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MORE ON THE CARDINALITY OF $S(n)$-SPACES*

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Abstract. In this paper, for a topological space X and any positive integer n, we define the cardinal functions $d_n(X)$, $t_n(X)$ and $bt_n(X)$, called respectively $S(n)$-density, $S(n)$-tightness and $S(n)$-bitightness, and using them and recently introduced in [10] cardinal functions $\chi_n(X)$, $\psi_n(X)$, and $s_n(X)$, called respectively $S(n)$-character, $S(n)$-pseudocharacter, and $S(n)$-spread, we prove some cardinal inequalities for $S(n)$-spaces, which extend to the class of $S(n)$-spaces some results of Pospišil, Arhangel'skiĭ, Hajnal and Juhász, Shapirovskiĭ and Kočinac. Two representative results are: If X is an $S(n)$-space, then $|X| \leq 2^{2^{d_n(X)}}$ and $|X| \leq [d_n(X)]^{bt_n(X)}$.

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1. Introduction. The following two results of Pospišil [17], which are valid for every Hausdorff space X, are well known: $|X| \le 2^{2^{d(X)}}$ and $|X| \le [d(X)]^{\chi(X)}$. Kočinac in [15], for Urysohn spaces X, sharpened the first inequality to $|X| \le 2^{2^{d_\theta(X)}}$. As it was shown by Arhangel'skiĭ in [2] for Hausdorff spaces and by Cammaroto and Kočinac in [4] (see also [15]) for Urysohn spaces, the second inequality can be sharpened respectively to $|X| \le [d(X)]^{bt(X)}$ and $|X| \le [d_\theta(X)]^{bt_\theta(X)}$.

In this paper, for a topological space X and any positive integer n, we define the cardinal functions $S(n)$-density (denoted by $d_n(X)$), $S(n)$-tightness (denoted by $t_n(X)$), and $S(n)$-bitightness (denoted by $bt_n(X)$), and using them and recently introduced in [10] cardinal functions $S(n)$-character, $S(n)$-pseudo-character, and $S(n)$-spread, denoted respectively by $\chi_n(X)$, $\psi_n(X)$, and $s_n(X)$, we prove some cardinal inequalities for $S(n)$-spaces.

In particular, we extend the above-mentioned inequalities for the class of $S(n)$-spaces, where n is a positive integer, by showing that for every $S(n)$-space X we have $|X| \le 2^{2^{d_n(X)}}$ (Theorem 3.1) and $|X| \le [d_n(X)]^{bt_n(X)}$ (Theorem 3.5). Since $bt_n(X) \le t_n(X)\psi_{2n}(X)$, whenever X is an $S(n)$-space (Theorem 3.3), as a corollary we obtain Theorem 3.7: If X is an $S(n)$-space, then $|X| \le [d_n(X)]^{t_n(X)\psi_{2n}(X)}$. Extending in Theorems 3.13, 3.15, and 3.17 to $S(n)$-spaces a fundamental result about spread due to Shapirovskiĭ (see [19] or [12, Theorem 5.1]), in Theorems 3.19, 3.21 and 3.23 we obtain upper bounds for the $S(n)$-density of $S(n)$-spaces using the cardinal function $s_n(X)$. In the proofs of these theorems we use substantially Lemmas 3.10, 3.11 and 3.12 proved in [10]. As corollaries, in Theorem 3.20, 3.22 and 3.24 we find upper bounds of the cardinality of $S(n)$-spaces as functions of $s_n(X)$ and $bt_n(X)$.

2. Preliminaries. All spaces considered here are assumed to be at least T_1 and infinite. $\mathbb{N}^+$ denotes the set of all positive integers and $\mathbb{N} = \{0\} \cup \mathbb{N}^+$. α, β, γ and δ are ordinal numbers, while λ and κ denote infinite cardinals; κ^+ is the successor cardinal of κ. As usual, cardinals are assumed to be initial ordinals. If X is a set, then $\mathfrak{P}(X)$ and $[X]^{\le\kappa}$ denote the power set of X and the collection of all subsets of X having cardinality $\le \kappa$, respectively.

We begin with recalling some definitions that we need. (For additional topological definitions not given here see [9], [13], or [12].)

Definition 2.1. *Let X be a topological space, $A \subset X$ and $n \in \mathbb{N}^+$. A point $x \in X$ is $S(n)$-*separated from A *if there exist open sets U_i, $i = 1, 2, \dots, n$ such that $x \in U_1$, $\overline{U}_i \subset U_{i+1}$ for $i = 1, 2, \dots, n-1$ and $\overline{U}_n \cap A = \varnothing$; x is $S(0)$-*

separated from A if $x \notin \overline{A}$. X is an $S(n)$-space [21] *if every two distinct points in X are $S(n)$-separated.*

Now, let $n \in \mathbb{N}$. The set $\mathrm{cl}_{\theta^n} A = \{x \in X : x$ is not $S(n)$-separated from $A\}$ is called θ^n-closure *of A* [6]. *A is* θ^n-closed [16] *if $\mathrm{cl}_{\theta^n}(A) = A$; $U \subset X$ is* θ^n-open *if $X \backslash U$ is θ^n-closed; and A is* θ^n-dense *in X if $\mathrm{cl}_{\theta^n}(A) = X$.*

It is a direct corollary of Definition 2.1 that $S(1)$ is the class of Hausdorff spaces and $S(2)$ is the class of Urysohn spaces. Since we are going to consider here only T_1-spaces, for us the $S(0)$-spaces will be exactly the T_1-spaces. Also, $\mathrm{cl}_{\theta^0}(A) = \overline{A}$ and $\mathrm{cl}_{\theta^1}(A) = \mathrm{cl}_\theta(A)$ is the so called *θ-closure of A* [20].

It will be more convenient for us to consider the $S(n)$-spaces in more 'symmetric' way similar to the way how $S(n)$-spaces are defined in [7], [8] or [16] but we are going to use the terminology and notation introduced in [10].

