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VARIETIES OF BICOMMUTATIVE ALGEBRAS*

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Abstract. Bicommutative algebras are nonassociative algebras satisfying the polynomial identities of right- and left-commutativity $(x_1x_2)x_3 = (x_1x_3)x_2$ and $x_1(x_2x_3) = x_2(x_1x_3)$. Let $\mathfrak{B}$ be the variety of all bicommutative algebras over a field K of characteristic 0 and let $F(\mathfrak{B})$ be the free algebra of countable rank in $\mathfrak{B}$. We prove that if $\mathfrak{V}$ is a subvariety of $\mathfrak{B}$ satisfying a polynomial identity $f = 0$ of degree k, where $0 \neq f \in F(\mathfrak{B})$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \ldots$, is bounded by a polynomial in n of degree $k-1$. Since $c_n(\mathfrak{B}) = 2^n - 2$ for $n \geq 2$, and $\exp(\mathfrak{B}) = 2$, this gives that $\exp(\mathfrak{V}) \leq 1$, i.e., $\mathfrak{B}$ is minimal with respect to the codimension growth. When the field K is algebraically closed there are only three pairwise nonisomorphic two-dimensional bicommutative algebras A which are nonassociative. They are one-generated and with the property $\dim A^2 = 1$. We present bases of their polynomial identities and show that one of these algebras generates the whole variety $\mathfrak{B}$.

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1. Introduction. Bicommutative algebras are nonassociative algebras over a field K satisfying the polynomial identities of right- and left-commutativity

$$(x_1x_2)x_3 = (x_1x_3)x_2, \quad x_1(x_2x_3) = x_2(x_1x_3).$$

In the sequel we consider algebras over a field K of characteristic 0 only. One-sided commutative algebras appeared first in the paper by Cayley [7] in 1857. In the modern language this is the right-symmetric Witt algebra W_1^{rsym} in one variable. Maybe the most important examples of one-side commutative algebras are Novikov algebras which are left-commutative and right-symmetric. The latter means that the algebras satisfy the polynomial identity $(x_1, x_2, x_3) = (x_1, x_3, x_2)$ for the associator $(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$. The motivation to study Novikov algebras comes from the needs of the Hamiltonian operator in mechanics and the equations of hydrodynamics, see Dzhumadil'daev, Ismailov, and Tulenbaev [13] and Drensky and Zhakhayev [12] for details. By Kaygorodov and Volkov [22] when the base field K is algebraically closed of arbitrary characteristic up to isomorphism there are only seven two-dimensional bicommutative algebras A with nontrivial multiplication. Four of them are associative-commutative. Changing the notation and the bases of the algebras in [22] the three nonassociative two-dimensional bicommutative algebras

$$A_{\pi,\varrho}, \quad (\pi, \varrho) = (0, 1), (1, 0), (1, -1),$$

are generated by one element r and satisfy the condition $\dim A_{\pi,\varrho}^2 = 1$. Their multiplication rules are

$$rr^2 = \pi r^2, r^2r = \varrho r^2, r^2r^2 = \pi\varrho r^2. \tag{1}$$

In is easy to see that the same holds over an arbitrary field K of characteristic 0: Up to isomorphism the three algebras $A_{0,1}, A_{1,0}, A_{1,-1}$, are the only one-generated nonassociative two-dimensional bicommutative algebras A with $\dim A^2 = 1$.

The structure of the free bicommutative algebra and the most important numerical invariants of the T-ideal of the polynomial identities were described by Dzhumadil'daev, Ismailov, and Tulenbaev [13], see also the announcement [14]. In [12], jointly with Zhakhayev, we proved that finitely generated bicommutative algebras are weakly noetherian, i.e., satisfy the ascending chain condition for two-sided ideals, and answer into affirmative the finite basis problem for varieties of bicommutative algebras over a field of arbitrary characteristic.

One of the most important measures for the complexity of the polynomial identities of a variety $\mathfrak{V}$ of K-algebras is the codimension sequence $c_n(\mathfrak{V})$,

$n = 1, 2, \ldots$, where $c_n(\mathfrak{V})$ is the dimension of the multilinear polynomials of degree n in the free algebra $F_n(\mathfrak{V})$ of rank n. As a first approximation to the more precise estimate of the growth of the codimensions one studies the behaviour of $\sqrt[n]{c_n(\mathfrak{V})}$. In the special case when

$$\exp(\mathfrak{V}) = \lim_{n\to\infty} \sqrt[n]{c_n(\mathfrak{V})}$$

exists it is called the *exponent* of $\mathfrak{V}$, see Giambruno and Zaicev [16, 17] who proved that for associative PI-algebras the exponent always exists and is an integer. Following [18] the variety $\mathfrak{V}$ is *minimal of a given exponent* if $\exp(\mathfrak{W}) < \exp(\mathfrak{V})$ for all proper subvarieties $\mathfrak{W}$ of $\mathfrak{V}$. (In [10] we called such varieties *extremal.*)

It was shown in [13] that for the variety $\mathfrak{B}$ of all bicommutative algebras

$$c_1(\mathfrak{B}) = 1 \text{ and } c_n(\mathfrak{B}) = 2^n - 2, \quad n = 2, 3, \ldots.$$

Hence $\exp(\mathfrak{B}) = 2$. Our first main result is that the variety $\mathfrak{B}$ is minimal of exponent 2. More precisely we show that if $\mathfrak{V}$ is a subvariety of $\mathfrak{B}$ satisfying a polynomial identity $f = 0$ of degree k, where $0 \neq f \in F(\mathfrak{B}) = F_\infty(\mathfrak{B})$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \ldots$, is bounded by a polynomial in n of degree $k - 1$. The results of [13] give that the variety $\mathfrak{B}$ is generated by the free algebra $F_2(\mathfrak{B})$ of rank 2. We slightly improve this and show that $\mathfrak{B}$ is generated by the free algebra $F_1(\mathfrak{B})$ of rank 1. As a byproduct of our approach, starting with the basis of $F(\mathfrak{B})$ in [13] we give a new proof of the description of the cocharacter sequence $\chi_n(\mathfrak{B})$, $n = 1, 2, \ldots$. Finally we study the polynomial identities of the two-dimensional algebras $A_{\pi,\varrho}$ with multiplication defined by (1). We show that the algebra $A_{1,-1}$ generates the whole variety $\mathfrak{B}$. The varieties $\mathrm{var}(A_{0,1})$ and $\mathrm{var}(A_{1,0})$ generated respectively by the algebras $A_{0,1}$ and $A_{1,0}$ are defined as subvarieties of $\mathfrak{B}$ by the polynomial identities $x_1(x_2x_3) = 0$ and $(x_1x_2)x_3 = 0$, i.e., they are equal, respectively, to the varieties of left-nilpotent and right-nilpotent of class 3 bicommutative algebras.

