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LOCALLY FINITE MODULES
WITH NOETHER NORMALIZATION*

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Abstract. The aim of this note is to show that if a finite field $k$ with absolute Galois group $G$ acts on a set $M$ with finite orbits and for some $m$ there is a $G$-equivariant map $\xi : M \to k^m$, whose fibres are of bounded cardinality, then $M$ admits a $G$-equivariant embedding in an affine space $\overline{k}^n$ of sufficiently large dimension $n$.

1. Introduction. Grothendieck has noticed that the Galois theory of fields is related to the Galois theory of coverings through the bijective correspondence between the finite coverings $f : Y \to X$ of algebraic varieties over a field $k$ and the finite extensions $k(X) \subset k(Y)$ of function fields. This led him to the notion of a Galois category (cf. [1], [7], [4] or [3]). To any connected scheme $X$ Grothendieck associates a profinite group $\pi^\text{et}(X)$, called the etale fundamental group of $X$ and shows that the category of the finite etale coverings of $X$ is equivalent to the category of the finite sets with discrete topology, acted continuously by $\pi_1^\text{et}(X)$. In particular, if $k$ is a perfect field with algebraic closure $\overline{k}$ then the

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etale fundamental group \( \pi_1^{et}(k) = \text{Gal}(\overline{k}/k) \) coincides with the absolute Galois group of \( k \).

Note that \( \text{Gal}(\overline{k}/k) \) acts on any algebraic variety \( X \), defined over \( k \) and the finite extensions \( k_1 \supset k \) induce finite separable extensions \( k_1(X) \supset k(X) \) of function fields. For the interplay between \( k(X) \subset k_1(X) \) and the finite separable extensions \( k(X) \subset k(Y) \), arising from finite coverings \( Y \to X \) see [6] or [5]. In general, the absolute Galois group \( \text{Gal}(\overline{k}/k) \) of a perfect field \( k \) is quite complicated. However, for a finite field \( k = \mathbb{F}_q \), the group \( \mathfrak{G} = \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \mathbb{Z} \) is the profinite completion of the infinite cyclic group, generated by the Frobenius automorphism \( \Phi_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q \), \( \Phi_q(\alpha) = \alpha^q \). The \( \mathfrak{G} \)-orbits \( \text{Orb}_\mathfrak{G}(p) \) on a smooth projective curve \( C \ni p \), defined over \( \mathbb{F}_q \) correspond to the discrete valuation rings \( \mathcal{O}_p(C) \) of \( \mathbb{F}_q(X) \) in such a way that the cardinality of \( \text{Orb}_\mathfrak{G}(p) \) equals the degree \( [\mathcal{O}_p(C)/\mathbb{M}_p(C) : \mathbb{F}_q] \) of its associated valuation. Based on this fact, [2] introduces the Hasse-Weil \( \zeta \)-function \( \zeta_M(t) \) of a set \( M \), acted by \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) with finite orbits. By a combinatorial argument it derives a sufficient condition for the Riemann Hypothesis Analogue on \( \zeta_M(t) \).

The present note studies to what extent the \( \mathfrak{G} = \text{Gal}(\overline{k}/k) \)-action on an affine variety \( X \subseteq \overline{k}^n \), defined over \( k = \mathbb{F}_q \), determines the geometric properties of \( X \). We say that a set \( M \) with a \( \mathfrak{G} \)-action is a locally finite \( \mathfrak{G} \)-module if all \( \mathfrak{G} \)-orbits on \( M \) are finite and there are finitely many \( \mathfrak{G} \)-orbits of fixed cardinality. An arbitrary \( \mathfrak{G} \)-equivariant map \( f : M \to \overline{k}^m \) with fibres of cardinality \( \leq s \), \( s \in \mathbb{N} \) is called a Noether normalization of \( M \). By a combinatorial argument we prove that any locally finite \( \mathfrak{G} \)-module \( M \) with a Noether normalization admits a \( \mathfrak{G} \)-equivariant embedding \( M \hookrightarrow \overline{k}^n \) in an affine space of sufficiently large dimension \( n \). The affine varieties \( X \subset \overline{k}^n \), defined over \( k \) are locally finite \( \mathfrak{G} \)-modules with a Noether normalization, as well as all \( \mathfrak{G} \)-submodules \( M \subset X \). By specific examples we show that the category of the locally finite \( \mathfrak{G} \)-modules with a Noether normalization (whose morphisms are the \( \mathfrak{G} \)-equivariant maps) contains strictly the category of the quasi-affine varieties.