Definition 2.2 ([10]). *Let X be a topological space, $U \subseteq X$, $x \in U$ and $k \in \mathbb{N}^+$. We will say that U is an* $S(2k-1)$-neighborhood of x *if there exist open sets U_i, $i = 1, 2, \dots, k$, such that $x \in U_1$, $\overline{U}_i \subset U_{i+1}$, for $i = 1, 2, \dots, k-1$, and $U_k \subseteq U$. We will say that U is an* $S(2k)$-neighborhood of x *if there exist open sets U_i, $i = 1, 2, \dots, k$, such that $x \in U_1$, $\overline{U}_i \subset U_{i+1}$, for $i = 1, 2, \dots, k-1$, and $\overline{U}_k \subseteq U$.*

Let $n \in \mathbb{N}^+$. When a set U is an $S(n)$-neighborhood of a point x and it is an open (closed) set in X, we will refer to it as open (closed) $S(n)$-neighborhood of x. *A set U will be called $S(n)$-open ($S(n)$-closed) if U is open (closed) and there exists at least one point x such that U is an open (closed) $S(n)$-neighborhood of x.*

Remark 2.3 ([10]). We note that in what follows every $S(2k-1)$-open set U in a space X, where $k \in \mathbb{N}^+$, will be considered as a fixed chain of k nonempty sets U_i, $i = 1, 2, \dots, k$, such that $\overline{U}_i \subset U_{i+1}$, for $i = 1, 2, \dots, k-1$, and $U_k \subseteq U$. (In fact, most of the time we will assume that $U_k = U$).

Now, using the terminology and notation introduced in Definition 2.2 it is easy to see that the following propositions are true.

Proposition 2.4 ([10]). *Let X be a topological space, $x \in X$ and $k \in \mathbb{N}^+$.*

(a) *Every closed $S(2k-1)$-neighborhood of x is a closed $S(2k)$-neighborhood of x.*

(b) *Every $S(2k)$-neighborhood of x contains a closed $S(2k)$-neighborhood of x; hence it contains a closed (and therefore an open) $S(2k-1)$-neighborhood of x. Thus, every $S(2k)$-neighborhood of x is an $S(2k-1)$-neighborhood of x.*

(c) *Every $S(2k+1)$-neighborhood of x contains an open $S(2k+1)$-neighbor-*

hood of x; hence it contains an open (and therefore a closed) $S(2k)$-neighborhood of x. Thus, every $S(2k+1)$-neighborhood of x is an $S(2k)$-neighborhood of x.

Proposition 2.5 ([10]). *Let X be a topological space and $k \in \mathbb{N}^+$.*

(a) *X is an $S(2k-1)$-space if and only if every two distinct points of X can be separated by disjoint (open) $S(2k-1)$-neighborhoods.*

(b) *X is an $S(2k)$-space if and only if every two distinct points of X can be separated by disjoint closed $S(2k-1)$-neighborhoods.*

(c) *X is an $S(2k)$-space if and only if every two distinct points of X can be separated by disjoint (closed) $S(2k)$-neighborhoods.*

(d) *X is an $S(2k+1)$-space if and only if every two distinct points of X can be separated by disjoint open $S(2k)$-neighborhoods.*

Definition 2.6 ([10]). *Let X be a topological space, $A \subseteq X$ and $k \in \mathbb{N}^+$. We will say that a point x is in the $S(2k-1)$-closure of A if and only if every (open) $S(2k-1)$-neighborhood of x intersects A and we will say that a point x is in the $S(2k)$-closure of A if and only if every (closed) $S(2k)$-neighborhood (or equivalently, every closed $S(2k-1)$-neighborhood) of x intersects A. For $n \in \mathbb{N}^+$, the $S(n)$-closure of A will be denoted by $\theta_n(A)$. A is θ_n-*closed *if $\theta_n(A) = A$ and $U \subset X$ is θ_n-*open *if $X \setminus U$ is θ_n-closed, or equivalently, $U \subset X$ is θ_n-*open *if U is an $S(n)$-neighborhood of every $x \in U$. Finally, A is θ_n-dense in X if $\theta_n(A) = X$.*

It is immediate that, for every $n \in \mathbb{N}^+$, every θ_n-open set is open and every set of the form $\theta_n(A)$, where $A \subseteq X$, is a closed set in X. Also, $\theta_1(A) = \mathrm{cl}(A) = \overline{A}$ is the usual closure operator in X and $\theta_2(A) = \mathrm{cl}_\theta(A)$ is the θ-closure operator introduced by Veličko [20]. We also note that, for any integer $n > 1$, the θ_n-closure operator, in general, is not idempotent.

Definition 2.7 ([10]). *Let $k \in \mathbb{N}^+$ and X be a topological space.*

(a) *A family $\{U_\alpha : \alpha < \kappa\}$ of open $S(2k-1)$-neighborhoods of a point $x \in X$ will be called* an open $S(2k-1)$-neighborhood base at the point x *if for every open $S(2k-1)$-neighborhood U of x there is $\alpha < \kappa$ such that $U_\alpha \subseteq U$.*

(b) *An $S(2k-1)$-space X is of* $S(2k-1)$-character κ, *denoted by $\chi_{2k-1}(X) = \kappa$, if κ is the smallest infinite cardinal such that for each point $x \in X$ there exists an open $S(2k-1)$-neighborhood base at x with cardinality at most κ. In the case $k = 1$ the $S(1)$-character $\chi_1(X)$ coincides with the usual character $\chi(X)$.*

(c) *An $S(2k)$-space X is of* $S(2k)$-character κ, *denoted by $\chi_{2k}(X) = \kappa$, if κ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\mathcal{V}_x$ of closed $S(2k-1)$-neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and if W is an open $S(2k-1)$-neighborhood of x, then $\overline{W}$ contains a member of $\mathcal{V}_x$. In the* *case $k = 1$ the $S(2)$-character $\chi_2(X)$ coincides with the cardinal function $k(X)$ defined in* [1].

