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COMPARISON PRINCIPLE FOR WEAKLY-COUPLED NON-COOPERATIVE ELLIPTIC AND PARABOLIC SYSTEMS

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Abstract. In this review article is considered the comparison principle for linear and quasi-linear weakly coupled systems of elliptic and of parabolic PDE. It is demonstrated that a cooperativeness is a kind of a watershed quality for the comparison principle. Roughly speaking comparison principle holds for cooperative systems, while it does not hold for every non-cooperative one.

Considering a cooperative system one can apply the theory of a positive operator in a positive cone and prove the validity of the comparison principle. One particularly important result for cooperative systems is the existence of positive first eigenvalue and positive first eigenvector.

Investigation of the validity of the comparison principle for non-cooperative system is more complicated. In this paper is mentioned the idea of division of the non-cooperative system in a cooperative and competitive part. Then

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the spectral properties of the cooperative part are employed in order to derive conditions for validity of comparison principle for the non-cooperative system.

Some applications of comparison principle are given as well

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1. Introduction. The aim of this paper is to summarize the results obtained by the authors about the validity of comparison principle (CP) for elliptic and parabolic systems of PDEs. For the sake of completeness the definition of CP follows:

Definition. *Let $\tilde{O}$ be a differential operator in some domain D and let $\underline{u}$ and $\overline{u}$ satisfy the inequality $\tilde{O}\underline{u} \leq \tilde{O}\overline{u}$ in D. Comparison principle holds for $\tilde{O}$ if $\underline{u} \leq \overline{u}$ on ∂D yields $\underline{u} \leq \overline{u}$ in $\overline{D}$.*

Considering CP, a key feature of the system is cooperativeness. It is a kind of watershed for the validity of CP - generally speaking, CP holds for cooperative systems, whereas it does not for any the non-cooperative ones. We show that CP holds for some non-cooperative systems but far not for all of them.

Historically, maximum principle (MP) in its contemporary form was stated (formulated) in 1927 by E.Hopf in his paper [22], where the strong MP was proved for elliptic equations of the type

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u + \sum_{i=1}^{n} b^i(x)D^i u + c(x)u.$$

Hopfs maximum principle states that if L is strictly elliptic operator, $c = 0$ and $Lu \geq 0$ ($Lu \leq 0$) in some domain $\Omega \subseteq \mathbb{R}^n$, and if u reach its maximum (minimum) at some internal for Ω point, then u is a constant. Moreover, if $c \geq 0$ and $c/\lambda(x)$ is bounded, where $\lambda(x)$ is the function from the strong elliptic inequality

$$0 < \lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2,\ x \in \overline{\Omega},\ \xi \neq 0,$$

then u cannot reach non-negative maximum (non-positive minimum) in internal for Ω point, unless u is a constant. Earlier results on Hopf maximum principle under much more restrictive hypothesis are discussed in [37, p. 156].

Classical maximum principle for linear elliptic operators is completely studied by now. H. Berestycki, L. Nirenberg, S. R. S. Varadhan gave in [5] necessary and sufficient condition for validity of MP that is positiveness of the first eigenvalue of the operator with null Dirichlet boundary data. Natural extension of the classical Hopf CP for degenerated equations is derived by Denson Hill [21] for propagation of the maximum for quasi-linear strictly elliptic equations. Similar result for strictly parabolic equations is given by Nirenberg in [35].

Classical maximum principle is well described in the remarkable work of M. Protter and H. Weinberger [37], in the book of D.Gilbarg and N.Trudinger [19], as well in the survey paper of P. Pucci and J. P. Serrin [36], in which is given analysis of the classical Hopf.

CP is essential in the theory of viscosity solutions, introduced by M. Crandall and P. L. Lions in[8] and [9], for further references [23] and [25], where it is fundamental for a modified Peron method for existence of continuous viscosity solutions. CP for viscosity solutions is studied as well by B. Kawohl and N. Kutev in [26], and for anisotropic diffusion in [27].

Note that in the linear case maximum principle and CP are equivalent concepts. It is completely different story in the non-linear case. For non-linear operators positiveness of the solutions is weaker statement than CP with non-negative boundary data. In the non-linear case there could be positiveness of the solutions and no CP, even no uniqueness of the solutions.

Results for validity of CP for quasi-linear elliptic equations are far not as complete, as the ones for linear equations. The main reason is that in quasi-linear case one cannot apply directly the relation between CP and the first eigenvalue of the elliptic operator. That is why quasi-linear equations are linearized and sufficient conditions are obtained for the first eigenfunction of the linearized equation. As a result some structural conditions arise for the coefficients of non-linear equations (Theorem 9.5 in [19]). These conditions guarantee estimates for the first eigenfunction of the linearized equation, though the results are not sharp.

2. Linear and quasi-linear cooperative elliptic systems. Comparison principle ans existence. In this section is considered the diffraction problem for weakly coupled elliptic systems of the type

$$-\operatorname{div} a^l(x,u^l,Du^l) + F^l(x,u^1,\dots,u^N,Du^l) = f^l(x) \quad \text{in} \quad \Omega \tag{1}$$

$$u^l(x) = g^l(x) \quad \text{on} \quad \partial\Omega \quad \text{for every} \quad l = 1,\dots,N, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with at least C^2 smooth boundary $\partial\Omega$ and $D = (D_1,\dots,D_n)$, $D_i = \dfrac{\partial}{\partial x_i}$. For convenience we assume $a^l(x,u^l,0) = 0$, $F^l(x,0,0) = 0$, $l = 1,\dots,N$ for every $x \in \overline{\Omega}$, $u^l \in \mathbb{R}$.

Suppose $\Omega_\gamma \subset \Omega$, $\gamma = 1,\dots,m-1$, are finite number of strictly interior sub-domains of Ω with smooth boundaries $\partial\Omega_\gamma$ with no intersection points, i.e. $\Omega_\gamma \cap \Omega_\delta = \emptyset$ for $\gamma \neq \delta$. Assume Ω_γ do not intercept $\partial\Omega$ as well, i.e. $\partial\Omega_\gamma \cap \partial\Omega = \emptyset$. For simplicity we denote $\Omega_m = \Omega \backslash \{\cup \overline{\Omega}_\gamma\}$. A positive direction is fixed on $\partial\Omega_\gamma$ by means of the unit external to Ω_γ normal vector $\nu(x)$ of $\partial\Omega_\gamma$.