2. The absolute Galois group of a finite field and its action on the affine varieties. Let us start with some properties of the action of the absolute Galois group \( \mathfrak{G} = \text{Gal}(\overline{k},k) \) of a finite field \( k = \mathbb{F}_q \), on an affine variety \( X \subseteq \overline{k}^n \), defined over \( k \). If \( a = (a_1, \ldots, a_n) \in X \) then \( a_i \in \mathbb{F}_{q^m} \) for some \( m \in \mathbb{N} \) and all \( 1 \leq i \leq n \). An arbitrary \( \varphi \in \mathfrak{G} = \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) transforms \( a \) into \( \varphi(a) \in \mathbb{F}_{q^m}^n \), so that \( |\text{Orb}_\mathfrak{G}(a_1, \ldots, a_n)| \leq q^{mn} \) and all the \( \mathfrak{G} \)-orbits on \( X \) are finite. We refer to the number of elements of an orbit as of its degree. Since \( \mathbb{F}_{q^m} \supset \mathbb{F}_q \) is a normal extension, the orbits \( \text{Orb}_\mathfrak{G}(a_1, \ldots, a_n) = \text{Orb}_{\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)}^{\mathfrak{G}}(a_1, \ldots, a_n) \) coincide. The Galois group \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = \langle \Phi_q \rangle \rangle/\langle \Phi_q^m \rangle \) is cyclic of order \( m \).
and generated by the Frobenius automorphism $\Phi_q(x) = x^q$. If the $\mathfrak{G}$-orbit of $(a_1, \ldots, a_n) \in X$ is of degree $s$ then $(a_1^s, \ldots, a_n^s) = \Phi_q^s(a_1, \ldots, a_n) = (a_1, \ldots, a_n)$, whereas $(a_1, \ldots, a_n) \in F_{q^s}$. Thus, $X$ has finitely many $\mathfrak{G}$-orbits of fixed degree $s$.

The absolute Galois group $\mathfrak{G} = \text{Gal}(\overline{k}/k)$ is profinite as a projective limit of the finite Galois groups $\text{Gal}(L/k)$ of the finite Galois extensions $L \supset k$. In the case of a finite field $k$, any extension $L \supset k$ of degree $[L : k] = m$ is Galois and its Galois group $\text{Gal}(L/k) = \langle \Phi_q \rangle / \langle \Phi_q^m \rangle \simeq \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$ is a finite quotient group of the infinite cyclic group $\langle \Phi_q \rangle \simeq (\mathbb{Z}, +)$. That is why the absolute Galois group $\mathfrak{G} = \text{Gal}(\overline{k}/k) = \langle \Phi_q \rangle \simeq \hat{\mathbb{Z}}$ is the profinite completion of $\langle \Phi_q \rangle \simeq (\mathbb{Z}, +)$. Let us endow the finite Galois groups $\text{Gal}(L/k)$ with the discrete topology. Then the induced product topology on $\prod \text{Gal}(L/k)$ is compact and totally disconnected.

The closed subgroup $\mathfrak{G}$ of $\prod \text{Gal}(L/k)$ is compact and totally disconnected, as well. The next proposition establishes the continuity of the $\mathfrak{G}$-action on an affine variety $X$ with respect to the Zariski topology.

**Proposition 1.** If $X \subset \overline{k}^n$ is an affine variety, defined over a finite field $k$ then the action $\mu : \mathfrak{G} \times X \to X$ of $\mathfrak{G} = \text{Gal}(\overline{k}/k)$ on $X$ is continuous with respect to the Zariski topology on $X$.

**Proof.** The $\mathfrak{G}$-action on the algebraic closure $\overline{k}$ induces a $\mathfrak{G}$-action on the polynomials $\overline{k}[x_1, \ldots, x_n]$, which fixes all the variables $x_1, \ldots, x_n$. Let $\mu : \mathfrak{G} \times \overline{k}^n \to \overline{k}^n$ be the $\mathfrak{G}$-action on the affine space $\overline{k}^n$ and $V(f) = \{a \in \overline{k}^n | f(a) = 0\}$ for $f \in \overline{k}[x_1, \ldots, x_n]$. Since $X$ is a closed subset of $\overline{k}^n$, it suffices to show that $\mu^{-1}(V(f)) \subset \mathfrak{G} \times \overline{k}^n$ is a closed subset for any polynomial $f$, in order to conclude that $\mu^{-1}(V(f)) \cap (\mathfrak{G} \times X)$ is a closed subset of $\mathfrak{G} \times X$ and to prove the proposition.

Note that $f$ has finitely many coefficients and there is a finite extension $L \supset k$ with $f \in L[x_1, \ldots, x_n]$. The closed normal subgroup $\text{Gal}(\overline{k}/L)$ of $\mathfrak{G} = \text{Gal}(\overline{k}/k)$ of index $[\mathfrak{G} : \text{Gal}(\overline{k}/L)] = |\text{Gal}(L/k)| = [L : k] = m$ fixes $f$. If $\mathfrak{G} = \bigcup_{i=1}^m \text{Gal}(\overline{k}/L) \varphi_i$ is the decomposition of $\mathfrak{G}$ into a disjoint union of cosets modulo $\text{Gal}(\overline{k}/L)$ then

$$\mu^{-1}(V(f)) = \bigcup_{\varphi \in \mathfrak{G}} (\varphi \times V(\varphi^{-1}(f))) = \bigcup_{i=1}^m \text{Gal}(\overline{k}/L) \varphi_i \times V(\varphi_i^{-1}(f))$$

is a closed subset of $\mathfrak{G} \times \overline{k}^n$, as far as $\text{Gal}(\overline{k}/L) \varphi_i$ is a closed subset of $\mathfrak{G} = \text{Gal}(\overline{k}/k)$ and $V(\varphi_i^{-1}(f))$ is a closed subset of $\overline{k}^n$.