(d) *An $S(k-1)$-space X is of* $S(2k-1)$-pseudocharacter κ, *denoted by $\psi_{2k-1}(X) = \kappa$, if κ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha < \kappa\}$ of $S(2k-1)$-open neighborhoods of x such that $\{x\} = \bigcap\{U_\alpha : \alpha < \kappa\}$. In the case $k = 1$ the pseudocharacter $\psi_1(X)$ coincides with the usual pseudocharacter $\psi(X)$.*

(e) *An $S(k)$-space X is of* $S(2k)$-pseudocharacter κ, *denoted by $\psi_{2k}(X) = \kappa$, if κ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha < \kappa\}$ of $S(2k-1)$-open neighborhoods of x such that $\{x\} = \bigcap\{\overline{U}_\alpha : \alpha < \kappa\}$. In the case $k = 1$ the pseudocharacter $\psi_2(X)$ coincides with the closed pseudocharacter $\psi_c(X)$.*

It follows immediately from the previous definition that if $k \in \mathbb{N}^+$, then $\chi_{2k}(X) \leq \chi_{2k-1}(X)$ and $\psi_{2k-1}(X) \leq \psi_{2k}(X) \leq \psi_{2k+1}(X) \leq \psi_{2k+2}(X)$, whenever they are defined (see [10]).

In relation to Definition 2.7(c) we recall that for a topological space X, $k(X)$ is the smallest infinite cardinal κ such that for each point $x \in X$, there is a collection $\mathcal{V}_x$ of closed neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and if W is a neighborhood of x, then $\overline{W}$ contains a member of $\mathcal{V}_x$ [1]. As it was noted in [1], $k(X) \leq \chi(X)$ and that $k(X)$ is equal to the character of the semiregularization of X.

Definition 2.8. *Let $n \in \mathbb{N}$. We define the* θ_n-density *and* hereditary θ_n-density *of a space X (denoted, respectively, by $d_{\theta_n}(X)$ and $hd_{\theta_n}(X)$) by*

$$d_n(X) = \min\{|A| : A \text{ is a } \theta_n\text{-dense subset of } X\} + \aleph_0, \text{ and}$$
$$hd_n(X) = \sup\{d_{\theta_n}(Y) : Y \subset X\}.$$

Clearly, if $n = 1$, then $d_1(X) = d(X)$ and $hd_1(X) = hd(X)$ are the usual density and hereditary density functions. If $n = 2$, then $d_2(X) = d_\theta(X)$ and $hd_2(X) = hd_\theta(X)$ are the θ-density and hereditary θ-density functions defined in [15].

It is not difficult to see that for every space X and every $n \in \mathbb{N}^+$ we have

$$d_n(X) \leq d_{n-1}(X) \leq \cdots \leq d_2(X) = d_\theta(X) \leq d_1(X) = d(X), \text{ and}$$
$$hd_n(X) \leq hd_{n-1}(X) \leq \cdots \leq hd_2(X) = hd_\theta(X) \leq hd_1(X) = hd(X).$$

Definition 2.9 ([10]). *Let $k \in \mathbb{N}^+$ and X be a topological space.*

(a) *We shall call a subset D of X* $S(2k-1)$-discrete *if for every $x \in D$, there is an open $S(2k-1)$-neighborhood U of x such that $U \cap D = \{x\}$, and* *we define the* $S(2k-1)$-spread *of* X*, denoted by* $s_{2k-1}(X)$*, to be* $\sup\{|D| : D$ *is* $S(2k-1)$*-discrete subset of* $X\} + \aleph_0$.

(b) *We shall call a subset* D *of* X $S(2k)$-discrete *if for every* $x \in D$*, there is an open* $S(2k-1)$*-neighborhood* U *of* x *such that* $\overline{U} \cap D = \{x\}$*, and we define the* $S(2k)$-spread *of* X*, denoted by* $s_{2k}(X)$*, to be* $\sup\{|D| : D$ *is* $S(2k)$*-discrete subset of* $X\} + \aleph_0$.

It is easily seen that a set D in a topological space X is discrete if and only if D is $S(1)$-discrete and a set D is Urysohn-discrete if and only if D is $S(2)$-discrete. Hence, $s_1(X)$ is the usual spread $s(X)$ and $s_2(X)$ is the Urysohn spread $Us(X)$ defined in [18].

Definition 2.10. *Let* $n \in \mathbb{N}^+$ *and* X *be a topological space.*

(a) *The* $S(n)$-tightness *of a space* X*, denoted by* $t_n(X)$*, is the smallest cardinal* τ *such that for every* $A \subset X$ *and every* $x \in \theta_n(A)$ *there exists a set* $B \subset A$ *such that* $|B| \leq \tau$ *and* $x \in \theta_n(B)$.

(b) *The* $S(n)$-bitightness *of a space* X*, denoted by* $bt_n(X)$*, is the smallest cardinal* τ *such that for each non-*θ_n*-closed set* $A \subset X$ *there exists a point* $x \in X \backslash A$ *and a collection* $\mathcal{S} \in [[A]^{\leq\tau}]^{\leq\tau}$ *such that* $\{x\} = \bigcap\{\theta_n(S) : S \in \mathcal{S}\}$.

If $n = 1$, then $t_1(X) = t(X)$ and $bt_1(X) = bt(X)$ are the usual tightness and bitightness functions (see [2]) and if $n = 2$, then $t_2(X) = t_\theta(X)$ and $bt_2(X) = bt_\theta(X)$ are the θ-tightness and θ-bitightness functions defined in [5].

3. Cardinal inequalities for $S(n)$-spaces. We begin with extending for the class of $S(n)$-spaces, where n is any positive integer, the following two Pospišil's inequalities: $|X| \leq 2^{2^{d(X)}}$ and $|X| \leq [d(X)]^{\chi(X)}$ [17].