The following diffraction conditions are prescribed on $\partial\Omega_\gamma$

$$[u^l]\Big|_{\partial\Omega_\gamma} = 0, \quad \left[\sum_{i=1}^{n} a^{li}(x,u^l,Du^l)\nu^i(x)\right]\Bigg|_{\partial\Omega_\gamma} = 0, \tag{3}$$

where $[u^l]|_{\partial\Omega_\gamma}$ is the jump of the function $u^l(x)$ through $\partial\Omega_\gamma$ in direction of the normal vector $\nu(x)$.

The coefficients of system (1), (2) are smooth enough in $\overline{\Omega}_\gamma$, i.e. there are $\dfrac{\partial a^{li}}{\partial p_j}, \dfrac{\partial a^{li}}{\partial u^l}, \dfrac{\partial F^l}{\partial u^i}, \dfrac{\partial F^l}{\partial p_i} \in L_1(\overline{\Omega}_\gamma)$, $i,j = 1,\dots,n$, $l = 1,\dots,N$, $\gamma = 1,\dots,m$. Furthermore a^l and F^l are possibly jump discontinuous on $\cup\partial\Omega_\gamma$.

As for the right hand side of (1) – the function $f^l(x)$, we suppose $f^l(x) \in L_2(\Omega)$.

Comparison principle for cooperative systems is studied in many works, for instance in [1], [3], [11], [15], [16], [24] (for fully non-linear systems), [34], [39], [40], [42]. Comparison principle for diffraction problem is studied in [7]. In [32] CP is applied in the investigation of the classical solvability for the diffraction problem for elliptic end parabolic equations.

CP for viscosity semi-continuous upper and lower solutions of fully non-linear elliptic system $G^l(x,u^1,\dots,u^N,Du^l,D^2u^l) = 0$, $l = 1,\dots,N$ is proved in [24]. Systems, considered in [24] are degenerated elliptic ones and satisfy the same structural conditions for smoothness as the scalar equations. The first main assumption in [24] guarantees quasi-monotony of the system. Quasi-monotony condition in the non-linear case is equivalent to cooperativeness in the linear case. The second main assumption in [24] comes from the method of doubling the number of variables, which is crucial for the proof of CP for viscosity solutions of fully non-linear elliptic equations. This condition appears to be technical one.

One of the most important applications of the comparison principle is the method of sub- and super solutions. It maintains, roughly speaking, that validity of the comparison principle for some operator and existence of it's sub-solution and super-solution yield existence of solution of this operator in the corresponding functional class.

This approach is widely used for scalar differential equations and it transfer to systems of differential equations is natural. In particular we consider weakly coupled cooperative systems of elliptic equations in a bounded domain. Existence of a piece-wise solution of operator with linear principal and first order symbol is proved for the diffraction problem in [5]. In the same paper the existence of classical C^2 solution is derived as well for the problem with smooth coefficients and linear principal symbol.

Another application of the comparison principle is the derivation of a priori estimate of $|u(x)|$, where $u(x)$ is the solution of (1), (2), (3). Meanwhile in the proof of this a priori estimate we obtain the super-solution of (1), (2), (3), that is necessary for the method of sub- and super-solutions.

2.1. Comparison principle for cooperative system of quasi-linear elliptic equations. Suppose (1) is strictly elliptic system, i.e. there are monotone decreasing and continuous function $\lambda(|u|) > 0$ and monotone increasing continuous function $\Lambda(|u|) > 0$, depending only on $|u| = \left(\left(u^1\right)^2 + \cdots + \left(u^N\right)^2\right)^{1/2}$, such that

$$\lambda(|u|)\left|\xi^l\right|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a^{li}}{\partial p_j^l}(x, u^1, \ldots, u^N, p^l)\xi_i^l \xi_j^l \leq \Lambda(|u|)\left|\xi^l\right|^2 \tag{4}$$

for every u^l and $\xi^l = (\xi_1^l, \ldots, \xi_n^l) \in \mathbb{R}^n$, $l = 1, 2, \ldots, N$.

Suppose the coefficients $a^l(x, u, p)$, $F^l(x, u, p)$, $f^l(x)$, $g^l(x)$ of the system (1) are measurable functions on variables x in Ω and are Lipschitz continuous on variables u^l, u and p, i.e.

$$\begin{aligned}
&\left|F^l(x,u,p) - F^l(x,v,q)\right| \le C(K)\left(|u-v| + |p-q|\right), \\
(5) \qquad & \\
&\left|a^l(x,u^l,p) - a^l(x,v^l,q)\right| \le C(K)\left(\left|u^l - v^l\right| + |p-q|\right)
\end{aligned}$$

for every $x \in \Omega$, $|u| + |v| + |p| + |q| \le K$, $l = 1, \dots, N$.

In this section we consider cooperative systems, i.e.

(6) $F^l(x, u^1, \dots, u^N, p)$ are non-increasing functions on u^k for $l \ne k$, $l, k = 1, \dots, N$ where $x \in \overline{\Omega}_\gamma$, $u \in \mathbb{R}^N$, $p \in \mathbb{R}^n$.

Then the following theorem (Theorem 2 in [7]) holds:

Theorem 1. *Assume $u, v \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ are weak sub- and super-solution of* (1), (2), (3) *and* (4)–(6) *hold. Suppose that is fulfilled at least one of the conditions* (7)–(9) *for $l = 1, 2, \dots, N$:*

$$(7) \qquad \begin{gathered} F^l(x,u,p) \quad \textit{are independent on } p \textit{ and} \\ \sum_{j=1}^{N} \frac{\partial F^j}{\partial u^l}(x, u^1, \dots, u^N) \ge 0 \quad \textit{for} \quad x \in \overline{\Omega}_\gamma, u \in \mathbb{R}^N; \end{gathered}$$

$$(8) \qquad \begin{gathered} a^l(x,u^l,p) \quad \textit{are independent on } u^l \textit{ and} \\ \sum_{k=1}^{N} \frac{\partial F^l}{\partial u^k}(x, u^1, \dots, u^N, p) \ge 0 \quad \textit{for} \quad x \in \overline{\Omega}_\gamma, u \in \mathbb{R}^N, p \in \mathbb{R}^n; \end{gathered}$$