Note that the Zariski topology on an affine variety $X \subseteq \overline{k}^n$ is $T_1$ since the points are closed subsets of $X$. Generalizing the properties of the $\mathfrak{G}$-action on an affine variety $X \subseteq \overline{k}^n$, defined over $k$, we give the following

**Definition 2.** A set $M$ with an action of $\mathfrak{G}$ is called a $\mathfrak{G}$-module.

A $\mathfrak{G}$-module is locally finite if all $\mathfrak{G}$-orbits on $M$ are finite and for any $s \in \mathbb{N}$ there are finitely many $\mathfrak{G}$-orbits on $M$ of cardinality $s$. 
A $\mathcal{G}$-module $M$ is $T_1$-continuous if there is a $T_1$-topology on $M$, with respect to which the $\mathcal{G}$-action $\mathcal{G} \times M \to M$ is a continuous map.

3. Noether normalization. In the present section we start our study of the morphisms of $\mathcal{G}$-modules, i.e., of the $\mathcal{G}$-equivariant maps of $\mathcal{G}$-modules.

**Definition 3.** Let $\xi : M \to N$ be a morphism of $\mathcal{G}$-modules.

- If all the fibres of $\xi$ are finite sets then $\xi$ is called a finite morphism.
- If there exists $d \in \mathbb{N}$, such that all the fibres of $\xi$ are of cardinality $\leq d$ then $\xi$ is said to be of bounded degree $d$.
- A morphism $\xi : M \to N$ in a $\mathcal{G}$-submodule $N \subseteq \overline{k}^n$ of an affine space is dominant if the Zariski closure $\overline{\xi(M)} = N$ of the image of $\xi$ coincides with $N$.

**Definition 4.** If $M$ is a $\mathcal{G}$-module then any $\mathcal{G}$-equivariant map $\xi : M \to \overline{k}^d$ of bounded degree with Zariski dense image $\overline{\xi(M)} = \overline{k}^d$ is called a Noether normalization of $M$.

**Proposition 5.** Let $M \subseteq \overline{F_q^n}$ be a $\text{Gal}(\overline{F_q}/F_q)$-submodule of $\overline{F_q^n}$ with an irreducible Zariski closure $\overline{M} \subseteq \overline{F_q^n}$ of dimension $d$. Then there exist $m \in \mathbb{N}$, a $\text{Gal}(\overline{F_q}/F_q^m)$-submodule $M_1 \subseteq M$ with the same Zariski closure $\overline{M_1} = \overline{M}$ and a finite morphism $\xi : M_1 \to \overline{k}^d$ of $\text{Gal}(\overline{F_q}/F_q^m)$-modules of bounded degree with Zariski dense image $\overline{\xi(M_1)} = \overline{k}^d$.

**Proof.** The function field $\overline{k}(X)$ of the affine variety $X = \overline{M}$ is a finite extension of the function field $\overline{k}(y_1, \ldots, y_d)$ of $\overline{k}^d$ and there exists a non-empty Zariski open subset $U \subseteq X$ with a dominant regular map $\xi : U \to \overline{k}^d$, whose fibres are of cardinality $t := [\overline{k}(X) : \overline{k}(y_1, \ldots, y_d)]$. For a sufficiently small $U$ the map $\xi = \left( \frac{f_1}{g_1}, \ldots, \frac{f_d}{g_d} \right)$ is given by an ordered $d$-tuple of rational functions $\frac{f_i}{g_i} \in \overline{k}(x_1, \ldots, x_n)$. Any Zariski open subset $U \subseteq X$ is a finite union $U = \cup_{1 \leq j \leq s} U_{h_j}$ of principal Zariski open subsets $U_{h_j} = \{ (a_1, \ldots, a_n) \in X | h_j(a_1, \ldots, a_n) \neq 0 \}$, determined by polynomials $h_j \in \overline{k}[x_1, \ldots, x_n]$. If all the coefficients of $f_i, g_i$, $1 \leq i \leq n$ and of $h_j$, $1 \leq j \leq s$ are contained in $F_q^m \supseteq F_q = k$ for some $m \in \mathbb{N}$ then $\xi : U \to \overline{k}^d$ is a $\text{Gal}(\overline{k}/F_q^m)$-equivariant map of the $\text{Gal}(\overline{k}/F_q^m)$-submodule $U$ of $X$. The restriction $\xi |_{M \cap U} : M \cap U \to \overline{k}^d$ is a morphism of $\text{Gal}(\overline{k}/F_q^m)$-modules of degree $\leq t$. There remains to be shown that $\overline{M \cap U} = X$ and $\overline{\xi(M \cap U)} = \overline{k}^d$. 
An arbitrary non-empty open set \( \emptyset \neq W \subseteq X \) has non-empty open intersection with \( U \), due to the irreducibility of \( X \). Consequently, \( \emptyset \neq U \cap W \cap M \) since \( M \) is dense in \( X \). This proves the Zariski density of \( M \cap U \) in \( X \). Let us assume that \( \xi(M \cap U) \) is not Zariski dense in \( \overline{K}^d \). Then there is a non-empty Zariski open subset \( V \subseteq \overline{K}^d \) with \( \xi(M \cap U) \cap V = \emptyset \). The Zariski open subset \( \xi^{-1}(V) \subseteq X \) intersects the Zariski dense subset \( M \cap U \subseteq X \) and any \( x \in \xi^{-1}(V) \cap M \cap U \) maps to \( \xi(x) \in V \cap \xi(M \cap U) \). That contradicts the assumption \( \xi(M \cap U) \cap V = \emptyset \) and proves the Zariski density of \( \xi(M \cap U) \) in \( \overline{K}^d \). \( \square \)

The above proposition establishes that the submodules of affine spaces have a Noether normalization. We are going to show that any locally finite \( T_1 \)-continuous module with a Noether normalization admits an equivariant embedding in an affine space.