We note that the case $n = 1$ of the following theorem is exactly the first Pospišil's inequality mentioned above and the case $n = 2$ is [15, Theorem 2.1].

Theorem 3.1. *Let* $n \in \mathbb{N}^+$*. If* X *is an* $S(n)$*-space, then* $|X| \leq 2^{2^{d_n(X)}}$.

P r o o f. Let $d_n(X) \leq \kappa$ and let A be a θ_n-dense subset of X such that $|A| \leq \kappa$. We need to consider two cases: (a) $n = 2k-1$ and (b) $n = 2k$, where $k \in \mathbb{N}^+$. Since X is an $S(n)$-space, for every two distinct points x and y in X, there exist open $S(2k-1)$-neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$ in case (a) and $\overline{U} \cap \overline{V} = \emptyset$ in case (b). Hence, there exists a set $B_x \subset A$ such that $x \in \theta_{2k-1}(B_x)$ and $y \notin \theta_{2k-1}(B_x)$ in case (a) and $x \in \theta_{2k}(B_x)$ and $y \notin \theta_{2k}(B_x)$ in case (b). Therefore $x \to \{B_x \subset A : x \in \theta_{2k-1}(B_x)\}$ in case (a) and $x \to \{B_x \subset A : x \in \theta_{2k}(B_x)\}$ in case (b) is an one-to-one correspondence between X and a subset of the set $\mathfrak{P}(\mathfrak{P}(A))$, so $|X| \leq 2^{2^\kappa}$. $\square$

The case $k = 1$ of the following theorem can be found in [2] and for $k = 2$ it was observed in [11]. We note that in [5, Proposition 2.2] it was shown that $bt_\theta(X) \leq \chi(X)$.

Theorem 3.2. *Let $n \in \mathbb{N}^+$. If X is an $S(n)$-space, then $bt_n(X) \leq \chi_n(X)$.*

Proof. Let $\chi_n(X) = \kappa$ and let A be a non-θ_n-closed subset of X. Then there exists a point $x \in \theta_n(A) \setminus A$. We need to consider two cases: (a) $n = 2k - 1$ and (b) $n = 2k$, where $k \in \mathbb{N}^+$. In both cases let $\{U_\alpha : \alpha < \kappa\}$ be an open $S(2k-1)$-neighborhood base for x. Then for each $\alpha < \kappa$ we have $U_\alpha \cap A \neq \emptyset$ in case (a) and $\overline{U}_\alpha \cap A \neq \emptyset$ in case (b). In both cases we choose a point x_α in these nonempty intersections. Let $B = \{x_\alpha : \alpha < \kappa\}$. Then $x \in \theta_{2k-1}(B \cap U_\alpha)$ in case (a) and $x \in \theta_{2k}(B \cap \overline{U}_\alpha)$ in case (b). Since X is an $S(2k-1)$-space in case (a) and $S(2k)$-space in case (b) we have

$$\bigcap\{\theta_{2k-1}(B \cap U_\alpha) : \alpha < \kappa\} \subset \bigcap\{\theta_{2k-1}(U_\alpha) : \alpha < \kappa\} = \{x\}$$

in case (a), and

$$\bigcap\{\theta_{2k}(B \cap \overline{U}_\alpha) : \alpha < \kappa\} \subset \bigcap\{\theta_{2k}(\overline{U}_\alpha) : \alpha < \kappa\} = \{x\}$$

in case (b). Therefore the collection $\{B \cap U_\alpha : \alpha < \kappa\}$ in case (a) and $\{B \cap \overline{U}_\alpha : \alpha < \kappa\}$ in case (b) witness that $bt_n(X) \leq \kappa$. □

Another estimation of the $S(n)$-bitightness is contained in our next theorem. In [2, Proposition 1] it was observed that $t(X) \leq bt(X) \leq \chi(X)$, whenever X is a Hausdorff space. The case $n = 1$ of Theorem 3.3 gives the following better estimation: $t(X) \leq bt(X) \leq t(X)\psi_c(X) \leq \chi(X)$.

Theorem 3.3. *Let $n \in \mathbb{N}^+$. If X is an $S(n)$-space, then $bt_n(X) \leq t_n(X)\psi_{2n}(X)$.*

Proof. Let $t_n(X)\psi_{2n}(X) = \kappa$ and let A be a non-θ_n-closed subset of X. Then there is a point $x \in \theta_n(A) \setminus A$. Since $t_n(X) \leq \kappa$, we can fix a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \theta_n(B)$. We need to consider two cases: (a) $n = 2k - 1$ and (b) $n = 2k$, where $k \in \mathbb{N}^+$. Let $\{V^\alpha : \alpha < \kappa\}$ be a collection of open $S(4k-3)$-neighborhoods of x in case (a) and a collection of open $S(4k-1)$-neighborhoods of x in case (b) such that $\bigcap\{\overline{V}^\alpha : \alpha < \kappa\} = \{x\}$. Since for each $\alpha < \kappa$, V^α is an open $S(4k-3)$-neighborhood of x in case (a) and an open $S(4k-1)$-neighborhood of x in case (b), there exist open neighborhoods of x such that

$$x \in V_1^\alpha \subset \overline{V}_1^\alpha \subset \cdots \subset V_k^\alpha \subset \overline{V}_k^\alpha \subset \cdots \subset V_{2k-1}^\alpha$$

in case (a) and

$$x \in V_1^\alpha \subset \overline{V}_1^\alpha \subset \cdots \subset V_k^\alpha \subset \overline{V}_k^\alpha \subset \cdots \subset V_{2k}^\alpha$$

in case (b).