(9) *symmetrized $(n+1)N \times (n+1)N$ matrix A defines non-negative quadratic form, where $\mathcal{A} = \boldsymbol{A} + \boldsymbol{A}^*$ and*

$$\boldsymbol{A} = \begin{pmatrix} \dfrac{\partial a^{li}}{\partial p^s_j} & \dfrac{\partial F^l}{\partial p^s_j} \\ \dfrac{\partial a^{li}}{\partial u^s} & \dfrac{\partial F^l}{\partial u^s} \end{pmatrix} \quad \textit{for } i, j = 1, \dots, n,\ l, s = 1, \dots, N$$

If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

For sake of completeness we recall the definition of super- (sub-) solution:

Definition. *Weak solution of* (1), (2), (3) *(sub-solution, super-solution) is a vector-function $u(x,t) \in \left(V_2(Q) \cap C(\overline{Q})\right)^N$ such that for every non-negative* *vector test function* $\eta \in \left(W_c^{1,1}(Q) \cap C(\overline{Q}))\right)^N$ *equality is (inequalities are) is fulfilled*

$$\int_\Omega u^l(x,t).\eta^l(x,t)dx + \int_{Q_t} \left(-u^l\eta_t^l + a^{li}\eta_{x_i}^l + F^l\eta^l - f^l\eta^l\right) dxdt = 0\,(\leq 0, \geq 0)$$

$$\left.\left[\sum_{i=1}^n a^{li}(x,t,u^l,Du^l).\nu^i\right]\right|_{S_t} = 0\,(\geq 0, \leq 0)$$

for every $l = 1,\dots,N$, *and every* $0 \leq t \leq T$, $Q_t = \Omega\times(0,t), S_t = \cup_{i=1}^{m-1}\partial\Omega_i\times(0,t)$.

2.2. Existence of classical and piece-wise classical solution for cooperative system of quasi-linear elliptic equations. In this sub-section is briefly given one of the most useful applications of CP – the method of sub- and super-solutions. All proofs are given in details in [5].

Employing the method of sub- and super-solutions requires some additional conditions on the coefficients of the system (1), (2), (3), namely:

(10) $\quad |f^l(x)| \leq C$ in Ω for every $l = 1,\dots,N$; $C > 0$ is a constant.

(11) $$F^l(x,u^1,\dots,u^N,p^l)u^l \geq c_1|u|^2 - c_2, \quad c_1 = \text{const} > 0,\ c_2 \geq 0$$

for every $x \in \Omega$, $l = 1,\dots,N$ and arbitrary vectors u and p,

(12) $$|F^l(x,u^1,\dots,u^N,p^l)| \leq [\varepsilon(M) + P(p,M)(1+|p|^2)],$$

The parameter $\varepsilon(M)$ in (12) is small enough and depends only on n, N, M, $\lambda(M)$, $\Lambda(M)$, where $\lambda(M)$ and $\Lambda(M)$ are the functions from the definition of strongly elliptic operator (4), and $P(p,M) \to 0$ when $|p| \to \infty$.

The following estimation of $|u(x)|$ holds.

Theorem 2. *Assume the coefficients of system* (1) *satisfy* (4)–(6), *and one of* (7)–(9). *Suppose* $u(x) \in W_2^1(\Omega)$ *is a weak solution of the diffraction problem* (1), (2), (3). *Then*

(13) $$\operatorname*{ess\,sup}_\Omega |u(x)| \leq \sqrt{n}M$$

where

(13′). $$M = \max\left\{\max_{\partial\Omega}|g(x)|, \frac{2\max|f(x)|}{c_1 n}, \sqrt{\frac{2c_2}{c_1 n}}\right\}$$

Inequality (13) *is true if* $u(x) \in W_2^1(\Omega)$ *is a piecewise classical solution of* (1), (2), (3) *as well.*

Note that existence of piecewise classical sub- and super-solution of system (1), (2), (3) is a corollary of Theorem 2. The function $\overline{m} = (M, M, \dots, M)$, where the constant M is defined in $(13')$, is one super-solution of (1). A sub-solution of (1), (2), (3) is for instance $\underline{m} = (m_0, m_0, \dots, m_0)$, where $m_0 = \min\{-M, m_{\partial\Omega}\}$, M is the constant from the proof of Theorem 2, and $m_{\partial\Omega} = \min\limits_{\partial\Omega} |g(x)|$.

If we consider the diffraction problem we need linearity of the first-order symbol in (1), in particular

$$\operatorname{div} a^l(x, u^l, Du^l) = \sum_{i,j=1}^{N} D_i(a_{ij}^l(x) D_j u^l),$$

$$F^l(x, u^1, \dots, u^N, p^l) = \sum_{i=1}^{N} a_i^l(x) p_i^l + F_1^l(x, u^1, \dots, u^N)$$

for every $l = 1, \dots, N$, i.e. we investigate the solvability of the system

$$-\sum_{i,j=1}^{N} D_i(a_{ij}^l(x) D_j u^l) + \sum_{i=1}^{N} a_i^l(x) D_i u^l + F_1^l(x, u^1, \dots, u^N) = f^l(x) \tag{14}$$

for $l = 1, \dots, N$, with null boundary data (2) and diffraction conditions (3).

Assume the following smoothness of the coefficients of (14) in every set $\overline{\Omega_\gamma}$, $\gamma = 1, \dots, m$: $a_{ij}^l(x) \in C^{1+\alpha}(\overline{\Omega_\gamma})$; a_i^l, F_1^l and f^l are Holder continuous with Holder constant $\alpha' \in (0,1)$ and $\|(a_i^l)^2\|_{L_{q/2}(\Omega)} < \infty$ for $q > n$. Moreover we suppose a finite jump of the coefficients of (14) on the diffraction surfaces $\partial\Omega_\gamma$, i.e. in every neighbourhood Ω^0 of $\partial\Omega_\gamma$

$$\left\| \frac{\partial a_{ij}^l}{\partial x_m}, a_i^l, (f^l(x) - F_1^l(x, M, \dots, M) + \sigma M) \right\|_{L_q(\Omega^0 \cap \Omega_\gamma)} \le \mu < \infty \tag{15}$$

holds for $q > n$, $\gamma = 1, \dots, m-1$ and $m, i, j = 1, \dots, n$, where M is the constant from $(13')$.