4. Affine embeddings of locally finite \( T_1 \)-continuous modules with a Noether normalization. We claim that if \( M \) is a locally finite \( T_1 \)-continuous module over \( \mathfrak{G} = \langle \Phi_q \rangle \), then the orbits \( \text{Orb}_\mathfrak{G}(x) = \text{Orb}_{\langle \Phi_q \rangle}(x) \) coincide. On one hand, \( \langle \Phi_q \rangle \) is residually finite and embeds in \( \mathfrak{G} \), so that \( \text{Orb}_{\langle \Phi_q \rangle}(x) \subseteq \text{Orb}_\mathfrak{G}(x) \). If \( \text{Orb}_{\langle \Phi_q \rangle}(x) \cap \text{Orb}_\mathfrak{G}(x) = \emptyset \) then \( \text{Stab}_{\langle \Phi_q \rangle}(x) \) is of index \( [\langle \Phi_q \rangle : \text{Stab}_{\langle \Phi_q \rangle}(x)] = m \), whereas \( \text{Stab}_\mathfrak{G}(x) = \langle \Phi_q^m \rangle \). The continuity of the action \( \mu : \mathfrak{G} \times M \to M \) with respect to a \( T_1 \)-topology on \( M \) implies the continuity of the maps \( \mu_y : \mathfrak{G} \to M, \mu_y(\varphi) = \varphi(y) \) for all \( y \in M \). The points \( y \in M \) form closed subsets \( \{ y \} \subseteq M \) with respect to any \( T_1 \)-topology on \( M \), so that \( \mu_y^{-1}(y) = \text{Stab}_\mathfrak{G}(y) \) are closed subgroups of \( \mathfrak{G} \). The closure of \( \langle \Phi_q^m \rangle \) in \( \mathfrak{G} = \text{Gal} (\overline{F}_q/F_q) \) coincides with the profinite completion \( \mathfrak{G}_m = \text{Gal} (\overline{F}_q/F^{\overline{m}}_q) \) of \( \langle \Phi_q^m \rangle \), so that \( \langle \Phi_q^m \rangle \subseteq \text{Stab}_\mathfrak{G}(x) \) implies \( \mathfrak{G}_m \subseteq \text{Stab}_\mathfrak{G}(x) \). As a result, \( m = [\mathfrak{G} : \mathfrak{G}_m] \geq [\mathfrak{G} : \text{Stab}_\mathfrak{G}(x)] = [\text{Orb}_\mathfrak{G}(x)] \geq [\text{Orb}_{\langle \Phi_q \rangle}(x)] = m \), whereas \( \text{Orb}_\mathfrak{G}(x) = \text{Orb}_{\langle \Phi_q \rangle}(x) \). Thus, the degree of \( \text{Orb}_\mathfrak{G}(x) \) is the minimal natural number \( m \) with \( \Phi_q^m(x) = x \).

**Definition 6.** Let \( M \) be a locally finite \( T_1 \)-continuous \( \mathfrak{G} \)-module. Then

- \( M^{\Phi_q^k} := \left\{ x \in M \mid \Phi_q^k(x) = x \right\} \) is called the set of the \( F_q^k \)-rational points of \( M \);
- \( N_k(M) := \left| M^{\Phi_q^k} \right| \) is the number of the \( F_q^k \)-rational points of \( M \);
- \( \mathfrak{B}_k(M) := \{ x \in M \mid \left| \text{Orb}_\mathfrak{G}(x) \right| = k \} \) is the set of the points of \( M \), whose \( \mathfrak{G} \)-orbits are of degree \( k \) and
- \( B_k(M) := \frac{1}{k} \left| \mathfrak{B}_k(M) \right| \) is the number of the \( \mathfrak{G} \)-orbits on \( M \) of degree \( k \).
Note that $\mathcal{B}_k(M)$ and $M^{\Phi_q^k}$ are $\mathfrak{G}$-modules, as far as $\mathfrak{G}$ is an abelian group and all the points from some $\mathfrak{G}$-orbit have coinciding stabilizers. Moreover, $\mathcal{B}_k(M) \subseteq M^{\Phi_q^k}$, so that $kB_k(M) \leq N_k(M)$.

**Proposition 7.** Let $L$ be a locally finite $\mathfrak{G}$-module and $k, n \in \mathbb{N}$ be natural numbers. Then for any $1 \leq i \leq n$ the set

$$L^{(i)}_k := (L^{\Phi_q^k})^{i-1} \times \mathcal{B}_k(L) \times (L^{\Phi_q^k})^{n-i} \subset (L^{\Phi_q^k})^n = (L^n)^{\Phi_q^k}$$

is contained in the $\mathfrak{G}$-submodule $\mathcal{B}_k(L^n)$ of $L^n$ and there holds the inequality

$$(1) \quad kB_k(L^n) \geq \bigg| \bigcup_{1 \leq i \leq n} L^{(i)}_k \bigg| = N_k(L)^n - [N_k(L) - kB_k(L)]^n$$