Since $x \in \theta_n(B)$, for each $\alpha < \kappa$, $V_k^\alpha \cap B \neq \emptyset$ in case (a) and $\overline{V}_k^\alpha \cap B \neq \emptyset$ in case (b). Thus, for every $\alpha < \kappa$ we have $x \in \theta_{2k-1}(B \cap V_k^\alpha)$ in case (a) and $x \in \theta_{2k}(B \cap \overline{V}_k^\alpha)$ in case (b).

Therefore

$$x \in \bigcap\{\theta_{2k-1}(B \cap V_k^\alpha) : \alpha < \kappa\} \subset \bigcap\{\theta_{2k-1}(V_k^\alpha) : \alpha < \kappa\} \subset \bigcap\{\overline{V}_{2k-1}^\alpha : \alpha < \kappa\} = \{x\}$$

in case (a) and

$$x \in \bigcap\{\theta_{2k}(B \cap \overline{V}_k^\alpha) : \alpha < \kappa\} \subset \bigcap\{\theta_{2k}(\overline{V}_k^\alpha) : \alpha < \kappa\} \subset \bigcap\{\overline{V}_{2k}^\alpha : \alpha < \kappa\} = \{x\}$$

in case (b).

This shows that $\bigcap\{\theta_{2k-1}(B \cap V_k^\alpha) : \alpha < \kappa\} = \{x\}$ in case (a) and $\bigcap\{\theta_{2k}(B \cap \overline{V}_k^\alpha) : \alpha < \kappa\} = \{x\}$ in case (b). The existence of the collections $\{B \cap V_k^\alpha : \alpha < \kappa\}$ in case (a) and $\{B \cap \overline{V}_k^\alpha : \alpha < \kappa\}$ in case (b) proves that $bt_n(X) \leq \kappa$. □

The case $n = 1$ of our next theorem is Lemma 1 in [2]. In [3] the authors proved that if X is a Urysohn space and $A \subset X$, then $|\operatorname{cl}_\theta(A)| \leq |A|^{\chi(X)}$ and it was sharpened in [4] to $|\operatorname{cl}_\theta(A)| \leq |A|^{bt_\theta(X)}$, which is the case $n = 2$ of the following theorem.

Theorem 3.4. *Let $n \in \mathbb{N}^+$. If A is a subset of an $S(n)$-space X, then $|\theta_n(A)| \leq |A|^{bt_n(X)}$.*

P r o o f. Let $|A| = \kappa$ and $bt_n(X) = \lambda$. Using transfinite recursion we define a family $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X such that:

(i) $A_\alpha \subset A_\beta$ for $\alpha < \beta < \lambda^+$; and

(ii) $|A_\alpha| \leq \lambda^\kappa$ for each $\alpha < \lambda^+$.

Let $A_0 = A$. Suppose we have already defined the sets A_β for all $\beta < \alpha$. We shall define A_α:

(1) If α is a limit ordinal, then $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$;

(2) If $\alpha = \gamma + 1$, for some γ, then $A_\alpha = \{x \in X :$ there exists $\mathcal{S} \in [[A_\gamma]^{\leq\lambda}]^{\leq\lambda}$ such that $\{x\} = \bigcap\{\theta_n(S) : S \in \mathcal{S}\}\}$.

The construction of the sets A_α is completed. The condition (i) is obviously satisfied since for every $x \in X$, $\{x\} = \theta_n(\{x\})$ for X is an $S(n)$-space. We are going to check (ii). Suppose that (ii) is not true and let β be the first ordinal for which $|A_\beta| > \kappa^\lambda$. Note that $\beta > 0$ and β is not a limit ordinal (otherwise $|A_\beta| \leq \sum\{|A_\delta| : \delta < \beta\} \leq \kappa^\lambda$). Hence, $\beta = \gamma + 1$ for some $\gamma < \lambda^+$. For each $x \in A_\beta$ there exists a collection $\mathcal{S}_x \in [[A_\gamma]^{\leq\lambda}]^{\leq\lambda}$ such that $\{x\} = \bigcap\{\theta_n(S) : S \in \mathcal{S}_x\}$. The correspondence $x \to \mathcal{S}_x$ is one-to-one. Therefore, we have $|A_\beta| \leq \left|[[A_\gamma]^{\leq\lambda}]^{\leq\lambda}\right| \leq ((\kappa^\lambda)^\lambda)^\lambda = \kappa^\lambda$. This contradiction proves (ii).

Let $F = \bigcup\{A_\alpha : \alpha < \lambda^+\}$. We shall show that F is θ_n-closed. Assume, to the contrary, that F is not θ_n-closed. Since $bt_n(X) = \lambda$, there is a point $x \in X \backslash F$ and a family $\mathcal{C} \in [[F]^{\leq\lambda}]^{\leq\lambda}$ such that $\{x\} = \bigcap\{\theta_n(C) : C \in \mathcal{C}\}$. Since λ^+ is regular, there is some $\alpha < \lambda^+$ such that $\bigcup\{C : C \in \mathcal{C}\} \subset \bigcup\{A_\beta : \beta < \alpha\} \subset A_\alpha$. Then, it follows from the definition of $A_{\alpha+1}$ that $x \in A_{\alpha+1}$ and we have a contradiction. Therefore A is θ_n-closed and the theorem is proved. □

The following result is a direct corollary of Theorem 3.4.

Theorem 3.5. *If $n \in \mathbb{N}^+$, then $|X| \leq [d_n(X)]^{bt_n(X)}$, whenever X is an $S(n)$-space.*

Theorem 3.4 and Theorem 3.3 imply immediately the following two results:

Theorem 3.6. *Let $n \in \mathbb{N}^+$. If A is a subset of an $S(n)$-space X, then $|\theta_n(A)| \leq |A|^{t_n(X)\psi_{2n}(X)}$.*

Theorem 3.7. *If $n \in \mathbb{N}^+$, then $|X| \leq [d_n(X)]^{t_n(X)\psi_{2n}(X)}$, whenever X is an $S(n)$-space.*

We note that if $n = 1$ in Theorem 3.6, then we obtain Bella and Cammaroto's result that if X is a Hausdorff space and A is a subset of X, then $|\overline{A}| \leq |A|^{t(X)\psi_c(X)}$ [3]. The case $n = 2$ of Theorem 3.6 states that if X is a Urysohn space and A is a subset of X, then $|\operatorname{cl}_\theta(A)| \leq |A|^{t_\theta(X)\psi_4(X)}$. Under the same assumptions it was shown in [11] that $|\operatorname{cl}_\theta(A)| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}$. Since $\psi_{\theta^2}(X) \leq \psi_4(X)$, for every Urysohn space X, the latter estimation is better. (For the definition of $\psi_{\theta^2}(X)$ see [11]).