Since we can build barrier functions for (14), (2), (3) and (CP) holds for system (14), we apply the method of sub-solutions and super-solutions and derive the following

Theorem 3. *Suppose (CP) holds for system* (14) *as well as* (15)*. Assume* $v(x)$ *is a piecewise super-solution and* $w(x)$ *is a piecewise sub-solution of* (14) *with null boundary conditions. Then exists a piecewise classical solution* $u(x)$ *of the diffraction problem* (14), (3) *with null boundary data.*

The method of sub- and super-solutions is applicable as well ff we consider the problem (1), (2) with smooth coefficients, i.e. $a^l(x, u^l, p) \in C^{1,\alpha}(\Omega \times \mathbb{R}^N \times \mathbb{R}^n)$ and $F^l(x, u^1, \ldots, u^N, p^l) = F_0^l(x, u^l, p^l) + F_1^l(x, u^1, \ldots, u^N)$ is locally Lipschitz continuous on x, u and p^l for every $l = 1, \ldots, N$. In other words we consider the system

$$-\operatorname{div} a^l(x, u^l, Du^l) + F_0^l(x, u^l, Du^l) + F_1^l(x, u^1, \ldots, u^N) = f^l(x) \tag{16}$$

for $l = 1, \ldots, N$, with null boundary data (2).

Let

$$\lambda(1 + |p|^2)^{\frac{\mu-2}{2}} |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a^i}{\partial p_j}(x, u, p)\xi_i\xi_j \leq \Lambda(1 + |p|^2)^{\frac{\mu-2}{2}} |\xi|^2 , \tag{17}$$

where λ and Λ are constants, and $\mu > 1$.

Suppose there are constants $b_1 > 0, b_2 \geq 0$ such that

$$\left[F_0^l(x, u, p) + \sigma u - \widetilde{f}(x)\right] u \geq b_1 u - b_2 \tag{18}$$

and

$$\sum_{i=1}^{n} \left(|a_i^l| + \left| \frac{\partial a_i^l}{\partial u} \right| \right) (1 + |p|)^{\frac{1}{2}} + \sum_{i,j=1}^{n} \left| \frac{\partial a_i^l}{\partial x_j} \right| + |F_0^l(x, u, p) + \sigma u - \widetilde{f}(x)| \leq \\ \leq \Lambda(1 + |p|^2)^{\frac{\mu}{2}} \tag{19}$$

for $\widetilde{f} = f^l(x) - F_1^l(x, M, \ldots, M) + \sigma M$.

Then the following Theorem holds:

Theorem 4. *Suppose CP holds for the system* (16) *and* (17)–(19) *are fulfilled. Assume $v(x)$ is weak super-solution and $w(x)$ is weak sub-solution of* (16) *with null boundary data. Then there is a classical C^2 solution $u(x)$ of* (16) *with null boundary data* (2).

3. Comparison principle for linear and quasi-linear non-cooperative elliptic systems. The theory of the positive operators in positive cone, which proved to be so useful in the cooperative case, is not applicable for non-cooperative elliptic system. So far there are no general criteria for validity of CP for non-cooperative elliptic operators. Some results are obtained by G. Sweers [41] and G. Caristi, E. Mitidieri [10] for non-cooperative elliptic systems, obtained by small perturbations of cooperative ones. In this case is used the theory of the positive operators in positive cone and the fact that small perturbations preserve some key features of the operator.

In this section is presented a different, "spectral" approach. Analogously to the scalar case, we are interested in the relation between the sign of the first eigenfunction and validity of the CP. Unfortunately, not any non-cooperative system has first eigenfunction, an example is given by Hess in [20]). Furthermore, the spectrum of non-cooperative systems is still not investigated. On the other hand, the spectral properties of cooperative systems are well studied (see [40]. Theorem 1.1). In this section we divide the non-cooperative system into two parts – "cooperative" and "competitive" one. By means of the well known spectral properties of the "cooperative" part we obtain conditions for validity of CP for the original non-cooperative system. The result is formulated in Theorem 5 below.

One particularly important model example for non-cooperative systems are "predator – prey" systems that often are described by parabolic systems, which stabilize in time to elliptic ones (see the introduction of [30]). In this paper is introduced a generalization of the concept "CP", considering the cone $C_U = P_U \times (-P_U)$, where P_U is the cone of the positive functions in $W^{1,\infty}(\Omega)$. CP applies for the order in C_U, i.e. $(u_1, u_2) \prec (v_1, v_2)$ if and only if $u_1 \leq v_1$ and $u_2 \geq v_2$. This approach is sensible since in the "predator prey" system is observed delay in time of the recovery of the system balance after decrease of the number of one or another species in the model. Applying this non-standard approach it is obtained an interesting result in [30] for two-dimensional system (20) with $m_{11} = m_{22} = 0$ and $m_{ij} = p_i(x) > 0$ for $i \neq j$, $i = 1, 2$. This system is competitive one and one cannot expect for it results similar to the ones for cooperative system. Nevertheless, in Theorem 6.5 in [30] is proved the existence of the first eigenfunction with positive corresponding eigenvector in the cone $C_U = P_U \times (-P_U)$. The proof follows the technique used for cooperative systems for a good reason – after multiplication by -1 of the second equation the system transfers to cooperative one and the cone C_U transfers to the standard one $P_U \times P_U$.

In [29] are considered existence and local stability of the positive solutions of systems with $L_k = -d_k\Delta$, linear cooperative and non-linear competitive part, and Neumann boundary conditions. Theorem 2.4 in [29] is similar to Theorem 5 for $L_k = -d_k\Delta$.

Since "predator – prey" systems are basic model example for non-cooperative systems, in Theorem 7 is adapted the main idea of Theorem 6 to systems with triangle cooperative part that could become diagonal one in a certain point. If all of the species dwell in whole of the habitat, the cooperative part is not a diagonal one in all points and Theorem 5 is applicable. If some species could extinct in some sub-domain of the habitat, then one sufficient condition for validity of CP is given in Theorem 7.