2. Preliminaries. We fix a field K of characteristic 0. All algebras, vector spaces, and tensor products will be over K. Traditionally, one states the results on polynomial identities and cocharacter sequences in the language of representation theory of the symmetric group S_n. Instead, we shall work with representation theory of the general linear group $\mathrm{GL}_d = \mathrm{GL}_d(K)$. Then using the approach developed by Berele [6] and the author [9] we shall translate easily the results in terms of representations of S_n. We start with the necessary background on representation theory of GL_d acting canonically on the d-dimensional vector space KX_d with basis $X_d = \{x_1, \ldots, x_d\}$. We refer, e.g., to the books by Macdonald [24] for general facts and by the author [11] for applications in the spirit of the problems considered here. All GL_d-modules which appear in this paper are completely reducible and are direct sums of irreducible polynomial modules. The irreducible polynomial representations of GL_d are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$. We denote by $W(\lambda) = W_d(\lambda)$ the corresponding irreducible GL_d-module. The action of GL_d on KX_d is extended diagonally on the tensor algebra of KX_d and, up to isomorphism, all $W(\lambda)$ can be found there. The tensor algebra of KX_d is isomorphic, also as a GL_d-module, to the free associative algebra $K\langle X_d\rangle = K\langle x_1, \ldots, x_d\rangle$. Since the diagonal action of GL_d on the tensor algebra is not affected by the parentheses, we may work also in the absolutely free algebra $K\{X_d\}$ and in the relatively free algebra $F_d(\mathfrak{V})$ of any variety $\mathfrak{V}$.

The module $W(\lambda) \subset K\{X_d\}$ is generated by a unique, up to a multiplicative constant, multihomogeneous element w_λ of degree $\lambda = (\lambda_1, \ldots, \lambda_d)$, i.e., homogeneous of degree λ_k with respect to each variable x_k, called the *highest weight vector* of $W(\lambda)$. In order to state the characterization of the highest weight vectors we recall that for an algebra R the linear operator $\delta : R \to R$ is a derivation if $\delta(r_1r_2) = \delta(r_1)r_2 + r_1\delta(r_2)$ for all $r_1, r_2 \in R$. If $\delta : KX_d \to KX_d$ is any linear operator of the d-dimensional vector space, then δ can be extended in a unique way to a derivation of $K\langle X_d\rangle, K\{X_d\}$, and of any relatively algebra $F_d(\mathfrak{V})$. The following lemma is a partial case of a result by De Concini, Eisenbud, and Procesi [8], see also Almkvist, Dicks, and Formanek [2]. In the version which we need, the first part of the lemma was established by Koshlukov [23].

Lemma 2.1 (see, e.g., Benanti and Drensky [5]). *Let $1 \leq i < j \leq d$ and let $\Delta_{x_j\to x_i}$ be the derivation of $K\{X_d\}$ defined by $\Delta_{x_j\to x_i}(x_j) = x_i$, $\Delta_{x_j\to x_i}(x_k) = 0$, $k \neq j$. If $w(X_d) = w(x_1, \ldots, x_d) \in K\{X_d\}$ is multihomogeneous of degree λ_k with respect to x_k, then $w(X_d)$ is a highest weight vector for some $W(\lambda)$ if and only if $\Delta_{x_j\to x_i}(w(X_d)) = 0$ for all $i < j$. Equivalently, $w(X_d)$ is a highest weight vector for $W(\lambda)$ if and only if*

$$g_{ij}(w(X_d)) = w(X_d), \quad 1 \leq i < j \leq d,$$

where g_{ij} is the linear operator of the KX_d which sends x_j to $x_i + x_j$ and fixes the other x_k.

If W_i, $i = 1, \ldots, m$, are m isomorphic copies of the GL_d-module $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1 \oplus \cdots \oplus W_m$ has the form $\xi_1 w_1 + \cdots + \xi_m w_m$ for some $\xi_i \in K$. Any m linearly independent highest weight vectors can serve as a set of generators of the GL_d-module $W_1 \oplus \cdots \oplus W_m$. The algebra $F_d(\mathfrak{V})$ decomposes as a GL_d-module as

$$F_d(\mathfrak{V}) = \bigoplus_{\lambda} m_\lambda(\mathfrak{V}) W(\lambda), \tag{2}$$

where the summation runs on all partitions λ in not more than d parts and the nonnegative integer $m_\lambda(\mathfrak{V})$ is the multiplicity of $W(\lambda)$ in the decomposition of $F_d(\mathfrak{V})$. The canonical multigrading of $F_d(\mathfrak{V})$ which counts the degree of each variable in X_d agrees with the action of GL_d in the following way. Let

$$H(F_d(\mathfrak{V}), T_d) = H(F_d(\mathfrak{V}), t_1, \ldots, t_d)$$

$$= \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) T_d^n = \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) t_1^{n_1} \cdots t_d^{n_d}$$

be the Hilbert series of $F_d(\mathfrak{V})$ as a multigraded vector space, where $F_d^{(n)}(\mathfrak{V})$ is the multihomogeneous component of $F_d(\mathfrak{V})$ of degree $n = (n_1, \ldots, n_d)$. Then

$$H(F_d(\mathfrak{V}), T_d) = \sum_{\lambda} m_\lambda(\mathfrak{V}) s_\lambda(T_d) = \sum_{\lambda} m_\lambda(\mathfrak{V}) s_\lambda(t_1, \ldots, t_d),$$

where $s_\lambda(T_d)$ is the Schur function corresponding to the partition λ.

There is another group action which is important for the theory of algebras with polynomial identities. The symmetric group S_n acts on the vector space $P_n(\mathfrak{V})$ of the multilinear polynomials of degree n in $F_n(\mathfrak{V})$ by

$$\sigma(f(x_1, \ldots, x_n)) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n, f \in P_n(\mathfrak{V}).$$

The S_n-character of $P_n(\mathfrak{V})$ is called the S_n-*cocharacter* of $\mathfrak{V}$. It is known that the decomposition of the n-th cocharacter

$$\chi_n(\mathfrak{V}) = \sum_{\lambda \vdash n} m_\lambda(\mathfrak{V}) \chi_\lambda, \tag{3}$$

where χ_λ is the irreducible S_n-character indexed with the partition λ of n, is determined by the Hilbert series of $F_n(\mathfrak{V})$. The multiplicities $m_\lambda(\mathfrak{V})$ are the same for $F_n(\mathfrak{V})$ in (2) and for $\chi_n(\mathfrak{V})$ in (3). Finally, we recall a special case of the Young rule (and of the Littlewood-Richardson rule) for the product of two Schur functions $s_{(p)}(T_d)$ and $s_{(q)}(T_d)$ (and also for the tensor product $W(p) \otimes W(q)$

of the GL_d-modules $W(p)$ and $W(q)$). We assume that $p \geq q$. The case $p < q$ is similar.

$$s_{(p)}(T_d)s_{(q)}(T_d) = \sum_{k=0}^{q} s_{(p+q-k,k)}(T_d),$$
$$W(p) \otimes W(q) \cong \bigoplus_{k=0}^{q} W(p+q-k,k). \tag{4}$$

We shall need also estimates for the degree of the irreducible S_n-characters.