Definition 3.8. *Denote by $C_n(X)$ the family of all θ_n-closed subsets of a space X.*

The case $n = 2$ of our next result is [15, Theorem 2.4].

Theorem 3.9. *Let $n \in \mathbb{N}^+$. If X is an $S(n)$-space, then $|C_n(X)| \leq 2^{hd_n(X)bt_n(X)}$.*

Proof. Let $hd_n(X)bt_n(X) = \kappa$ and let F be a θ_n-closed subset of X. Take a set $D_F \subset F$ such that $\theta_n(D_F) = F$ and $|D_F| \leq \kappa$. So the set $C_n(X)$ of all θ_n-closed subsets of X is contained in the set $\{\theta_n(D) : D \subset X, |D| \leq \kappa\}$, which means $|C_n(X)| \leq |X|^\kappa$. By Theorem 3.5 and the fact that $d_n(X) \leq \kappa$ we have $|C_n(X)| \leq (\kappa^\kappa)^\kappa = 2^\kappa$. The theorem is proved. □

Before we continue we recall some results from [10], which we will use later.

Lemma 3.10 ([10])**.** *Let $k \in \mathbb{N}^+$, X be a topological space, $\kappa = s_{2k-1}(X)$ and $C \subseteq X$. For each $x \in C$ let U^x be an open $S(2k-1)$-neighborhood of x and let $\mathcal{U} = \{U^x : x \in C\}$. Then there exist an $S(2k-1)$-discrete subset A of C such that $|A| \leq \kappa$ and $C \subseteq \theta_{2k-1}(A) \cup \bigcup\{U^x : x \in A\}$.*

Lemma 3.11 ([10])**.** *Let $k \in \mathbb{N}^+$, X be a topological space, $\kappa = s_{2k}(X)$ and $C \subseteq X$. For each $x \in C$ let U^x be an open $S(2k-1)$-neighborhood of x and let $\mathcal{U} = \{U^x : x \in C\}$. Then there exist an $S(2k)$-discrete subset A of C such that $|A| \leq \kappa$ and $C \subseteq \theta_{2k}(A) \cup \bigcup\{\overline{U}^x : x \in A\}$.*

Lemma 3.12 ([10])**.** *Let $k \in \mathbb{N}^+$.*
(a) *For every $S(3k)$-space X, $\psi_{2k}(X) \leq 2^{s_{2k}(X)}$;*
(b) *For every $S(3k-2)$-space X, $\psi_{2k-1}(X) \leq 2^{s_{2k-1}(X)}$;*
(c) *For every $S(3k-1)$-space X, $\psi_{2k-1}(X) \leq 2^{s_{2k}(X)}$;*
(d) *For every $S(3k-1)$-space X, $\psi_{2k}(X) \leq 2^{s_{2k-1}(X)}$.*

Our next three theorems are versions of the fundamental result on spread due to Shapirovskiĭ (see [19] or [12, Theorem 5.1]). We note that the case $k = 1$ of Theorem 3.13 was stated in [15, Proposition 3.3] for Urysohn spaces X and hereditary spread $hs_\theta(X)$ but its proof was based on [18, Lemma 11], which proof has a gap (see also [10]). Here we state and prove Proposition 3.3 from [15] for $S(3)$-spaces and we use the spread $s_\theta(X)$, instead (see Corollary 3.14).

Theorem 3.13. *Let $k \in \mathbb{N}^+$ and X be an $S(3k)$-space with $s_{2k}(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and*

$$\bigcup\left\{\theta_{2k}(B) : B \in [A]^{\leq\kappa}\right\} = X.$$

P r o o f. Since X is an $S(3k)$-space, according to Lemma 3.12(a) $\psi_{2k}(X) \leq 2^{s_{2k}(X)}$ and therefore for every $x \in X$ one can choose a collection $\mathcal{U}_x$ of $S(2k)$-neighborhoods of x such that $|\mathcal{U}_x| \leq 2^\kappa$ and $\bigcap\left\{\overline{U} : U \in \mathcal{U}_x\right\} = \{x\}$. Using transfinite recursion we will construct a sequence $\left\{A_\alpha : \alpha \in \kappa^+\right\}$ of subsets of X and a sequence $\left\{\mathcal{U}_\alpha : \alpha < \kappa^+\right\}$ of families of open $S(2k)$-subsets of X satisfying the following conditions:

(a) $|A_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$;

(b) $|\mathcal{U}_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$; and

(c) If $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa}$, $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$, and $\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\} \neq X$, then $A_\alpha \setminus \left(\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\}\right) \neq \emptyset$.

