \mathrm{Cov}_M(X),\ \mathcal{V} \in \mathrm{Cov}(Y)\}$ is a proximate net, then there must be a member $f_{\mathcal{U}} \times g_{\mathcal{V}}$ which is $\mathcal{W}$-continuous. We will prove that this is not possible. For fixed $y \in Y'$ the point $\left(\frac{1}{2}, y\right)$ is mapped in the point $\left(\frac{1}{2} + \frac{1}{n_{\mathcal{U}}}, y\right)$. If we fix $y_0 < 1$, near to 1, such that:

$$y_0 \in \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}} + \frac{1}{2^n}, \quad \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right)$$

and $\frac{1}{2} + \frac{1}{n_{\mathcal{U}}} > 1/2 + 1/(n+2)$ for n large enough. Then for every neighborhood $O \times M$ of $(1/2, y_0)$, the set $f_{\mathcal{U}}(O)$ consists of numbers less than $1/2$ and also contains the number $\frac{1}{2} + \frac{1}{n_{\mathcal{U}}} = f_{\mathcal{U}}(1/2)$. So, the set $f_{\mathcal{U}}(O) \times g_{\mathcal{V}}(M)$ is not inscribed in any set of the covering $\mathcal{W}$. We couldn't find a member $f_{\mathcal{U}} \times g_{\mathcal{V}}$ which would be $\mathcal{W}$-continuous, hence $\underline{h} = \{\ f_{\mathcal{U}} \times g_{\mathcal{V}} | \ \ \mathcal{U} \in \mathrm{Cov}_M(X),\ \mathcal{V} \in \mathrm{Cov}(Y)\}$ is not a proper proximate net. $\square$

Conclusion. Proper shape over finite coverings, as theory that classifies noncompact spaces, fulfills the following universal property:

If $Sh_{pF}(X) = Sh_{pF}(X')$ and $Sh_{pF}(Y) = Sh_{pF}(Y')$, then

$$Sh_{pF}(X \times Y) = Sh_{pF}(X' \times Y').$$

The following question arises:

Does the theory of proper shape from [2] have the same property without the assumption one of the spaces to be compact?

The example constructed in the end of the paper ensures us that, in the category of proper shape, we couldn't always obtain a morphism by product of two morphisms.

REFERENCES

[1] Y. Akaike, K. Sakai. Describing the proper n-shape category by using non-continuous functions. *Glas. Mat. Ser. III* **33(53)**, *2* (1998), 299–321.

[2] B. J. Ball, R. B. Sher. A theory of proper shape for locally compact metric spaces. *Bull. Amer. Math. Soc.* **79**, *5* (1973), 1023–1026.

[3] K. Borsuk. Theory of Shape. Mathematical Monographs vol. **59**. Warsaw, PWN – Polish Scientific Publishers, 1975.

[4] Y. Kodama. On product of shape and a question of Sher. *Pacific J. Math.* **72**, *1* (1977), 115–134.

[5] Gj. Markoski, A. Buklla. On product of spaces of quasicomponents. *Filomat* **31**, *20* (2017), 6307–6311.

[6] N. Shekutkovski, G. Markoski. Equivalence of the inverse system approach and the intrinsic approach to proper shape. *God. Zb. Inst. Mat. Prir.-Mat. Fak. Univ. Kiril Metodij Skopje* **41** (2008), 43–56.

[7] N. Shekutkovski, G. Markoski. Intrinsic definition of proper shape and spaces with compact components. Proceedings of IV Congress of the mathematicians of Republic of Macedonia, Struga, October 18–22, 2008, 272–286.

[8] N. Shekutkovski, G. Markoski. Proper shape and shape of compactificatiion. *God. Zb. Inst. Mat. Prir.-Mat. Fak. Univ. Kiril Metodij Skopje* **42** (2013), 27–38.

[9] N. Shekutkovski, G. Markoski. Proper shape over finite coverings. Topology Appl. **158**, *15* (2011), 2016–2021.

[10] N. Shekutkovski. Intrinsic shape – the proximate approach. *Filomat* **29**, *10* (2015), 2199–2205.

[11] J. Steprans, A. Shostak. Restricted compactness properties and their preservation under products. *Topology Appl.* **101**, *3* (2000), 213–229.

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