**Proof.** If $(a_1, \ldots, a_n) \in L^{(i)}_k$ then $d = |\text{Orb}_\mathfrak{G}(a_1, \ldots, a_n)|$ is the minimal natural number with $\Phi_q^d(a_1, \ldots, a_n) = (a_1^d, \ldots, a_n^d) = (a_1, \ldots, a_n)$, so that $d \leq k$. Since $k$ is the minimal natural number with $\Phi_q^k(a_i) = a_i$, there follow $k = d$ and $L^{(i)}_k \subseteq \mathcal{B}_k(L^n)$. Combining $\bigcup_{1 \leq i \leq n} L^{(i)}_k \subseteq \mathcal{B}_k(L^n)$ with

$$\bigcup_{1 \leq i \leq n} L^{(i)}_k = (L^{\Phi_q^k})^n \setminus \left[ (L^{\Phi_q^k})^n \setminus \bigcup_{1 \leq i \leq n} L^{(i)}_k \right] =$$

$$= (L^{\Phi_q^k})^n \setminus \left\{ \cap_{1 \leq i \leq n} [(L^{\Phi_q^k})^n \setminus L^{(i)}_k] \right\} =$$

$$= (L^{\Phi_q^k})^n \setminus \left\{ \cap_{1 \leq i \leq n} (L^{\Phi_q^k})^{i-1} \times [L^{\Phi_q^k} \setminus \mathcal{B}_k(L)] \times (L^{\Phi_q^k})^{n-i} \right\} =$$

$$= (L^{\Phi_q^k})^n \setminus \left\{ [L^{\Phi_q^k} \setminus \mathcal{B}_k(L)]^{n} \right\},$$

one derives (1). □

For an arbitrary morphism $\xi : M \to L$ of $\mathfrak{G}$-modules and an arbitrary point $x \in M$ one has $\text{Stab}_\mathfrak{G}(x) \leq \text{Stab}_\mathfrak{G}(\xi(x))$. Moreover, if the $\mathfrak{G}$-action on $M$ has finite orbits then one defines the inertia map

$$e_\xi : M \to \mathbb{Q},$$

$$e_\xi(x) := \frac{\deg \text{Orb}_\mathfrak{G}(x)}{\deg \text{Orb}_\mathfrak{G}(\xi(x))} = \frac{[\mathfrak{G} : \text{Stab}_\mathfrak{G}(x)]}{[\mathfrak{G} : \text{Stab}_\mathfrak{G}(\xi(x))]} = [\text{Stab}_\mathfrak{G}(\xi(x)) : \text{Stab}_\mathfrak{G}(x)] \in \mathbb{N}$$

and notes that it takes natural values. As far as the inertia map is constant on the $\mathfrak{G}$-orbits of $M$, the set $M^{[t]} = \{x \in M \mid e_\xi(x) = t\}$ is a $\mathfrak{G}$-submodule of $M$. 

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Let \( \xi : M \rightarrow L \) be a morphism of bounded degree \( d \) between locally finite \( T_1 \)-continuous \( \mathcal{G} \)-modules. Then

\[
\mathcal{B}_k(M^s) = \{ x \in M | k = \deg \text{Orb}_\mathcal{G}(x) = s \deg \text{Orb}_\mathcal{G}(\xi(x)) \} \neq \emptyset
\]

only when \( s \in \mathbb{N} \) divides \( k \in \mathbb{N} \). If so, then \( \xi(\mathcal{B}_k(M^s)) \subseteq \mathcal{B}_\frac{k}{s}(L) \cap \xi(M^s) = \mathcal{B}_\frac{k}{s}(\xi(M^s)). \) Conversely, if \( y \in \mathcal{B}_\frac{k}{s}(\xi(M^s)) \) then \( y = \xi(x) \) for some \( x \in M^s \).

As a result, \( \deg \text{Orb}_\mathcal{G}(x) = s \deg \text{Orb}_\mathcal{G}(\xi(x)) = k \), so that \( x \in \mathcal{B}_k(M^s) \). That justifies \( \mathcal{B}_\frac{k}{s}(\xi(M^s)) \subseteq \xi(\mathcal{B}_k(M^s)) \) and

\[
\xi(\mathcal{B}_k(M^s)) = \mathcal{B}_\frac{k}{s}(\xi(M^s)).
\]

In particular, \( \xi(\mathcal{B}_k(M^s)) \subseteq \mathcal{B}_\frac{k}{s}(L) \), so that \( \mathcal{B}_k(M^s) \subseteq \xi^{-1}(\mathcal{B}_\frac{k}{s}(L)) \) and there holds \( k B_k(M^s) \leq d \frac{k}{s} B_\frac{k}{s}(L) \). Therefore

\[
B_k(M^s) \leq d \frac{k}{s} B_\frac{k}{s}(L).
\]

Note that \( \xi(\text{Orb}_\mathcal{G}(x)) \subseteq \text{Orb}_\mathcal{G}(\xi(x)) \) implies \( \text{Orb}_\mathcal{G}(x) \subseteq \xi^{-1}(\text{Orb}_\mathcal{G}(\xi(x))) \), whereas \( \deg \text{Orb}_\mathcal{G}(x) \leq d \deg \text{Orb}_\mathcal{G}(\xi(x)) \). Therefore \( e_\xi(x) \leq d \). That allows to split \( M \) into a disjoint union \( M = \bigcup_{1 \leq i \leq d} M^i \) and to observe that

\[
B_k(M) = \sum_{1 \leq i \leq d} B_k(M^i) = \sum_{i \leq d; \ i/k} B_k(M^i) \leq \sum_{i \leq d; \ i/k} \frac{d}{i} B_\frac{k}{i}(L) = \frac{d}{k} \sum_{i \leq d; \ i/k} \frac{k}{i} B_\frac{k}{i}(L) \leq \frac{d}{k} N_k(L).
\]

In such a way, we have derived

\[
B_k(M) \leq \frac{d}{k} N_k(L).
\]

The inequalities (1) and (2) will be used for showing that an arbitrary locally finite \( T_1 \)-continuous \( \mathcal{G} \)-module with a Noether normalization admits a \( \mathcal{G} \)-equivariant embedding in an affine space of sufficiently large dimension. Prior to that, we derive a lower bound on \( B_k(\mathbb{F}_q^s) \).