Lemma 2.2. *The degree d_λ of the irreducible S_n-character χ_λ, $\lambda = (\lambda_1, \lambda_2) \vdash n$, is a polynomial in n of degree λ_2.*

Proof. By the hook formula

$$d_\lambda = \frac{n!}{\prod h_{ij}},$$

where h_{ij} is the length of the hook at the (i,j)-position of the Young diagram of λ. For $\lambda = (\lambda_1, \lambda_2) \vdash n$ the lengths of the hooks of the first row are equal, reading them from right to left, to

$$1, 2, \ldots, n - 2\lambda_2, n - 2\lambda_2 + 2, \ldots, n - \lambda_2 + 1$$

and those of the second row are $1, 2, \ldots, \lambda_2$. Hence

$$d_\lambda = \frac{n(n-1)\cdots(n-\lambda_2+1)(n-2\lambda_2+1)}{\lambda_2!},$$

which is a polynomial of degree λ_2 in n. □

Let $\mathfrak{B}$ be the variety of all bicommutative algebras. We assume that the free bicommutative algebras $F = F(\mathfrak{B})$ and $F_d = F_d(\mathfrak{B})$ are freely generated, respectively, by the sets $X = \{x_1, x_2, \ldots\}$ and $X_d = \{x_1, \ldots, x_d\}$. By [13] the basis of the square F_d^2 of the algebra F_d as a K-vector space consists of the following polynomials:

$$u_{i,j} = x_{i_1}(\cdots(x_{i_{p-1}}((\cdots((x_{i_p}x_{j_1})x_{j_2})\cdots)x_{j_q}))\cdots), \tag{5}$$

where $p, q \geq 1$, $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq i_p \leq d$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_q \leq d$. For any permutations $\sigma \in S_p$ and $\tau \in S_q$ the element $u_{i,j}$ from (5) satisfy the equality

$$u_{i,j} = x_{i_{\sigma(1)}}(\cdots(x_{i_{\sigma(p-1)}}((\cdots((x_{i_{\sigma(p)}}x_{j_{\tau(1)}})x_{j_{\tau(2)}})\cdots)x_{j_{\tau(q)}}))\cdots). \tag{6}$$

The properties and the multiplication rules of $F_d(\mathfrak{B})$ from [13] are restated in [12] in the following way. We consider the polynomial algebra

$$K[Y_d, Z_d] = K[y_1, \ldots, y_d, z_1, \ldots, z_d]$$

in $2d$ commutative and associative variables. We associate to each monomial $u_{i,j}$ in (5) the monomial

$$\psi(u_{i,j}) = y_{i_1} \cdots y_{i_{p-1}} y_{i_p} z_{j_1} z_{j_2} \cdots z_{j_q} \in K[Y_d, Z_d]$$

and extend ψ by linearity to a linear map $\psi : F_d^2 \to K[Y_d, Z_d]$. The image $\psi(F_d^2)$ is spanned by all monomials

$$Y_d^\alpha Z_d^\beta = y_1^{\alpha_1} \cdots y_d^{\alpha_d} z_1^{\beta_1} \cdots z_d^{\beta_d}, \quad |\alpha| = \sum_{i=1}^{d} \alpha_i > 0, |\beta| = \sum_{j=1}^{d} \beta_j > 0.$$

Then we define an algebra G_d generated by X_d with basis

$$X_d \cup \{Y_d^\alpha Z_d^\beta \mid |\alpha|, |\beta| > 0\}$$

and multiplication rules

$$x_i x_j = y_i z_j,$$

$$x_i(Y_d^\alpha Z_d^\beta) = y_i Y_d^\alpha Z_d^\beta,$$

$$(Y_d^\alpha Z_d^\beta) x_j = Y_d^\alpha Z_d^\beta z_j,$$

$$(Y_d^\alpha Z_d^\beta)(Y_d^\gamma Z_d^\delta) = Y_d^{\alpha+\gamma} Z_d^{\beta+\delta}.$$

The algebras F_d and G_d are isomorphic both as algebras and as multigraded vector spaces with isomorphism $\psi : F_d \to G_d$ which sends $x_i \in F_d$ to $x_i \in G_d$ and acts on F_d^2 in the same way as the linear map $\psi : F_d^2 \to K[Y_d, Z_d]$ defined above.

3. Free bicommutative algebras. In this section we give an alternative proof of the formula for the cocharacter sequence of $\mathfrak{B}$ given in [13] and describe the highest weight vectors of the irreducible GL_d-submodules of $F_d = F_d(\mathfrak{B})$. The action of GL_d on the d-dimensional vector space KX_d induces a similar action on KY_d and KZ_d which is extended diagonally on the polynomial algebras $K[Y_d]$ and $K[Z_d]$ and on the square G_d^2 of the algebra G_d. In the sequel we equip $K[Y_d]$, $K[Z_d]$, and G_d^2 with this action of GL_d.

Lemma 3.1. *As a multigraded vector space the square F_d^2 of the free bicommutative algebra F_d is isomorphic to the tensor product $\omega(K[Y_d])\otimes\omega(K[Z_d])$, where ω is the augmentation ideal of the polynomial algebra, i.e., the ideal of polynomials without constant term. As a GL_d-module F_d^2 is isomorphic to the direct sum of tensor products*

$$\bigoplus_{p,q\geq 1} W(p)\otimes W(q).$$

Proof. We identify the monomial $Y_d^\alpha Z_d^\beta \in K[Y_d,Z_d]$ with $Y_d^\alpha\otimes Z_d^\beta\in K[Y_d]\otimes K[Z_d]$. Then the first part of the lemma is simply a restatement of the fact that the image of F_d^2 under the action of ψ has a basis $\{Y_d^\alpha Z_d^\beta \mid |\alpha|,|\beta|>0\}$. The second part of the lemma holds because the GL_d-module $K[Y_d]^{(p)}$ of the homogeneous polynomials of degree p in $K[Y_d]$ is isomorphic to $W(p)$ and similarly for $K[Z_d]^{(q)}$. □

Proposition 3.2 ([13]). *The cocharacter sequence of the variety $\mathfrak{B}$ of all bicommutative algebras is*

$$\chi_n(\mathfrak{B})=\sum_{(\lambda_1,\lambda_2)\vdash n} m_{(\lambda_1,\lambda_2)}(\mathfrak{B})\chi_{(\lambda_1,\lambda_2)},$$

where

$$\begin{aligned} m_{(1)}(\mathfrak{B})&=1,\\ m_{(n)}(\mathfrak{B})&=n-1,\quad n>1,\\ m_{(\lambda_1,\lambda_2)}(\mathfrak{B})&=n-2\lambda_2+1,\quad \lambda_2>0.\end{aligned}$$

Proof. The multiplicities of the irreducible S_n-characters in the cocharacter sequence (3) and of the irreducible GL_d-modules of the homogeneous component $F_d^{(n)}$ of degree n of the free algebra F_d in (2) are the same for $d\geq n$. Hence we may work in F_d instead of with $\chi_n(\mathfrak{B})$. Since the case $n=1$ is trivial, we shall assume that $n>1$. By Lemma 3.1 and the Young rule (4) we derive that the only nontrivial multiplicities $m_\lambda(\mathfrak{B})$ are for $\lambda=(\lambda_1,\lambda_2)$. Then $m_\lambda(\mathfrak{B})$ is equal to the number of tensor products $W(p)\otimes W(q)$ which contain an isomorphic copy of $W(\lambda)$ as a submodule. For $\lambda=(n)$ there are $n-1$ possibilities

$$W(1)\otimes W(n-1), W(2)\otimes W(n-2),\ldots, W(n-1)\otimes W(1),$$

i.e., $m_{(n)}(\mathfrak{B})=n-1$. For $\lambda=(\lambda_1,\lambda_2)$ with $\lambda_2>0$ the possibilities are

$$W(\lambda_2)\otimes W(n-\lambda_2), W(\lambda_2+1)\otimes W(n-\lambda_2-1),\ldots, W(n-\lambda_2)\otimes W(\lambda_2),$$

which gives $m_{(\lambda_1,\lambda_2)}(\mathfrak{B})=n-2\lambda_2+1$. □

Lemma 3.3. *The following polynomials* $w_\lambda^{(k)}$ *form a maximal linearly independent system of highest weight vectors of the* GL_d*-submodules* $W(\lambda)$ *in* G_d^2*:*

$$w_{(n)}^{(j)} = y_1^j z_1^{n-j}, \quad j = 1, 2, \dots, n-1, \tag{7}$$
$$w_\lambda^{(j)} = y_1^j (y_1z_2 - y_2z_1)^{\lambda_2} z_1^{\lambda_1-\lambda_2-j}, \quad j = 0, 1, \dots, \lambda_1 - \lambda_2, \ \textit{if}\ \lambda_2 > 0.$$