3.1. Comparison principle for linear non-cooperative elliptic systems. Suppose elliptic system (1) is a linear one with smooth coefficients, i.e.

$$L_M u = f(x) \text{ in } \Omega \tag{20}$$

where $L_M = L + M$, L is a matrix operator with null off-diagonal elements $L = \operatorname{diag}(L_1, L_2, \ldots, L_N)$, and matrix $M = \{m_{ki}(x)\}_{k,i=1}^{N}$. Scalar operators

$$L_k u^k = -\sum_{i,j=1}^{n} D_j \left(a_{ij}^k(x) D_i u^k\right) + \sum_{i=1}^{n} b_i^k(x) D_i u^k + c^k u^k \text{ in } \Omega$$

are supposed uniformly elliptic ones for $k = 1, 2, \ldots, N$. Coefficients c^k and m_{ik} in (20) are supposed continuous in $\overline{\Omega}$, $a_{ij}^k(x) \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\dfrac{\partial a_{ij}^k}{\partial x_j}, b_i^k(x)$ are Holder continuous with Holder constant $\alpha \in (0.1)$.

In formulation of Theorem 5 below we need a definition for "irreducible" matrix.

Definition. *Irreducible matrix is one that can not be decomposed to matrices of lower rank, and respectively, the reducible matrix can be decomposed.*

Let us recall the following Theorem (Theorem 3 in [4]):

Theorem 5. *Let* (20) *be a weakly coupled elliptic system with irreducible cooperative part of* $L_{M^-}^*$. *Then the comparison principle holds for the classical solutions of system* (20) *if there is* $x_0 \in \Omega$ *such that*

$$\lambda + \sum_{k=1}^{N} m_{kj}^+(x_0) > 0 \quad \textit{for} \quad j = 1, \ldots, N \tag{21}$$

and

$$\lambda + m_{jj}^+(x) \geq 0 \quad \textit{for every} \quad x \in \Omega \quad \textit{and} \quad j = 1, \ldots, N, \tag{22}$$

where λ *is the principal eigenvalue of the operator* L_{M^-} *in* Ω*.*

The same result holds if the cooperative part of $L_{M^-}^*$ has structure with Jordan cells on the main diagonal and zeroes otherwise (Theorem 4 in [4]).

Theorem 6. *Assume $m^-_{ij} \equiv 0$ for $i \neq j$ and (2) is satisfied. Then the comparison principle holds for the classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ solutions of system (1) if there is $x_0 \in \Omega$ such that*

$$\lambda_j + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \quad \text{for every} \quad j = 1, \ldots, N, \tag{23}$$

and

$$\lambda_j + m^+_{jj}(x) \geq 0 \quad \text{for every} \quad x \in \Omega \quad \text{and} \quad j = 1, \ldots, N, \tag{24}$$

where λ_j is the principal eigenvalue of $\widetilde{L}_j = L_j + m^-_{jj}$ in Ω.

Theorem 6 is formulated for diagonal matrix M^-, but the statement is valid with obvious modification if M^- has Jordan cells on the main diagonal.

Finally (Theorem 5 in [4]), in case that the cooperative part M^- is triangular, we have

Theorem 7. *Assume the cooperative part M^- of system (20) is triangular, i.e. $m^-_{ij} = 0$ for $i = 1, \ldots, N$, $j > i$. Then the comparison principle holds for the classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ solutions of system (1), if there is $\varepsilon > 0$ such that*

$$\lambda_j - (1 - \delta_{1j})\varepsilon + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \tag{25}$$

for $j = 1, \ldots, N$ and some $x_0 \in \Omega$ and

$$\lambda_j - (1 - \delta_{1j})\varepsilon + m^+_{jj}(x) \geq 0 \quad \text{for every} \quad x \in \Omega \quad \text{and} \quad j = 1, \ldots, N, \tag{26}$$

where λ_j is the principal eigenvalue of the operator $L_j + m^-_{jj}$ and δ_{1j} is Kronecker delta.

These results can be transferred to the quasi-linear case.

3.2. Comparison principle for quasi-linear non-cooperative elliptic systems. Let $\underline{u}(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be weak sub-solution of (1), (2), i.e.

$$\int_\Omega \left(a^{li}(x, \underline{u}^l, D\underline{u}^l)\eta^l_{x_i} + F^l(x, \underline{u}^1, \ldots, \underline{u}^N, D\underline{u}^l)\eta^l - f^l(x)\eta^l \right) dx \leq 0$$

for $l = 1, \ldots, N$ and for every non-negative vector-function $\eta \in \left(W^1_c(\Omega) \cap C(\overline{\Omega})\right)^N$ (i.e. $\eta = (\eta^1, \ldots, \eta^N)$, $\eta^l \geq 0$, $\eta^l \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ and $\eta^l = 0$ on $\partial\Omega$).

Analogously, let $\overline{u}(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be a weak super-solution of (1), (2), i.e.

$$\int_\Omega \left(a^{li}(x, \overline{u}^l, D\overline{u}^l)\eta^l_{x_i} + F^l(x, \overline{u}^1, \dots, \overline{u}^N, D\overline{u}^l)\eta^l - f^l(x)\eta^l\right) dx \geq 0$$

for $l = 1, \dots, N$ and every non-negative vector-function $\eta \in \left(W^1_c(\Omega) \cap C(\overline{\Omega})\right)^N$.

Recall that the comparison principle holds for (1), (2), if $Q(\underline{u}) \leq Q(\overline{u})$ in Ω and $\underline{u} \leq \overline{u}$ on $\partial\Omega$ yields $\underline{u} \leq \overline{u}$ in Ω.

Since $\underline{u}(x)$ and $\overline{u}(x)$ are sub- and super-solutions, then $\widetilde{w}(x) = \underline{u}(x) - \overline{u}(x)$ is weak sub-solution of the following problem

$$-\sum_{i,j=1}^{n} D_i \left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right) + \sum_{k=1}^{N} E^l_k \widetilde{w}^k + \sum_{i=1}^{n} H^l_i D_i \widetilde{w}^l = 0 \text{ in } \Omega$$

with non-positive boundary data on $\partial\Omega$, i.e.

$$\int_\Omega \left(\sum_{i,j=1}^{n} \left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right)\eta^l_{x_i} + \sum_{k=1}^{N} E^l_k \widetilde{w}^k \eta^l + \sum_{i=1}^{n} H^l_i D_i \widetilde{w}^l \eta^l\right) dx \leq 0 \text{ in } \Omega$$

Here

$$\begin{aligned}
B^{li}_j &= \int_0^1 \frac{\partial a^{li}}{\partial p_j}(x, P^l) ds, \quad B^{li}_0 = \int_0^1 \frac{\partial a^{li}}{\partial u^l}(x, P^l) ds, \\
P^l &= \left(v^l + s(u^l - v^l), Dv^l + sD(u^l - v^l)\right) \\
E^l_k &= \int_0^1 \frac{\partial F^l}{\partial u^k}(x, S^l) ds, \quad H^l_i = \int_0^1 \frac{\partial F^l}{\partial p_i}(x, S^l) ds, \\
S^l &= \left(v + s(u - v), Dv^l + sD(u^l - v^l)\right).
\end{aligned}$$

Therefore $\widetilde{w}_+(x) = \max(\widetilde{w}(x), 0)$ is weak sub-solution of

$$-\sum_{i,j=1}^{n} D_i \left(B^{li}_j D_j \widetilde{w}^l_+ + B^{li}_0 \widetilde{w}^l_+\right) + \sum_{k=1}^{N} E^l_k \widetilde{w}^k_+ + \sum_{i=1}^{n} H^l_i D_i \widetilde{w}^l_+ = 0 \text{ in } \Omega \tag{27}$$

with null boundary data on $\partial\Omega$.