**Proposition 8.** For any \( k \in \mathbb{N} \) there holds

\[
k B_k(\mathbb{F}_q^s) \geq q^{k/2}
\]

**Proof.** Let \( a \) be a generator of the multiplicative group \( \mathbb{F}_q^* = \langle a \rangle \). Then \( q^k - 1 \in \mathbb{N} \) is the minimal natural number with \( a^{q^k-1} = 1 \) and \( k \in \mathbb{N} \).
is the minimal natural number with \( a^q = a \), so that \( \text{Stab}_q(\Phi_q)(a) = \langle \Phi_q^k \rangle \) and \( \text{Orb}_q(\Phi_q)(a) = \text{Orb}_q(a) \) is of degree \( \deg \text{Orb}_q(a) = [\langle \Phi_q \rangle : \langle \Phi_q^k \rangle] = k \). For an arbitrary natural number \( 1 \leq s \leq q^k - 1 \), if \( \deg \text{Orb}_q(a^s) = \deg \text{Orb}_q(\Phi_q)(a^s) = d \) then

\[
\langle \Phi_q^d \rangle = \text{Stab}_q(\Phi_q)(a^s) \geq \text{Stab}_q(\Phi_q)(a) = \langle \Phi_q^k \rangle,
\]

whereas \( \Phi_q^k \in \langle \Phi_q^d \rangle \) and \( d \) divides \( k \). In particular, \( d \leq k \) and \( q^d - 1 \) divides \( q^k - 1 \). On the other hand, \( \Phi_q^d \in \text{Stab}_q(\Phi_q)(a^s) \) implies \( (a^s)^q = a^l \), whereas \( a^{s(q^d-1)} = 1 \). Therefore the order \( q^k - 1 \) of \( a \) divides \( s(q^d - 1) \) and, in particular, \( q^k - 1 \leq s(q^d - 1) \). As a result,

\[
s \geq \frac{q^k - 1}{q^d - 1} = q^{k-d} + q^{k-2d} + \ldots + q^d + 1 \geq q^{k-d} + 1.
\]

If \( d < k \) then \( k/d \in \mathbb{N} \), \( k/d > 1 \), whereas \( k/d \geq 2 \), which is equivalent to \( k/2 \geq d \). Therefore

\[
s \geq q^{k-d} + 1 \geq q^{k-k/2} + 1 > q^{k/2}
\]

whenever \( d < k \). In other words, for any \( 1 \leq s \leq q^{k/2} \) the orbit \( \text{Orb}_q(a^s) \) is of degree \( \deg \text{Orb}_q(a^s) = k \) and \( a^s \in \mathcal{B}_k(\mathbb{F}_q) \). That implies (3). □

Now, we are ready to prove our main result:

**Theorem 9.** Let \( M \) be a locally finite \( T_1 \)-continuous \( \mathcal{G} \)-module with a \( \mathcal{G} \)-equivariant map \( \xi : M \rightarrow \mathbb{F}_q^m \) of bounded degree \( d \) (i.e \( \xi \) is a Noether normalization of \( M \)). Then there exists a \( \mathcal{G} \)-equivariant embedding \( \mu : M \rightarrow \mathbb{F}_q^n \) for a sufficiently large \( n \in \mathbb{N} \).

**Proof.** For any \( k \in \mathbb{N} \) inequality (2) implies that

\[
B_k(M) \leq \frac{d}{k} N_k(\mathbb{F}_q^n) = \frac{d}{k} N_k(\mathbb{F}_q)^m = \frac{d}{k} (q^k)^m = \frac{d}{k} q^{km}.
\]

On the other hand, by (3) from Proposition 8 and (1) there follows

\[
B_k(\mathbb{F}_q^n) \geq \frac{N_k(\mathbb{F}_q) - [N_k(\mathbb{F}_q) - kB_k(\mathbb{F}_q)]^n}{k} = \frac{q^{kn} - (q^k - kB_k(\mathbb{F}_q))^n}{k} \geq \frac{q^{kn} - (q^k - q^{k/2})^n}{k}.
\]

We are going to show the existence of a natural number \( n \in \mathbb{N} \) with

\[
dq^{km} \leq q^{kn} - (q^k - q^{k/2})^n \text{ for all } k \in \mathbb{N},
\]

in order to have \( \mathcal{G} \)-equivariant embeddings \( \mu_k : \mathcal{B}_k(M) \rightarrow \mathcal{B}_k(\mathbb{F}_q^n) \) for all \( k \in \mathbb{N} \), which give rise to a \( \mathcal{G} \)-equivariant embedding \( \mu : M \rightarrow \mathbb{F}_q^n \). Note that (4) is
equivalent to
\[ q^{k(n-m)} - q^{k(n/2-m)} \left( q^{k/2} - 1 \right)^n - d \geq 0 \]
and consider the function
\[ f(x) := q^{x(n-m)} - q^{x(n/2-m)} \left( q^{x/2} - 1 \right)^n - d. \]