Suppose we have already defined all A_β and $\mathcal{U}_\beta$ for $\beta < \alpha$. Let us define A_α and $\mathcal{U}_\alpha$. For every $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa}$ and every $\mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq\kappa}$ choose a point $x(\mathcal{S},\mathcal{V}) \in X \setminus \left(\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\}\right)$ whenever the last set is not empty (otherwise the construction has been finished). Let

$$A_\alpha = \left\{x(\mathcal{S},\mathcal{V}) : \mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa} \text{ and } \mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq\kappa}\right\}, \text{ and}$$
$$\mathcal{U}_\alpha = \bigcup\{\mathcal{U}_x : x \in A_\alpha\}.$$

It is easy to check that A_α and $\mathcal{U}_\alpha$ satisfy (a), (b), and (c). Now, let $A = \bigcup\left\{A_\alpha : \alpha < \kappa^+\right\}$. We shall prove that A is as it is required. Take a point $p \in X \setminus A$. We shall show that $p \in \theta_{2k}(B)$ for some $B \in [A]^{\leq\kappa}$. For every $x \in A$ pick $U_x \in \mathcal{U}_x$ such that $p \notin \overline{U}_x$. Applying now Lemma 3.11 (to the set A and the collection $\{U_x : x \in A\}$) we find a set B in $[A]^{\leq\tau}$ such that

$$(*) \qquad A \subset \theta_{2k}(B) \cup \bigcup\left\{\overline{U}_y : y \in B\right\}.$$

Let us show that $p \in \theta_{2k}(B)$. Suppose not. Then one can choose $\alpha < \kappa^+$ for which $B \subset \bigcup\{A_\beta : \beta < \alpha\}$. By (c), then $A_\alpha \setminus \left(\theta_{2k}(B) \cup \bigcup\left\{\overline{U}_y : y \in Y\right\}\right) \neq \varnothing$ which contradicts $(*)$. The theorem is proved. □

The case $k = 1$ of the previous theorem gives us the following:

Corollary 3.14. *Let X be an $S(3)$-space with $s_\theta(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and $\bigcup\left\{\mathrm{cl}_\theta(B) : B \in [A]^{\leq\kappa}\right\} = X$.*

Theorem 3.15. *Let $k \in \mathbb{N}^+$ and X be an $S(3k-2)$-space with $s_{2k-1}(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and*

$$\bigcup \left\{ \theta_{2k-1}(B) : B \in [A]^{\leq \kappa} \right\} = X.$$

Proof. Since X is an $S(3k-2)$-space, according to Lemma 3.12(b), $\psi_{2k-1}(X) \leq 2^{s_{2k-1}(X)}$ and therefore for every $x \in X$ one can choose a collection $\mathcal{U}_x$ of $S(2k-1)$-neighborhoods of x such that $|\mathcal{U}_x| \leq 2^\kappa$ and $\bigcap\{U : U \in \mathcal{U}_x\} = \{x\}$. Using transfinite recursion we will construct a sequence $\left\{A_\alpha : \alpha \in \kappa^+\right\}$ of subsets of X and a sequence $\left\{\mathcal{U}_\alpha : \alpha < \kappa^+\right\}$ of families of open $S(2k-1)$-subsets of X satisfying the following conditions:

(a) $|A_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$;

(b) $|\mathcal{U}_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$;

(c) If $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq \kappa}$, $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq \kappa}$, and $\theta_{2k-1}(\mathcal{S}) \cup \bigcup\{V : V \in \mathcal{V}\} \neq X$, then $A_\alpha \setminus \left(\theta_{2k-1}(\mathcal{S}) \cup \bigcup\{V : V \in \mathcal{V}\}\right) \neq \emptyset$.

Suppose we have already defined all A_β and $\mathcal{U}_\beta$ for $\beta < \alpha$. Let us define A_α and $\mathcal{U}_\alpha$. For every $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq \kappa}$ and every $\mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq \kappa}$ choose a point $x(\mathcal{S}, \mathcal{V}) \in X \setminus \left(\theta_{2k-1}(\mathcal{S}) \cup \bigcup\{V : V \in \mathcal{V}\}\right)$ whenever the last set is not empty (otherwise the construction has been finished). Let

$$A_\alpha = \left\{ x(\mathcal{S}, \mathcal{V}) : \mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq \kappa} \text{ and } \mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq \kappa} \right\}, \text{ and}$$

$$\mathcal{U}_\alpha = \bigcup\{\mathcal{U}_x : x \in A_\alpha\}.$$

It is easy to check that A_α and $\mathcal{U}_\alpha$ satisfy (a), (b), and (c). Now, let $A = \bigcup\left\{A_\alpha : \alpha < \kappa^+\right\}$. We shall prove that A is as it is required. Take a point $p \in X \setminus A$. We shall show that $p \in \theta_{2k-1}(B)$ for some $B \in [A]^{\leq \kappa}$. For every $x \in A$ pick $U_x \in \mathcal{U}_x$ such that $p \notin U_x$. Applying now Lemma 3.10 (to the set A and the collection $\{U_x : x \in A\}$) we find a set B in $[A]^{\leq \tau}$ such that

$$(*) \qquad A \subset \theta_{2k-1}(B) \cup \bigcup\{U_y : y \in B\}.$$

Let us show that $p \in \theta_{2k-1}(B)$. Suppose not. Then one can choose $\alpha < \kappa^+$ for which $B \subset \bigcup\{A_\beta : \beta < \alpha\}$. By (c), then

$$A_\alpha \setminus \left(\theta_{2k-1}(B) \cup \bigcup\{U_y : y \in Y\}\right) \neq \varnothing$$

which contradicts $(*)$. The theorem is proved. □

The case $k = 1$ of the previous theorem is the well-known Shapirovskiĭ's result on spread (see [19]).

Corollary 3.16. *Let X be a Hausdorff space with $s(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and* $\bigcup\left\{\overline{B} : B \in [A]^{\leq\kappa}\right\} = X$.