Equation (27) is equivalent to

$$B_E \widetilde{w}_+ = (B + E)\widetilde{w}_+ = 0 \text{ in } \Omega, \tag{28}$$

where $B = \mathrm{diag}(B_1, B_2, \dots, B_N)$, $B_l = -\sum_{i,j=1}^{n} D_i \left(B_j^{li} D_j \widetilde{w}_+^l + B_0^{li} \widetilde{w}_+^l \right) + \sum_{i=1}^{n} H_i^l D_i \widetilde{w}_+^l$ and $E = \{E_k^l\}_{l,k+1}^N$.

Then the following theorem (Theorem (8) in [4]) holds:

Theorem 8. *Let* (1), (2) *be quasi-linear uniformly elliptic system. Then comparison principle holds for system* (1), (2) *if*

either B_{E^-} *is irreducible one and for every* $j = 1, \dots, n$ *hold*

$$(i) \quad \lambda + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x, p, q^l) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, q^j) \right)^+ > 0 \quad \textit{for some} \quad x_0 \in \Omega,$$

$$(ii) \quad \lambda + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, q^j) + \frac{\partial F^j}{\partial p^j}(x, p, q^j) \right)^+ \geq 0 \quad \textit{for every} \quad x \in \Omega,$$

where $p, q \in \mathbb{R}^n$ *and* λ *is the first eigenvalue of operator* B_{E^-} *in* Ω*;*

or *if* B_{E^-} *is reducible one and for every* $j = 1, \dots, n$ *hold*

$$(i') \quad \lambda_j + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x, p, q^j) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, q^j) \right)^+ > 0 \ \textit{for some} \ x_0 \in \Omega,$$

$$(ii') \quad \lambda_j + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, q^j) + \frac{\partial F^j}{\partial p^j}(x, p, q^j) \right)^+ \geq 0 \ \textit{for every} \ x \in \Omega,$$

where $p, q \in \mathbb{R}^n$ *and* λ_l *is the first eigenvalue of operator* B_l *in* Ω*.*

4. Linear and quasi-linear parabolic systems. Most results for positiveness of the classical solutions, or validity of the comparison principle on classical sense, are obtained for cooperative systems (see [1], [2], [13], [17], [18], [33], [38] and [45] for optimal control problems, [12] for diffraction-diffusion systems arising in medicine, [28] for stabilized non-linear system type "heat transfer", in [7] for general quasi-linear cooperative reaction-diffusion systems and many others). One can summarize all these results as "cooperativeness is sufficient condition for the validity of the comparison principle for parabolic systems" (see for instance [7]). On the other hand, the following simple example shows that comparison principle is not a feature of every parabolic system.

Example 1. Let $Q = (0, \pi) \times (0, T)$. Consider the problem

$$\left| \begin{array}{l} u_t^1 - u_{xx}^1 - u^1 + u^2 = 0 \\ u_t^2 - u_{xx}^2 - u^2 = 0 \end{array} \right. \quad \text{in } Q$$

with initial and boundary conditions $u^1(x,0) = u^2(x,0) = 0$ for $x \in [0,\pi]$, $u^1(0,t) = u^2(0,t) = u^1(\pi,t) = u^2(\pi,t) = 0$ for $t \in [0,T]$.

This system is non-cooperative, since $m_{12}(x,t) = 1 > 0$. One solution of this system is the trivial one $v^1 = v^2 = 0$, which is a sub-solution as well. One super-solution is $w^1 = -t.\sin x$, $w^2 = \sin x$. Since the inequality $-t.\sin x = w^1 > v^1 = 0$ is not satisfied in Q there is no comparison principle for the above system.

So the very reasonable question arises: is there comparison principle for some non-cooperative system of parabolic equations? In fact Theorem 10 below shows the strong correlation between the global on time comparison principle for linear non-cooperative parabolic systems and the existence of positive solution of the L^2-adjoint operator. Furthermore, comparison principle holds as well if the first eigenvalue and the corresponding first eigenfunction are positive. Note that unlike the cooperative elliptic systems, which first eigenfunction is positive one, in the parabolic case this is not always true. For instance, Theorem 12 gives some conditions such that comparison principle does not hold for system (34), and therefore its first eigen-function is not positive one.

In the last section of this section another approach is applied to the problem of validity of the comparison principle. In Theorem 11 are given some conditions on the coefficients of the system such that in a small neighbourhood of some point t_0 comparison principle holds. The result is useful to investigate the maximal interval $(0, t_m)$ in which the comparison principle holds for system (34). The result is based on the validity of the comparison principle for the elliptic system $Pu(t_0, x) = f$.

One application of CP for existence theorems is given in [43] for Lotka-Volterra cooperation model, applying the method of sub- and super-solutions. Similar technique is used in [44], studying n-dimensional diffraction problem for weakly coupled quasilinear parabolic system on a bounded domain Ω, where the inner diffraction interfaces may intersect the outer boundary $\partial\Omega$ and the coefficients of the equations may to be discontinuous on these interfaces.