It suffices to prove that \( f(x) \) is an increasing function of a real variable \( x \in [1, +\infty) \) with \( f(1) \geq 0 \) for a sufficiently large \( n \in \mathbb{N} \), in order to establish that \( f(k) \geq 0 \) for all \( k \in \mathbb{N} \) and to conclude the proof of the theorem. To this end, let us introduce \( t := q^{x/2} \) and note that
\[ f(x) = t^{2(n-m)} - t^{n-2m}(t-1)^n - d = t^{n-2m}[t^n - (t-1)^n] - d. \]

The function \( h(t) := t^n - (t-1)^n \) takes positive values and increases for \( t \geq q^{1/2} \), as far as its derivative \( h'(t) = n[t^{n-1} - (t-1)^{n-1}] \geq 0 \). For \( n > 2m \) the function \( t^{n-2m} \) is non-negative and increasing, as well. Therefore \( f(x) \) is a non-negative increasing function on \( t \geq q^{1/2} \) and according to
\[ \frac{d}{dx} f(x) = \frac{d}{dt} f(x) \frac{dt}{dx} \geq 0 \]
on all \( x \geq 1 \). That suffices for \( f(x) \) to be an increasing function on \( x \in [1, +\infty) \), whenever \( n > 2m \).

There remains to be shown the existence of \( n \in \mathbb{N}, n > 2m \) with
\[ f(1) = q^{n-m} - q^{n/2-m} \left( q^{1/2} - 1 \right)^n - d \geq 0. \]

To this end, it suffices to prove that the auxiliary function
\[ g(x) := q^{x-m} - q^{x/2-m} \left( q^{1/2} - 1 \right)^x = q^{x/2-m} \left[ q^{x/2} - \left( q^{1/2} - 1 \right)^x \right] \]
tends to \( +\infty \) as \( x \to +\infty \). We denote by \( r \) the constant \( q^{1/2} \) and show that
\[ G(x) := \frac{r^x}{q^m} [r^x - (r-1)^x] \]
has \( \lim_{x \to +\infty} G(x) = +\infty \) for any fixed \( r > 1 \). The function \( g_1(x) := r^x - (r-1)^x \)
is strictly increasing, as far as it has a strictly positive derivative
\[ \frac{d}{dx} g_1(x) = \log(r) r^x - \log(r-1)(r-1)^x = \log(r)[r^x - (r-1)^x] + [\log(r) - \log(r-1)](r-1)^x > 0. \]

Therefore \( \lim_{x \to +\infty} g_1(x) = +\infty \), whereas
\[ \lim_{x \to +\infty} G(x) = \left( \lim_{x \to +\infty} \frac{r^x}{q^m} \right) \left( \lim_{x \to +\infty} g_1(x) \right) = +\infty. \]
for any fixed \( r > 1 \). In particular, for a sufficiently large \( n \in \mathbb{N} \) one has \( f(1) = g(n) \geq 0 \). \( \square \)

5. Some distinctions between the morphisms of \( \mathcal{G} \)-modules and the morphisms of affine varieties. It is well known that if \( f : X \to \overline{\mathbb{F}_q} \) is a finite morphism of affine varieties then \( X \) is a curve, \( f \) is of bounded degree \( d \) and \( f \) has a finite branch locus

\[
R := \{ z \in f(X) | |f^{-1}(z)| < d \}.
\]

The present section provides an example of a finite morphism \( \xi : M \to \overline{\mathbb{F}_q} \) of locally finite \( \mathcal{G} \)-modules of unbounded degree and an example of a finite morphism \( \eta : N \to \overline{\mathbb{F}_q} \) of locally finite \( \mathcal{G} \)-modules of bounded degree \( d \) with an infinite branch locus \( R \). These examples reveal that the locally finite \( T_1 \)-continuous \( \mathcal{G} \)-action allows a larger diversity of morphisms than the Zariski topology.

Let us consider the \( \mathcal{G} \)-submodules

\[
M := \{(a, b) \in \overline{\mathbb{F}_q}^2 | \deg Orb_\mathcal{G}(a) \neq \deg Orb_\mathcal{G}(b)\}
\]

of \( \overline{\mathbb{F}_q}^2 \) and \( \overline{\mathbb{F}_q} := \mathbb{F}_q \setminus \bigcup_{i \geq 2} \mathcal{B}_i(\overline{\mathbb{F}_q}) \) of \( \overline{\mathbb{F}_q} \). The map

\[
\xi : M \to \overline{\mathbb{F}_q}, \quad \xi(a, b) = \begin{cases} a & \text{for } \deg Orb_\mathcal{G}(a) > \deg Orb_\mathcal{G}(b), \\ b & \text{for } \deg Orb_\mathcal{G}(b) > \deg Orb_\mathcal{G}(a) \end{cases}
\]

is \( \mathcal{G} \)-equivariant and has finite fibres

\[
\xi^{-1}(a) = \bigcup_{1 \leq i < \deg Orb_\mathcal{G}(a)} \mathcal{B}_i(\overline{\mathbb{F}_q}) \times \{a\} \cup \{a\} \times \bigcup_{1 \leq i < \deg Orb_\mathcal{G}(a)} \mathcal{B}_i(\overline{\mathbb{F}_q})
\]

of unbounded degree.