Theorem 3.17. *Let $k \in \mathbb{N}^+$ and X be an $S(3k-1)$-space with $s_{2k-1}(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and*

$$\bigcup\left\{\theta_{2k}(B) : B \in [A]^{\leq\kappa}\right\} = X.$$

P r o o f. Since X is an $S(3k-1)$-space, according to Lemma 3.12(d), $\psi_{2k}(X) \leq 2^{s_{2k-1}(X)}$ and therefore for every $x \in X$ one can choose a collection $\mathcal{U}_x$ of $S(2k-1)$-neighborhoods of x such that $|\mathcal{U}_x| \leq 2^\kappa$ and $\bigcap\left\{\overline{U} : U \in \mathcal{U}_x\right\} = \{x\}$. Using transfinite recursion we will construct a sequence $\left\{A_\alpha : \alpha \in \kappa^+\right\}$ of subsets of X and a sequence $\left\{\mathcal{U}_\alpha : \alpha < \kappa^+\right\}$ of families of open $S(2k-1)$-subsets of X satisfying the following conditions:

(a) $|A_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$;

(b) $|\mathcal{U}_\alpha| \leq 2^\kappa$, $\alpha < \kappa^+$; and

(c) If $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa}$, $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$, and $\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\} \neq X$, then $A_\alpha \backslash \left(\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\}\right) \neq \emptyset$.

Suppose we have already defined all A_β and $\mathcal{U}_\beta$ for $\beta < \alpha$. Let us define A_α and $\mathcal{U}_\alpha$. For every $\mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa}$ and every $\mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq\kappa}$ choose a point $x(\mathcal{S}, \mathcal{V}) \in X \backslash \left(\theta_{2k}(\mathcal{S}) \cup \bigcup\left\{\overline{V} : V \in \mathcal{V}\right\}\right)$ whenever the last set is not empty (otherwise the construction has been finished). Let

$$A_\alpha = \left\{x(\mathcal{S}, \mathcal{V}) : \mathcal{S} \in \left[\bigcup\{A_\beta : \beta < \alpha\}\right]^{\leq\kappa} \text{ and } \mathcal{V} \in \left[\bigcup\{\mathcal{U}_\beta : \beta < \alpha\}\right]^{\leq\kappa}\right\}, \text{ and}$$

$$\mathcal{U}_\alpha = \bigcup\{\mathcal{U}_x : x \in A_\alpha\}.$$

It is easy to check that A_α and $\mathcal{U}_\alpha$ satisfy (a), (b), and (c). Now, let $A = \bigcup\left\{A_\alpha : \alpha < \kappa^+\right\}$. We shall prove that A is as it is required. Take a point $p \in X \backslash A$. We shall show that $p \in \theta_{2k}(B)$ for some $B \in [A]^{\leq\kappa}$. For every $x \in A$ pick $U_x \in \mathcal{U}_x$ such that $p \notin \overline{U}_x$. Applying now Lemma 3.11 (to the set A and the collection $\{U_x : x \in A\}$) we find a set B in $[A]^{\leq\tau}$ such that

$$(**) \qquad A \subset \theta_{2k}(B) \cup \bigcup\left\{\overline{U}_y : y \in B\right\}.$$

Let us show that $p \in \theta_{2k}(B)$. Suppose not. Then one can choose $\alpha < \kappa^+$ for which $B \subset \bigcup\{A_\beta : \beta < \alpha\}$. By (c), then $A_\alpha \setminus \left(\theta_{2k}(B) \cup \bigcup\{\overline{U}_y : y \in Y\}\right) \neq \varnothing$ which contradicts $(**)$. The theorem is proved. □

The case $k = 1$ of the previous theorem could be restated as follows:

Corollary 3.18. *Let X be a Urysohn space with $s(X) \leq \kappa$. Then there exists a subset A of X such that $|A| \leq 2^\kappa$ and $\bigcup\left\{\mathrm{cl}_\theta(B) : B \in [A]^{\leq\kappa}\right\} = X$.*

Our next result follows from Theorem 3.13:

Theorem 3.19. *Let $k \in \mathbb{N}^+$. If X is an $S(3k)$-space, then $d_{2k}(X) \leq 2^{s_{2k}(X)}$.*

Using the last theorem and Theorem 3.5 we get:

Theorem 3.20. *Let $k \in \mathbb{N}^+$. If X is an $S(3k)$-space, then $|X| \leq 2^{s_{2k}(X) bt_{2k}(X)}$.*

P r o o f. By Theorem 3.5 and Theorem 3.19 we have
$|X| \leq (d_{2k}(X))^{bt_{2k}(X)} \leq \left(2^{s_{2k}(X)}\right)^{bt_{2k}(X)} = 2^{s_{2k}(X) bt_{2k}(X)}$. □

As a corollary of Theorem 3.15 we obtain:

Theorem 3.21. *Let $k \in \mathbb{N}^+$. If X is an $S(3k-2)$-space, then $d_{2k-1}(X) \leq 2^{s_{2k-1}(X)}$.*

Using the previous theorem and Theorem 3.5 we get:

Theorem 3.22. *Let $k \in \mathbb{N}^+$. If X is an $S(3k-2)$-space, then $|X| \leq 2^{s_{2k-1}(X) bt_{2k-1}(X)}$.*

P r o o f. By Theorem 3.5 and Theorem 3.21 we have
$|X| \leq (d_{2k-1}(X))^{bt_{2k-1}(X)} \leq \left(2^{s_{2k-1}(X)}\right)^{bt_{2k-1}(X)} = 2^{s_{2k-1}(X) bt_{2k-1}(X)}$. □

As a consequence of Theorem 3.17 we have:

Theorem 3.23. *Let $k \in \mathbb{N}^+$. If X is an $S(3k-1)$-space, then $d_{2k}(X) \leq 2^{s_{2k-1}(X)}$.*

Using the last theorem and Theorem 3.5 we get:

Theorem 3.24. *Let $k \in \mathbb{N}^+$. If X is an $S(3k-1)$-space, then $|X| \leq 2^{s_{2k-1}(X) bt_{2k}(X)}$.*

P r o o f. By Theorem 3.5 and Theorem 3.23 we have
$|X| \leq (d_{2k}(X))^{bt_{2k}(X)} \leq \left(2^{s_{2k-1}(X)}\right)^{bt_{2k}(X)} = 2^{s_{2k-1}(X) bt_{2k}(X)}$. □

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