P r o o f. For a fixed λ the elements (7) are linearly independent because are nonzero and of pairwise different degree in y_1. They are of degree λ_1 with respect to y_1, z_1 and of degree λ_2 with respect to y_2, z_2. By Proposition 3.2 the multiplicities of $W(n)$ and $W(\lambda)$, $\lambda = (\lambda_1, \lambda_2) \vdash n$, in G_d^2 are, respectively,

$$m_{(n)}(\mathfrak{B}) = n - 1 \text{ and } m_{(\lambda_1,\lambda_2)}(\mathfrak{B}) = n - 2\lambda_2 + 1 = \lambda_1 - \lambda_2 + 1.$$

Hence their number coincides with the number of polynomials in (7). Now, it is sufficient to show that all $w_\lambda^{(j)}$ are highest weight vectors. Applying Lemma 2.1, this is obvious for $w_{(n)}^{(j)}$. Let $\lambda_2 > 0$. The analogue $\Delta_{y_2\to y_1, z_2\to z_1}$ of the derivation $\Delta_{x_2\to x_1}$ acting on $K[Y_d, Z_d]$ sends y_1, z_1 to 0 and y_2, z_2 to y_1, z_1, respectively. Obviously

$$\Delta_{y_2\to y_1,z_2\to z_1}(w_\lambda^{(j)}) = \lambda_2 y_1^j (y_1z_2 - y_2z_1)^{\lambda_2-1} z_1^{\lambda_1-\lambda_2-j} \Delta_{y_2\to y_1,z_2\to z_1}(y_1z_2 - y_2z_1) = 0$$

and all $w_\lambda^{(j)}$ are highest weight vectors. □

4. Subvarieties. In this section we assume that $\mathfrak{V}$ is a proper subvariety of $\mathfrak{B}$ and $\mathfrak{V}$ satisfies a nontrivial polynomial identity $f = 0$ of degree k, where $0 \neq f(X_d) \in F = F(\mathfrak{B})$. Since the case $k = 1$ is trivial we shall assume that $k \geq 2$. In the sequel we shall work mainly in the isomorphic copies G and G_d of the algebras F and F_d instead of in F and F_d. Identifying F and F_d with their isomorphic copies, we shall denote the corresponding elements with the same symbols. In particular, if $f(X_d) \in F_d^2$ we shall write $f(Y_d, Z_d) \in G_d^2$ and vise versa. Since the GL_d-module generated by f contains an irreducible submodule $W(\lambda)$, there exists a highest weight vector w_λ such that the polynomial identity $w_\lambda = 0$ follows from the polynomial identity $f = 0$. Hence we may assume that $\mathfrak{V}$ satisfies some polynomial identity $w_\lambda(x_1, x_2) = 0$ for $\lambda \vdash k$. Then $w_\lambda(Y_2, Z_2) \in G_2$ is a linear combination of the highest weight vectors in (7) and for some $\xi_j \in K$

$$w_{(k)} = \sum_{j=1}^{k-1} \xi_j y_1^j z_1^{k-j}, \text{ for } \lambda = (k), \tag{8}$$
$$w_{(k)} = (y_1z_2 - y_2z_1)^{\lambda_2} \sum_{j=0}^{\lambda_1-\lambda_2} \xi_j y_1^j z_1^{\lambda_1-\lambda_2-j}, \text{ for } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0.$$

If $f(X_d) \in F_d$ is multihomogeneous then its partial linearization $\text{lin}_{x_i} f(X_d)$ in x_i is the component of degree $\deg_{x_i} -1$ with respect to x_i of the polynomial $f(x_1, \ldots, x_i + x_{d+1}, \ldots, x_d) \in F_{d+1}$. If $\Delta_{x_i \to x_{d+1}}$ is the derivation of F_{d+1} which sends x_i to x_{d+1} and the other x_j to 0, then

$$\text{lin}_{x_i} f(X_d) = (\text{lin}_{x_i} f)(X_{d+1}) = \Delta_{x_i \to x_{d+1}}(f(X_d)).$$

If $u \in F$ then $(\text{lin}_{x_i} f)(x_1, \ldots, x_d, u)$ can be expressed in terms of derivations as

$$(\text{lin}_{x_i} f)(x_1, \ldots, x_d, u) = \Delta_{x_i \to u}(f(X_d)),$$

where $\Delta_{x_i \to u}$ is the derivation of F sending x_i to u and all other generators x_j to 0. The action of the analogue of $\Delta_{x_i \to x_{d+1}}$ on G^2 is clear: It sends y_i and z_i, respectively, to y_{d+1} and z_{d+1} and all other variables y_j and z_j to 0. We denote this derivation by $\Delta_{y_i \to y_{d+1}, z_i \to z_{d+1}}$. Now we shall translate the action of $\delta_{x_i \to u}$ on F^2, $u \in F^2$, in the language of G and the usual partial derivatives.

Lemma 4.1. *Let $u \in K[Y, Z]$ be in the image $G^2 \subset K[Y, Z]$ of F^2. Let $\Delta_{y_i, z_i \to u}$ be the derivation of $K[Y, Z]$ which sends the variables y_i, z_i to u and the other variables to 0. If $f(X_d) \in F^2$ is multihomogeneous, then the image of $(lin_{x_i} f)(x_1, \ldots, x_d, u)$ in G^2 is*

$$\Delta_{y_i, z_i \to u}(f) = \left(\frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial z_i}\right) u.$$

Proof. It is sufficient to consider the case when f and u are monomials and $i = 1$:

$$f = (y_1^{\alpha_1} z_1^{\beta_1}) v_1 v_2, v_1 = y_2^{\alpha_2} \cdots y_d^{\alpha_d}, v_2 = z_2^{\beta_2} \cdots z_d^{\beta_d},$$

$$\alpha_1 + \beta_1 \geq 1, |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d > 0, |\beta| = \beta_1 + \beta_2 + \cdots + \beta_d > 0,$$

$$u = Y_d^{\gamma} Z_d^{\delta}, |\gamma| > 0, |\delta| > 0.$$

Then

$$\Delta_{y_1 \to y_{d+1}, z_1 \to z_{d+1}}(f) = (\alpha_1 y_1^{\alpha_1 - 1} y_{d+1} z_1^{\beta_1} + \beta_1 y_1^{\alpha_1} z_1^{\beta_1 - 1} z_{d+1}) v_1 v_2$$

$$= \frac{\partial f}{\partial y_1} y_{d+1} + \frac{\partial f}{\partial z_1} z_{d+1},$$

In virtue of (6) we may assume that the preimage $\psi^{-1}\left(\frac{\partial f}{\partial y_1} y_{d+1}\right)$ in F_{d+1}^2 is of the form $\alpha_1(\cdots(x_{d+1} x_{j_1}) \cdots)$, where the dots before and after $(x_{d+1} x_{j_1})$ correspond to the beginning and the end of the element in the form (5). Since $u \in F^2$ we obtain that

$$\psi(\alpha_1(\cdots(ux_{j_1})\cdots)) = \alpha_1(\cdots(Y_d^\gamma Z_d^\delta z_{j_1})\cdots) = \frac{\partial f}{\partial y_1}u.$$

Similarly

$$\psi(\beta_1(\cdots(x_{i_p}u)\cdots)) = \beta_1(\cdots(y_{i_p}Y_d^\gamma Z_d^\delta)\cdots) = \frac{\partial f}{\partial z_1}u. \qquad \square$$

Lemma 4.2. *If* $0 \neq f \in W(\lambda_1,\lambda_2) \subset F(\mathfrak{B})$, *then all polynomial identities* $w^{(j)}_{(\mu_1,\mu_2)} = 0$ *with* $\mu_2 \geq \lambda_1$ *are consequences of the polynomial identity* $f = 0$.