Let \( d \in \mathbb{N} \) be coprime to \( q \), \( X_o := \{(y^d, y) | y \in \mathbb{F}_q\} \) and \( \eta : X_o \to \overline{\mathbb{F}_q}, \eta(y^d, y) = y^d \) be the first canonical projection. Then \( X_o \) is a \( \mathcal{G} \)-submodule of \( \overline{\mathbb{F}_q}^2 \) and \( \eta \) is a morphism of \( X_o \) onto \( \overline{\mathbb{F}_q} \). All the fibres of \( \eta \) except \( \eta^{-1}(0) = (0, 0) \) are of cardinality \( d \). We are going to show that if \( \delta \in \mathbb{N}, \delta > \log_q(d-1) \) and \( \beta \) is a generator of \( \mathbb{F}_q^{*\delta} = \langle \beta \rangle \) then the inertia index of \( \eta : X_o \to \overline{\mathbb{F}_q} \) at \( (\beta^d, \beta) \in X_o \) is \( e_\eta(\beta^d, \beta) < d \). Therefore \( \eta^{-1} Orb_\mathcal{G}(\beta^d) \supsetneq Orb_\mathcal{G}(\beta^d, \beta) \) and

\[
N := X_o \setminus \bigcup_{(\beta) = \mathbb{F}_q^{*\delta}, \delta > \log_q(d-1)} Orb_\mathcal{G}(\beta^d, \beta)
\]
is a $G$-submodule of $X_o$ with a finite morphism $\eta : N \to \overline{\mathbb{F}}_q$, whose branch locus

$$R := \{ z \in \overline{\mathbb{F}}_q \mid |\eta^{-1}(z) \cap N| < d \} \supseteq \bigcup_{(\beta) = \mathbb{F}^*_{q^d \delta}, \delta > \log_q(d-1)} \text{Orb}_G(\beta^d)$$

is infinite. Note that there are infinitely many fibres of $\eta : N \to \overline{\mathbb{F}}_q$ of cardinality $d$. For instance, for any natural number $1 \leq r \leq d - 1$ and any generator $\gamma_{r, \delta}$ of $\mathbb{F}^*_{q^d + r} = \langle \gamma_{r, \delta} \rangle$ the fibre $\eta^{-1}(\gamma_{r, \delta}^d)$ is of cardinality $d$ and there are infinitely many such $\gamma_{r, \delta}$ with $\delta > \log_q(d - 1)$. Towards $e_\eta(\beta^d, \beta) < d$, note that if $\beta$ is a generator of $\mathbb{F}^*_{q^d} = \langle \beta \rangle$ then $\text{deg Orb}_G(\beta^d, \beta) = \text{deg Orb}_G(\beta) = d\delta$ and $\beta^d \in \mathbb{F}^*_{q^d \delta}$ is of order

$$\text{ord}(\beta^d) = \frac{\text{ord}(\beta)}{\text{GCD}(\text{ord}(\beta), d)} = \frac{q^{d\delta} - 1}{\text{GCD}(q^{d\delta} - 1, d)}.$$ 

If $e_\eta(\beta^d, \beta) = d$ then

$$\text{deg Orb}_G(\beta^d) = \frac{\text{deg Orb}_G(\beta^d, \beta)}{e_\eta(\beta^d, \beta)} = \frac{d\delta}{d} = \delta,$$

so that $\text{Stab}_G(\beta^d) = \langle \Phi^d \rangle$ and $(\beta^d)^q^\delta = \beta^d$. As a result, $(\beta^d)^q^\delta - 1 = 1$ and the order $\text{ord}(\beta^d)$ of $\beta^d \in \mathbb{F}^*_{q^d \delta}$ divides $q^\delta - 1$, i.e.,

$$\frac{q^{d\delta} - 1}{\text{GCD}(q^{d\delta} - 1, d)} r = q^\delta - 1 \quad \text{for some} \quad r \in \mathbb{N}.$$ 

Now,

$$q^\delta + 1 \leq q^{d\delta - \delta} + q^{d\delta - 2\delta} + \ldots + q^\delta + 1 = \frac{q^{d\delta} - 1}{q^\delta - 1} \leq \frac{q^{d\delta} - 1}{q^\delta - 1} r = \text{GCD}(q^{d\delta} - 1, d) \leq d$$

implies that $\delta \leq \log_q(d - 1)$. In such a way we have shown that if $e_\eta(\beta^d, \beta) = d$ for a generator $\beta$ of $\mathbb{F}^*_{q^d} = \langle \beta \rangle$ then $\delta \leq \log_q(d - 1)$. Bearing in mind that $e_\eta(\beta^d, \beta) \leq d$ for all $\beta \in \overline{\mathbb{F}}_q$, one concludes that $e_\eta(\beta^d, \beta) < d$ for any generator $\beta$ of $\mathbb{F}^*_{q^d} = \langle \beta \rangle$ with $\delta > \log_q(d - 1)$.

In the light of the previous example of a morphism $\eta : N \to \overline{\mathbb{F}}_q$ of bounded degree with infinite branch locus, one questions the existence of Noether normalizations $\xi_1 : M \to \overline{\mathbb{F}}_q^{m_1}, \xi_2 : M \to \overline{\mathbb{F}}_q^{m_2}$ of one and a same locally finite $G$-module $M$ with images of different dimensions $m_1 \neq m_2$. 
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