4.1. Diffraction problem for cooperative quasi-linear parabolic systems. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with at least C^2 smooth boundary $\partial\Omega$. In this section we consider the diffraction problem in $Q = \Omega \times (0,T)$ with parabolic boundary $\Gamma = (\partial\Omega \times [0,T]) \cup \{\Omega \times \{t = 0\}\}$. Let $\Omega_\gamma \subset \Omega$, $\gamma = 1, \ldots, m-1$, be finite number of sets with smooth boundaries such that

$\partial\Omega_\gamma \cap \partial\Omega = \emptyset$ and $\Omega_\gamma \cap \Omega_\delta = \emptyset$ for $\gamma \neq \delta$. For sake of simplicity we denote by $\Omega_m = \Omega \backslash \{\cup \overline{\Omega}_\gamma\}$. For the diffraction problem we define the diffraction surface $S = \cup S_\gamma$, $S_\gamma = \partial\Omega_\gamma \times [0,T]$, $\gamma = 1,\dots,m-1$, where $Q_\gamma = \Omega_\gamma \times (0,T)$.

In this section we consider the following system of uniformly parabolic equations in Q:

$$u_t^l - \operatorname{div} a^l(x,t,u^l,Du^l) + F^l(x,t,u^1,\dots,u^N,Du^l) = f^l(x,t) \tag{29}$$

with boundary conditions on Γ

$$u^l(x,t) = g^l(x,t). \tag{30}$$

We suppose $a^l(x,t,u^l,0) = 0$, $F^l(x,t,0,0) = 0$, $i = 1,\dots,n$, $l = 1,\dots,N$ for every $(x,t) \in \overline{Q}$, $u^l \in \mathbb{R}$.

The diffraction conditions on S are

$$[u^l]|_S = 0, \qquad \left.\left[\sum_{i=1}^n a^{li}(x,t,u^l,Du^l)\nu^i(x)\right]\right|_S = 0, \tag{31}$$

where $[u^l]|_S$ is the jump of function $u^l(x,t)$ on S toward the positive (outward-pointing with respect to Ω_γ) normal vector $\nu(x)$.

The definition of uniformly parabolic equation is similar to the one for elliptic equations, given by (4).

We suppose that the coefficients $a^l(x,t,u,p)$, $F^l(x,t,u,p)$, $f^l(x,t)$, $g^l(x,t)$ of (29) are measurable functions w.r.t. variables x,t and are locally Lipschitz continuous w.r.t. u^l, u and p, i.e.

$$\begin{aligned} &\left|F^l(x,t,u,p) - F^l(x,t,v,q)\right| \le C(K)\left(|u-v| + |p-q|\right), \\ &\left|a^l(x,t,u^l,p) - a^l(x,t,v^l,q)\right| \le C(K)\left(\left|u^l - v^l\right| + |p-q|\right) \end{aligned} \tag{32}$$

for every $(x,t) \in \overline{Q}$, $|u| + |v| + |p| + |q| \le K$, $l = 1,\dots,N$.

Let system (29) be cooperative, i.e.

(33) $F^l(x,t,u^1,...u^N,p)$ are non-increasing functions w.r.t. u^k for $l \neq k$, $l,k = 1,\dots,N$ and $(x,t) \in \overline{Q}_\gamma$, $u \in \mathbb{R}^N$, $p \in \mathbb{R}^n$.

Then the following theorem holds:

Theorem 9. *Let $u,v \in W_\infty^{1,1}(Q) \cap C(\overline{Q})$ be sub- and super-solutions of* (29), (30), (31). *If* (32)–(33) *hold and $u \le v$ on Γ, then $u \le v$ in Q.*

4.2. Non-cooperative linear parabolic systems. In this section is studied the validity of the comparison principle for non-cooperative weakly-coupled linear systems of uniformly parabolic PDE in Q, i.e. (29) has the form

$$Pu = f, \tag{34}$$

or component-wise

$$u_t^k - \sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x,t)D_i u^k\right) + \sum_{i=1}^{n} b_k^i(x,t)D_i u^k + c_k(x,t)u^k + \sum_{k\neq l=1}^{N} m_{lk}(x,t)u^l =$$
$$= f^k(x,t)$$

$k = 1,\dots,N$, with boundary conditions (30) on Γ. Note that for the sake of simplicity in notations we suppose $m_{kk} = 0$ for all $k = 1,\dots,N$.

The right-hand side of (34) is supposed bounded function, i.e. $|f^l(x,t)| \leq C$ in $\overline{Q}$ for every $l = 1,\dots,N$, where $C > 0$ is a constant. Coefficients c_k and m_{lk} in (34) are supposed continuous in $\overline{Q}$, $a_k^{ij}(x,t) \in C^{1+\alpha}(Q) \cap C(\overline{Q})$ and $b_k^i(x,t) \in C^1(Q) \cap C(\overline{Q})$. We assume in addition that for every $k = 1,\dots,N$

$$\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} D_j a_k^{ij} + b_k^i(x)\right)^2, |c_k|\right\} \leq b \tag{35}$$

holds, where $b > 0$ is a constant.

One approach to CP problem is the so called "spectral" one. The strong connection between the validity of the comparison principle and the first eigenvalue of the operator is well-known feature of elliptic equations and systems. Similarly, the following result gives the validity of global on t comparison principle for parabolic systems.

Theorem 10. *Let P^* be L^2-adjoint operator of P. Comparison principle holds for system* (34), (30) *if*

(*i*) *there is a positive solution of $P^*v = F(x,t)$ for some $F(x,t) > 0$*

or

(*ii*) *there is a positive eigenfunction λ of P^* and the corresponding eigenfunction u^* is positive one as well.*

Note that for parabolic systems the first eigenfunction may not be positive one, unlike the case of elliptic operators.

S k e t c h o f t h e p r o o f. Let $\underline{u}$ and $\overline{u}$ be sub- and super-solutions of (34). Denote $w = \overline{u} - \underline{u}$.

1. Let $F(x,t) > 0$ and there is a positive solution of $P^*v = F(x,t)$. If we suppose that there is no comparison principle for P, then $w_- = \min(w,0) \neq 0$. Let $Q^- = \operatorname{supp}\{w_- \leq 0\}$. Then $0 \leq (Pw^-, v) = (w^-, P^*v) = (w^-, F) \leq 0$. The first inequality above follows since w^- is a super-solution. Therefore $w^- \equiv 0$ and $w = \overline{u} - \underline{u} > 0$.

2. Let the first eigenvalue λ of P^* is positive one and the corresponding eigenfunction u^* is positive one as well. Suppose $w_- \leq 0$. Then u^* is suitable test-function and $0 \leq (Pw^-, u^*) = (w^-, P^*u^*) = (w^-, \lambda u^*) \leq 0$. Therefore $w^- \equiv 0$ and $w = \overline{u} - \underline{u} > 0$. □

Another approach to the comparison principle for non-cooperative systems employs the conditions for validity of the comparison principle for non-cooperative elliptic systems, obtained in [4]. The idea of transferring that results to the case of parabolic equations is very simple - if we fix t variable we reduce the parabolic system to the elliptic one and we can apply the conditions for the validity of comparison principle for elliptic systems.