Proof. As commented in the beginning of the section, we may assume that $f = w_\lambda$ is a highest weight vector in $W(\lambda_1,\lambda_2) \subset F_2$. Hence, working in G_2 instead of in F_2, w_λ has the form (8), i.e.,

$$w_\lambda = \sum_{j\geq p}\xi_j w_\lambda^{(j)} = (y_1z_2 - y_2z_1)^{\lambda_2}\sum_{j\geq p}\xi_j y_1^j z_1^{\lambda_1-\lambda_2-j}, \xi_p \neq 0.$$

First, let $p > 0$, i.e., w_λ is divisible by y_1^p. The partial linearizations of the identity $w_\lambda = 0$ are its consequences. Hence $\Delta_{y_1\to y_2, z_1\to z_2}(w_\lambda) = 0$ which has the form

$$(y_1z_2 - y_2z_1)^{\lambda_2}\sum_{j\geq p}\xi_j(jy_1^{j-1}y_2z_1^{\lambda_1-\lambda_2-j} + (\lambda_1-\lambda_2-j)y_1^j z_1^{\lambda_1-\lambda_2-j-1}z_2) = 0$$

is also a consequence of $w_\lambda = 0$ and the same holds for

$$w_{(\lambda_1,\lambda_2+1)} = \Delta_{y_1\to y_2, z_1\to z_2}(w_\lambda)z_1 - (\lambda_1-\lambda_2)w_\lambda z_2$$

$$= -(y_1z_2 - y_2z_1)^{\lambda_2+1}\sum_{j\geq p-1}(j+1)\xi_{j+1}y_1^j z_1^{\lambda_1-\lambda_2-j-1} = 0.$$

We obtained that $w_{(\lambda_1,\lambda_2+1)} = 0$ is a consequence of $w_{(\lambda_1,\lambda_2)} = 0$. It is divisible by y_1^{p-1} but is not divisible by y_1^p. Continuing in this way we shall reach a consequence

$$w_{(\lambda_1,\lambda_2+p)} = (y_1z_2 - y_2z_1)^{\lambda_2+p} j!\xi_p z_1^{\lambda_1-\lambda_2-p} = 0.$$

Now the consequence

$$y_1\Delta_{y_1\to y_2, z_1\to z_2}(w_{(\lambda_1,\lambda_2+p)}) - y_2(\lambda_1-\lambda_2+p)w_{(\lambda_1,\lambda_2+p)} = 0$$

is of the form $w_{(\lambda_1,\lambda_2+p+1)}=0$ and is divisible by $z_1^{\lambda_1-\lambda_2-p-1}$ only. Continuing the process we shall obtain as a consequence

$$w_{(\lambda_1,\lambda_1)}=(y_1z_2-y_2z_1)^{\lambda_1}=w^{(0)}_{(\lambda_1,\lambda_1)}.$$

Since all $w^{(j)}_{(\mu_1,\mu_2)}$ with $\mu_2\geq\lambda_1$ are divisible by $w^{(0)}_{(\lambda_1,\lambda_1)}$ and hence are its consequences, we complete the proof. □

Corollary 4.3. *If $0\neq f\in F$ is of degree k then all identities $w^{(j)}_{(\mu_1,\mu_2)}=0$ with $\mu_2\geq k$ follow from the identity $f=0$.*

Proof. The statement follows immediately from Lemma 4.2 because if $(\lambda_1,\lambda_2)\vdash k$, then $\lambda_1\leq k$. □

Lemma 4.4. *The polynomial identity $w^{(0)}_{(k,k)}=(y_1z_2-y_2z_1)^k=0$ has as consequences all identities*

$$(y_1z_1)^k(y_1-z_1)^kw^{(j)}_\mu=0$$

for all $\mu=(\mu_1,\mu_2)$ and all $j=0,1,\dots,\mu_1-\mu_2$.

Proof. We apply the derivation $\Delta_{y_2,z_2\to y_1z_1}$ and obtain as a consequence of the identity $w^{(0)}_{(k,k)}=0$ the identity

$$\Delta_{y_2,z_2\to y_1z_1}(w^{(0)}_{(k,k)})=y_1z_1\left(\frac{\partial}{\partial y_2}+\frac{\partial}{\partial z_2}\right)(y_1z_2-y_2z_1)^k$$

$$=ky_1z_1(y_1z_2-y_2z_1)^{k-1}\left(\frac{\partial}{\partial y_2}+\frac{\partial}{\partial z_2}\right)(y_1z_2-y_2z_1)$$

$$=ky_1z_1(y_1-z_1)(y_1z_2-y_2z_1)^{k-1}=0.$$

Continuing in this way we obtain

$$\Delta^k_{y_2,z_2\to y_1z_1}(w^{(0)}_{(k,k)})=k!(y_1z_1)^k(y_1-z_1)^k=0$$

which gives that $(y_1z_1)^k(y_1-z_1)^kw^{(j)}_\mu=0$ for all μ and all j. □

Corollary 4.5. *The variety $\mathfrak{B}$ is generated by its one-generated free algebra $F_1(\mathfrak{B})$.*

Proof. If $\mathrm{var}(F_1(\mathfrak{B})) \neq \mathfrak{B}$, then by Lemma 4.2 the algebra $F_1(\mathfrak{B})$ satisfies some identity $w^{(0)}_{(k,k)}$ and by Lemma 4.4 satisfies the identity $(y_1z_1)^k(y_1-z_1)^k = 0$ in one variable. This means that $(y_1z_1)^k(y_1-z_1)^k = 0$ in $F_1(\mathfrak{B})$ which is impossible. □

The following theorem is the first main result of our paper.

Theorem 4.6. *If $\mathfrak{V}$ is a proper subvariety of the variety $\mathfrak{B}$ of all bicommutative algebras such that $\mathfrak{V}$ satisfies a polynomial identity $f = 0$ of degree k, $0 \neq f \in F(\mathfrak{B})$, then $c_n(\mathfrak{V})$ is bounded by a polynomial of degree $k-1$.*

Proof. Let

$$\chi_n(\mathfrak{V}) = \sum_{\lambda\vdash n} m_\lambda(\mathfrak{V})\chi_\lambda, \quad n = 1,2,\ldots, \tag{9}$$

be the cocharacter sequence of $\mathfrak{V}$. By Proposition 3.2 the summation in (9) runs on $\lambda = (\lambda_1,\lambda_2) \vdash n$. By Corollary 4.3 we obtain that $m_{(\lambda_1,\lambda_2)}(\mathfrak{V}) = 0$ for $\lambda_2 \geq k$. If $\lambda_1 - \lambda_2 \leq 3k-1$, then

$$m_{(\lambda_1,\lambda_2)}(\mathfrak{V}) \leq m_{(\lambda_1,\lambda_2)}(\mathfrak{B}) \leq \lambda_1 - \lambda_2 + 1 \leq 3k.$$

Now, let $\lambda_1 - \lambda_2 \geq 3k$. By Lemma 4.4, the variety $\mathfrak{V}$ satisfies the identities

$$w_j = (y_1z_1)^k(y_1-z_1)^k(y_1z_2-y_2z_1)^{\lambda_2}y_1^jz_1^{\lambda_1-\lambda_2-3k-j} = 0, \quad j = 0,1,\ldots,\lambda_1-\lambda_2-3k.$$

All w_j, $j = 0,1,\ldots,\lambda_1-\lambda_2-3k$, are linearly independent in $F_2(\mathfrak{B})$ and are highest weight vectors for GL_2-submodules of $F_2(\mathfrak{B})$. Hence the multiplicity $m_\lambda(\mathfrak{V})$ satisfies the inequality

$$m_\lambda(\mathfrak{V}) \leq m_\lambda(\mathfrak{B}) - (\lambda_1-\lambda_2-3k+1) = (\lambda_1-\lambda_2+1) - (\lambda_1-\lambda_2-3k+1) = 3k.$$

Hence (9) satisfies the inequality

$$\chi_n(\mathfrak{V}) = \sum_{\substack{(\lambda_1,\lambda_2)\vdash n\\ \lambda_2<k}} m_\lambda(\mathfrak{V})\chi_{(\lambda_1,\lambda_2)} \leq \sum_{j=0}^{k-1} 3k\chi_{(n-j,j)}.$$

We obtain that the codimension sequence $c_n(\mathfrak{B})$, $n = 1,2,\ldots$, satisfies

$$c_n(\mathfrak{B}) \leq \sum_{j=0}^{k-1} 3kd_{(n-j,j)}$$

which by Lemma 2.2 is a polynomial of degree $k-1$. □

Remark 4.7. We may precise Corollary 4.3: If $\mathfrak{V} \subset \mathfrak{B}$ satisfies an identity $w_\lambda = 0$ of degree k and $\lambda_2 > 0$ in $\lambda = (\lambda_1, \lambda_2) \vdash k$, then $\lambda_1 \leq k-1$ and $\mathfrak{V}$ satisfies all identities $w_\mu^{(j)} = 0$ for $\mu = (\mu_1, \mu_2)$ and $\mu_2 \geq k-1$. Hence in this case $c_n(\mathfrak{V})$ is bounded by a polynomial of degree $k-2$.

Example 4.8. The bound by a polynomial of degree $k-2$ in Remark 4.7 is sharp. Let $\mathfrak{V}$ be the subvariety of $\mathfrak{B}$ defined by the polynomial identity of right nilpotency $(\cdots((x_1x_2)x_3)\cdots)x_k = 0$. It is easy to see that the image in G of the T-ideal $T(\mathfrak{V})$ of the identities of $\mathfrak{V}$ is generated as an ordinary two-sided ideal by the products $y_{i_1}z_{i_2}\cdots z_{i_k}$. Hence if $\mu_2 \geq k-1$, then all $w^{(j)}_{(\mu_1,\mu_2)}$ belong to this T-ideal and

$$w^{(n-2k+4)}_{(n-k+2,k-2)} = y_1^{n-2k+4}(y_1z_2 - y_2z_1)^{k-2}$$

does not belong to this ideal. Hence $c_n(\mathfrak{V}) \geq d_{(n-k+2,k-2)}$ which is a polynomial of degree $k-2$. We do not know whether there exists a variety $\mathfrak{V} \subset \mathfrak{B}$ satisfying a polynomial identity in one variable of degree k such that $c_n(\mathfrak{V})$ grows as a polynomial of degree $k-1$.

5. Two-dimensional algebras. The classification of all two-dimensional algebras can be traced back to the two-dimensional part of the classification project in the seminal book by B. Peirce [25] published lithographically in 1870 in a small number of copies for distribution among his friends and then reprinted posthumously in 1881 with addenda of his son C. S. Peirce. (See Grattan–Guinness [21] for the contributions of Peirce.) Starting in 2000 with the paper by Petersson [26] (which contains also the history of the classification) and the paper by Anan'in and Mironov [3] there are several papers containing different kinds of classification of two-dimensional algebras – by Goze and Remm [20], Ahmed, Bekbaev, and Rakhimov [1], Rausch de Traubenberg and Slupinski [27], Kaygorodov and Volkov [22]. Concerning the polynomial identities of two-dimensional algebras, Giambruno, Mishchenko, and Zaicev [15] proved that the growth of the codimension sequence $c_n(A)$ of such an algebra A over a field of characteristic 0 is either linear (and bounded by $n+1$) or grows exponentially as 2^n.

In this section we shall study the polynomial identities of two-dimensional bicommutative algebras over an arbitrary field of characteristic 0. It is well known that if A is an algebra over an infinite field K then the K-algebra A and the E-algebra $E \otimes_K A$ have the same bases of polynomial identities for any extension E of K. More precisely, if $\{f_i\} \subset K\{X\}$ is a basis of the polynomial identities of the K-algebra A, then $\{1 \otimes f_i\} \subset E \otimes_K K\{X\} \cong E\{X\}$ is a basis of the polynomial identities of the E-algebra $E \otimes_K A$. Hence in the sequel we may assume that the field K is algebraically closed.

The classification of all two-dimensional bicommutative algebras over an arbitrary algebraically closed field of any characteristic is given by Kaygorodov and Volkov in [22]:

Theorem 5.1. *When the base field K is algebraically closed any two-dimensional bicommutative algebra with nontrivial multiplication is isomorphic to one of the seven algebras*

$$\{\mathbf{A}_3, \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0), \mathbf{D}_2(1,1), \mathbf{D}_2(0,0), \mathbf{E}_1(0,0,0,0)\}, \tag{10}$$

where the algebras have bases $\{e_1, e_2\}$ and multiplication tables given below:

$$\begin{array}{lllll}
\mathbf{A}_3: & e_1e_1 = e_2, & e_1e_2 = 0, & e_2e_1 = 0, & e_2e_2 = 0;\\
\mathbf{B}_2(0): & e_1e_1 = 0, & e_1e_2 = e_1, & e_2e_1 = 0, & e_2e_2 = 0;\\
\mathbf{B}_2(1): & e_1e_1 = 0, & e_1e_2 = 0, & e_2e_1 = e_1, & e_2e_2 = 0;\\
\mathbf{D}_1(0,0): & e_1e_1 = e_1, & e_1e_2 = e_1, & e_2e_1 = 0, & e_2e_2 = 0;\\
\mathbf{D}_2(1,1): & e_1e_1 = e_1, & e_1e_2 = e_2, & e_2e_1 = e_2, & e_2e_2 = 0;\\
\mathbf{D}_2(0,0): & e_1e_1 = e_1, & e_1e_2 = 0, & e_2e_1 = 0, & e_2e_2 = 0;\\
\mathbf{E}_1(0,0,0,0): & e_1e_1 = e_1, & e_1e_2 = 0, & e_2e_1 = 0, & e_2e_2 = e_2.
\end{array}$$