Let denote by $L_M u$ the operator

$$L_M u = -\sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x,t) D_i u^k \right) + \sum_{i=1}^{n} b_k^i(x,t) D_i u^k + c_k(x,t) u^k + \sum_{k \neq l=1}^{N} m_{lk}(x,t) u^l \tag{36}$$

where $k = 1, \dots, N$. Let denote by $L_{M^-} u$ the cooperative part of (36), i.e. the operator

$$-\sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x,t) D_i u^k \right) + \sum_{i=1}^{n} b_k^i(x,t) D_i u^k + c_k(x,t) u^k + \sum_{l=1}^{N} m_{lk}^-(x,t) u^l,$$

where $k = 1, \dots, N$ and $m_{lk}^-(x,t) = \min\{m_{lk}(x,t), 0\}$. Let $L_{M^-}^*$ be the L^2-adjoint operator of L_{M^-}.

Theorem 11. *Let* (34) *be a weakly coupled, uniformly parabolic system and the cooperative part of* $L_{M^-}^*$ *is irreducible. Then the comparison principle holds for system* (34) *if for every* $t \in [0,T]$ *there is* $x_0(t) \in \Omega$ *such that*

$$\left(\lambda(t) + \sum_{k=1}^{N} m_{kj}^+(x_0, t) \right) > 0 \quad \text{for} \quad j = 1, \dots, N, \tag{37}$$

where $\lambda(t)$ *is the principal eigenvalue of the operator* $L_{M^-}(t)$ *in* Ω.

Theorem 11 is formulated for irreducible cooperative part of the system (34) and the result is based on Theorem 3 in [4]. If the cooperative part is reducible then in Theorem 11 above $\lambda(t)$ can be substituted by $\lambda_k(t)$ (see Theorem (5) in [4]). Here $\lambda_k(t)$ is the the principal eigenvalue of the operator

$$-\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x,t)D_iu^k\right)+\sum_{i=1}^{n} b_k^i(x,t)D_iu^k+c_k(x,t)u^k.$$

S k e t c h o f t h e p r o o f. Let $\underline{u}$ and $\overline{u}$ be sub- and super-solutions of (34). Then $w=\overline{u}-\underline{u}$ is a super-solution of (34) and $Pw\geq 0$. In other words in Q we have

$$L_Mw\geq -w_t \tag{38}$$

Suppose there is no comparison principle for P, i.e. $\min\{w(x,t)\} = w(x_0,t_0)<0$. If the point $(x_0,t_0)\in Q$, i.e. (x_0,t_0) is internal point for the domain, then $w_t(x_0,t_0)=0$. If $(x_0,t_0)\in\Omega\times|T|$, i.e. (x_0,t_0) belongs to the upper base of the parabolic cylinder, then $w_t(x_0,t_0)\leq 0$. Therefore $w_t(x_0,t_0)\leq 0$.

On the other hand, following the proof of Theorem 3 in [6], one can prove that $0>L_Mw(x_0,t_0)$. Actually, in the proof of Theorem 3 in [6], which concerns elliptic systems, we need one additional condition

$$\lambda(t)+c_k^+(x,t)\geq 0 \tag{39}$$

for every $x\in\Omega$ and $k=1,\dots,N$. But in the case of parabolic systems, one can substitute in system (34)

$$u^k=v^k.e^{(\lambda_0+b)t},$$

where $\lambda_0=\sup|\lambda(t)|$ and b is defined in (35) and we obtain that (39) is fulfilled.

Substitution of $w_t(x_0,t_0)\leq 0$ and $0>L_Mw(x_0,t_0)$ in (38) yields the contradiction

$$0>L_Mw(x_0,t_0)\geq -w_t(x_0,t_0)\geq 0$$

and therefore comparison principle holds for system (34). □

Example 2. Let the coefficients of system (34) depend only on x, i.e. consider systems of the type

$$u_t^k-\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x)D_iu^k\right)+\sum_{i=1}^{n} b_k^i(x)D_iu^k+c_k(x)u^k+\sum_{l=1}^{N} m_{lk}(x)u^k=f^l(x)$$

$l = 1, \ldots, N$, with boundary conditions on Γ

$$u^k(x,t) = g^k(x).$$

Let

$$\left(\lambda + \sum_{k=1}^{N} m_{kj}^{+}(x_0)\right) > 0 \quad \text{for} \quad j = 1, \ldots, N$$

and

$$\lambda + m_{jj}^{+}(x) \geq 0 \quad \text{for every} \quad x \in \Omega \quad \text{and} \quad j = 1, \ldots, N,$$

where λ is the principal eigenvalue of the operator L_{M^-} in Ω. Then comparison principle holds for system (34), (30).

If $f^l(x) = g^k(x) \equiv 0$ we can write explicitly the solution in the form

$$u(x,t) = \exp^{-\lambda t} v(x),$$

where v is the principal eigenfunction of

$$-\sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x) D_i v^k\right) + \sum_{i=1}^{n} b_k^i(x) D_i v^k + c_k(x) v^k + \sum_{l=1}^{N} m_{lk}(x) v^k = \lambda v(x).$$

If the system is cooperative, then $v(x) > 0$.

The following theorem gives some sufficient conditions when comparison principle fails. It is based on Theorem 38 in [4]. The idea is that if we fix t_0 and there is no comparison principle for the elliptic system $Pu(t_0, x) = f$ then there is no comparison principle for system (34), (30).

Theorem 12. *Let* (34) *be a weakly coupled, uniformly parabolic system with fully coupled cooperative part of* $L^*_{M^-}$. *Suppose there is* t_0 *and index* $j \in \{1, \ldots, N\}$ *such that* $\left(\lambda + m_{jj}^{+}(x, t_0)\right) < 0$ *for some point* $x \in \Omega$, *where* λ *is the principal eigenvalue of* L_{M^-}, *and* $m_{jl}^{+}(x, t_0) = 0$ *for* $l \neq j$, $l = 1, \ldots, N$. *Then the comparison principle does not hold for system* (34).

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