Remark 5.2. In the classification of two-dimensional bicommutative algebras given in the preliminary version of [22, Section 7.2] there is one more algebra $\mathbf{D}_1(1,0)$. This algebra is isomorphic to the algebra $\mathbf{D}_1(0,0)$ because in [22, Table 1] the pairs $(0,0)$ and $(1,0)$ belong to the same orbit of the cyclic group of order 2 generated by ϱ, where

$$^{\varrho}(\alpha, \beta) = (1 - \alpha + \beta, \beta), \quad (\alpha, \beta) \in K^2.$$

The algebra $\mathbf{A}_3$ is commutative and nilpotent of class 3. Hence the T-ideal of its polynomial identities is generated by

$$x_1x_2 = x_2x_1, \quad (x_1x_2)x_3 = 0$$

and the cocharacter sequence of $\mathbf{A}_3$ is

$$\chi_1(\mathbf{A}_3) = \chi_{(1)}, \quad \chi_2(\mathbf{A}_3) = \chi_{(2)}, \quad \chi_n(\mathbf{A}_3) = 0, \quad n = 3, 4, \ldots.$$

Similarly, the algebras $\mathbf{D}_2(1,1), \mathbf{D}_2(0,0)$, and $\mathbf{E}_1(0,0,0,0)$ are associative-commutative, have the same bases of polynomial identities consisting of

$$x_1x_2 = x_2x_1, \quad (x_1x_2)x_3 = x_1(x_2x_3)$$

and their cocharacter sequence is

$$c_n(\mathbf{D}_2(1,1)) = c_n(\mathbf{D}_2(0,0)) = c_n(\mathbf{E}_1(0,0,0,0)) = \chi_{(n)}, \quad n = 1,2,\ldots.$$

Hence to complete the description of the polynomial identities of two-dimensional bicommutative algebras it is sufficient to handle the cases $A = \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0)$. These three algebras satisfy the condition $\dim A^2 = 1$. For our purposes it is more convenient to have another presentation of the algebras. The next proposition shows that over any field of characteristic 0 the two-dimensional bicommutative algebras A with $\dim A^2 = 1$ are in the list of Theorem 5.1.

Proposition 5.3. *Over an arbitrary field K of characteristic 0 there are only five nonisomorphic two-dimensional bicommutative algebras A such that $\dim A^2 = 1$. They are one-generated and isomorphic to the algebras*

$$A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1}, \tag{11}$$

where the algebra $A_{\pi,\varrho}$ is with multiplication given in (1). *The five algebras in* (11) *are isomorphic, respectively, to the algebras*

$$\mathbf{A}_3, \mathbf{D}_2(0,0), \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0)$$

from the list in (10).

Proof. Let A have a basis $\{a,b\}$, where $a \in A \setminus A^2$, $b \in A^2$. If $a^2 \neq 0$, then $a^2 = \alpha b$, $0 \neq \alpha \in K$. Hence A is generated by a and has a basis $\{a, a^2\}$.

Now, let $a^2 = 0$. If $ab = \beta b \neq 0$, $0 \neq \beta \in K$, then the identity of right-commutativity gives

$$\beta ba = (ab)a = (aa)b = 0,$$

and hence $ba = 0$. Let $b^2 = \gamma b$, $\gamma \in K$. Then for $\eta \in K$

$$(a + \eta b)^2 = \eta(\beta + \eta\gamma)b$$

and we always can choose η in a way to have $(a + \eta b)^2 \neq 0$. Again A is one generated and has a basis $\{a + \eta b, (a + \eta b)^2\}$. Similarly, if $ba \neq 0$, then $ab = 0$ and again A is one-generated.

Hence we may assume that A has a basis $\{r, r^2\}$. Let

$$rr^2 = \pi r^2, r^2 r = \varrho r^2, \quad \pi, \varrho \in K.$$

Then the right-commutativity implies

$$r^2r^2 = (rr)r^2 = (rr^2)r = \pi r^2 r = \pi\varrho r^2.$$

Hence the multiplication of the algebra A is as of the algebra $A_{\phi,\varrho}$ in (1). Let $\pi = 0$, $\varrho \neq 0$. If we replace the generator r by $r = \varrho r_1$, then

$$r^2r = \varrho r^2, \quad \varrho^3 r_1^2 r_1 = \varrho\varrho^2 r_1^2, \quad r_1^2 r_1 = r_1^2,$$

i.e., $A_{0,\varrho} \cong A_{0,1}$. Similarly, $A_{\pi,0} \cong A_{1,0}$. If $\pi = \varrho \neq 0$, then the change of the generator r with $r = \pi r_1$ gives that

$$r^2r = \pi r^2, \quad \pi^3 r_1^2 r_1 = \pi\pi^2 r_1^2, \quad r_1^2 r_1 = r_1^2, \quad r_1 r_1^2 = r_1^2,$$

and $A_{\pi,\pi} \cong A_{1,1}$. Finally, let $\pi \neq \varrho$ be different from 0. We fix solutions ξ and η of the linear system

$$\pi(\xi + \varrho\eta) = 1, \quad \varrho(\xi + \pi\eta) = -1.$$

Then $r_1 = \xi r + \eta r^2$ satisfies the conditions

$$r_1^2 = (\xi + \pi\eta)(\xi + \varrho\eta)r^2 = -\frac{1}{\pi\varrho}r^2,$$

$$r_1 r_1^2 = -\frac{1}{\pi\varrho}(\xi r + \eta r^2)r^2 = -\frac{1}{\pi\varrho}\pi(\xi + \eta\varrho)r^2 = \pi(\xi + \eta\varrho)r_1^2 = r_1^2,$$

$$r_1^2 r_1 = \varrho(\xi + \pi\eta)r_1^2 = -r_1^2,$$

i.e., $A_{\pi,\varrho} \cong A_{1,-1}$.

The isomorphisms between the algebras $A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1}$ and, respectively, the algebras $\mathbf{A}_3, \mathbf{D}_2(0,0), \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0)$ are given as follows:

$$\begin{array}{lll} A_{0,0} \cong \mathbf{A}_3 : & r \to e_1, & r^2 \to e_2; \\ A_{1,1} \cong \mathbf{D}_2(0,0) : & r \to e_1 + e_2, & r^2 \to e_1; \\ A_{0,1} \cong \mathbf{B}_2(0) : & r \to e_1 + e_2, & r^2 \to e_1; \\ A_{1,0} \cong \mathbf{B}_2(1) : & r \to e_1 + e_2, & r^2 \to e_1; \\ A_{1,-1} \cong \mathbf{D}_1(0,0) : & r \to e_1 - 2e_2, & r^2 \to -e_1. \end{array}$$

□

The next theorem gives bases for the polynomial identities and the cocharacter sequences of the three nonassociative algebras $A_{0,1}, A_{1,0}, A_{1,-1}$.

Theorem 5.4. (i) *As subvarieties of the variety* $\mathfrak{B}$ *of all bicommutative algebras the varieties* $\mathrm{var}(A_{0,1})$ *and* $\mathrm{var}(A_{1,0})$ *generated by the algebras* $A_{0,1}$ *and* $A_{1,0}$ *are defined by the identities of left-nilpotency* $x_1(x_2x_3)=0$ *and right-nilpotency* $(x_1x_2)x_3=0$*, respectively. Their cocharacter and codimension sequences coincide and are*

$$\chi_1(A_{0,1})=\chi_1(A_{1,0})=\chi_{(1)},\chi_n(A_{0,1})=\chi_{(n)}+\chi_{(n-1,1)},\quad n=2,3,\dots,$$

$$c_n(A_{0,1})=c_n(A_{1,0})=n,\quad n=1,2,\dots.$$

(ii) *The algebra* $A_{1,-1}$ *generates the whole variety* $\mathfrak{B}$*.*

Proof. (i) Clearly the algebra $A_{0,1}$ satisfies the polynomial identity $x_1(x_2x_3)=0$. The origins in $F=F(\mathfrak{B})$ of the polynomials $w_\lambda^{(j)}$ from (7) have the form

$$w_{(n)}^{(j)}(x_1)=\underbrace{x_1(\cdots(x_1(((x_1}_{j\text{ times}}\underbrace{x_1)\cdots)x_1}_{n-j\text{ times}}))\cdots),$$

$$w_{(\lambda_1,\lambda_2)}^{(j)}(x_1,x_2)=\underbrace{x_1(\cdots x_1}_{j\text{ times}}((\cdots((x_1x_2-x_2x_1)^{\lambda_2}\underbrace{x_1)\cdots)x_1}_{\lambda_1-\lambda_2-j\text{ times}})\cdots).$$

Obviously $w_{(\lambda_1,\lambda_2)}^{(j)}$ follows from $x_1(x_2x_3)=0$ for $\lambda=(n)$, $j=2,\dots,n-1$, $n\geq 3$, for $\lambda=(n-1,1)$, $j=1,\dots,n-2$, and for $\lambda=(\lambda_1,\lambda_2)$, $\lambda_2\geq 2$. On the other hand $w_{(n)}^{(1)}(r)=r^2\neq 0$, $w_{(n-1,1)}^{(0)}(r,r^2)=-r^2\neq 0$. This shows that the identities of $A_{0,1}$ follow from $x_1(x_2x_3)=0$, $\chi_1(A_{0,1})=\chi_{(1)}$, $\chi_n(A_{0,1})=\chi_{(n)}+\chi_{(n-1,1)}$, $n=2,3,\dots$, and $c_n(A_{0,1})=n$, $n=1,2,\dots$. The proof for $A_{1,0}$ is similar.

(ii) By Corollary 4.5 it is sufficient to show that the algebra $A_{1,-1}$ does not satisfy an identity in one variable. Let

$$w_{(n)}(y_1,z_1)=\sum_{j=1}^{n-1}\xi_jw_{(n)}^{(j)}(y_1,z_1),\quad \xi_j\in K,$$

be a polynomial in G which corresponds to a homogeneous polynomial identity $f(x_1)=0$ in one variable and of degree $n\geq 2$, $0\neq f(x_1)\in F(\mathfrak{B})$. We shall evaluate $f(x_1)$ on all $\gamma r+\delta r^2\in A_{1,-1}$, $\gamma,\delta\in K$. Since

$$(\gamma r+\delta r^2)^2=(\gamma^2-\delta^2)r^2,$$

$$(\gamma r+\delta r^2)\cdot(\gamma r+\delta r^2)^2=(\gamma-\delta)(\gamma^2-\delta^2)r^2,$$

$$(\gamma r+\delta r^2)^2\cdot(\gamma r+\delta r^2)=-(\gamma+\delta)(\gamma^2-\delta^2)r^2,$$

we obtain that the evaluation of the proimage of $w^{(j)}_{(n)}(y_1,z_1)$ on $\gamma r+\delta r^2$ is equal to

$$(-1)^{n-j-1}(\gamma^2-\delta^2)(\gamma-\delta)^{j-1}(\gamma+\delta)^{n-j-1}r^2=(-1)^{n-1}(\delta-\gamma)^j(\delta+\gamma)^{n-j}.$$

Hence

$$f(\gamma r+\delta r^2)=(-1)^{n-1}w_{(n)}(\delta-\gamma,\delta+\gamma)r^2=0.$$

When γ and δ run on the whole field K the same holds for $\delta-\gamma$ and $\delta+\gamma$. Therefore the polynomial $w_{(n)}(y_1,z_1)$ vanishes evaluated on the infinite field K and hence is identically equal to 0. This means that $A_{1,-1}$ does not satisfy any polynomial identity in one variable and hence generates the whole variety $\mathfrak{B}$. □

The following easy lemma gives an upper bound for the codimensions of a finite dimensional algebra. It makes more precise the bound for the codimensions established for graded algebras in [4] and independently in [19].

Lemma 5.5. *Let A be a finite dimensional algebra and let k be a positive integer. Then for all $n\geq k$*

$$c_n(A)\leq \dim(A^k)\dim^n(A).$$

Proof. Let $\dim(A)=p$ and $\dim(A^k)=q$. We fix a basis $\{r_1,\ldots,r_q\}$ of A^k and extend it to a basis $\{r_1,\ldots,r_p\}$ of A. We consider the multilinear identity

$$f(x_1,\ldots,x_n)=\sum_{(\sigma)}\xi_{(\sigma)}(x_{\sigma(1)}\cdots)(\cdots x_{\sigma(n)})=0,\quad \xi_{(\sigma)}\in K,$$

where the summation runs on all permutations $\sigma\in S_n$ and all possible bracket decompositions. Clearly, $f(x_1,\ldots,x_n)=0$ is a polynomial identity for A if and only if $f(r_{i_1},\ldots,r_{i_n})=0$ for all possible choices of the basis elements $r_{i_1},\ldots,r_{i_n}$. Since $\deg(f)=n$ and $n\geq k$ the evaluations of $f(x_1,\ldots,x_n)$ on R belong to A^k. Let

$$f(r_{i_1},\ldots,r_{i_n})=\sum_{j=1}^{q}f_j(r_{i_1},\ldots,r_{i_n})r_j,$$

where $f_j(r_{i_1},\ldots,r_{i_n})\in K$ are linear functions in the coefficients $\xi_{(\sigma)}$. Considering $\xi_{(\sigma)}$ as unknowns, we obtain the linear homogeneous system

$$f_j(r_{i_1},\ldots,r_{i_n})=0,\quad r_{i_1},\ldots,r_{i_n}\in\{r_1,\ldots,r_p\},j=1,\ldots,q.$$

The system has $n!C_n$ unknowns, where C_n is the n-th Catalan number (equal to the number of the bracket decompositions). Since the codimension $c_n(A)$ is equal to the rank of the system and the system has qp^n equations, its rank is less or equal to qp^n and the same holds for the n-th codimension $c_n(A)$. □

Remark 5.6. It was shown in [15] that if the two-dimensional algebra A has a one-dimensional nilpotent ideal, then $c_n(A) \leq n+1$. The algebras $A_{0,1}$ and $A_{1,0}$ satisfy this condition and Theorem 5.4 (i) shows that their codimensions are very close to the upper bound. For the algebra $A_{1,-1}$ the results in [15] give that

$$\frac{2^n}{n^2} \leq c_n(A_{1,-1}) \leq 2^{n+1}.$$

Since $\dim(A_{1,-1}) = 2$ and $\dim(A^2_{1,-1}) = 1$ Lemma 5.5 implies $c_n(A_{1,-1}) \leq 2^n$. By [13] and Theorem 5.4 (ii) we have that $c_n(A_{1,-1}) = c_n(\mathfrak{B}) = 2^n - 2$. Again, this is very close to the upper bound 2^n.

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