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ISOTHERMIC SURFACES AND SOLUTIONS OF THE CALAPSO EQUATION

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Abstract. In this paper, we introduce a class of surfaces called *radial inverse mean curvature surface* (RIMC-surfaces) and we show that there is a correspondence between these surfaces and Bryant surfaces in the hyperbolic space $\mathbb{H}^3$, therefore, the RIMC-surfaces are isothermic. We obtain a Weierstrass type representation for RIMC-surfaces which depends on a meromorphic function and a holomorphic function and we obtain a characterization so that these surfaces are parametrized by lines of curvature. In [3] it is shown that for each isothermic surface parametrized by lines of curvature in the Euclidean space a solution of the Calapso equation is associated, in this work we show that for these surfaces we can associate another solution of the Calapso equation. Moreover, we give explicit solutions of the Calapso equation that depend on holomorphic functions.

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1. Introduction. Ribaucour transformations for surfaces parametrized by lines of curvature, were classical studied by Bianchi [1], they can be applied to obtain surfaces of constant Gaussian curvature from a given such surface. Similarly, by using Ribaucour transformations, one may obtain surfaces of constant mean curvature from such a given surface. In [10] was showed that an n-dimensional sphere or hyperplane can be locally associated by a Ribaucour transformation to any given hypersurface M^n of $\mathbb{R}^{n+1}$. Since, two surfaces are related by a Ribaucour transformation if they are envelopes of a congruence of spheres preserving lines of curvature, in [9] the author show that locally all surface is an envelope of a congruence of spheres where the other envelope is contained in a plane and obtain an explicit parametrization for this surface.

We say that an oriented surface $M \subset \mathbb{R}^3$ is a *Laguerre minimal surface* if

$$\triangle_{III}\left(\frac{H}{K}\right) = 0$$

where H, K are the mean and Gaussian curvature of M and III is the third fundamental form of M. Also, M is a *surface of the spherical type* if the set of spheres of centers $p + \dfrac{H(p)}{K(p)}N(p)$ and radius $\dfrac{H(p)}{K(p)}$ are tangent to a fixed oriented plane, $p \in M$. In [17], was showed that all surface of the spherical type is a Laguerre minimal surface.

A regular surface M is *isothermic* if locally, near each non umbilic point of M there exist curvature line coordinates which are conformal with respect to the first fundamental form of M.

The research of isothermic surfaces is one of the most common more difficult problems of differential geometry and depends on the integration of an equation with fourth-order partial derivatives (see [23]). Particular classes of these surfaces are known and some transformations by means of which it is possible to deduce from isothermic surfaces other isothermic surfaces. All this is known indirectly and independently of the fourth-order differential equation, because it is difficult to integrate.

The theory of isothermic surfaces has a great development for eminent geometers as Christoffel [7], Darboux [12]–[13] and Bianchi [1] among others. In the last decades, the theory woke up interest by his connection with the modern theory of integrated systems, see [5], [6], [18], [21] and [22]. Particular classes of isothermic surfaces are the constant mean curvature surfaces, quadrics, surfaces whose lines of curvature has constant geodesic curvature, in particular, the cyclides of Dupin. Trasformations of $\mathbb{R}^3$ that preserve isothermic surfaces are isometries, dilations and inversions.

In [2], the authors study surfaces with harmonic inverse mean curvature (HIMC surfaces), they distinguish a subclass of θ-isothermic surfaces, which is a generalization of the isothermic HIMC surfaces, and classify all the θ-isothermic HIMC surfaces, note that when $\theta = 0$, the surfaces are isothermic.

In [6], the author show that theory of soliton surfaces, modified in an appropriate way, can be applied also to isothermic immersions in $\mathbb{R}^3$. In this case the so called Sym's formula gives an explicit expression for the isothermic immersion with prescribed fundamental forms. The complete classification of the isothermic surfaces is an open problem.

In [3], the author establishes a new equation for the equation with fourth-order partial derivatives from which the problem of obtaining isothermic surfaces apparently becomes much simpler. The Calapso equation defined in [3] given by

$$\left(\frac{\omega_{,u_1u_2}}{\omega}\right)_{,u_1u_1} + \left(\frac{\omega_{,u_1u_2}}{\omega}\right)_{,u_2u_2} + (\omega^2)_{,u_1u_2} = 0,$$

describes the isothermic surfaces $\omega = \omega(u_1, u_2)$ in $\mathbb{R}^3$. As was shown in the paper [18], this PDE has solitonic sense. This equation is very difficult to solve and is strongly connected to the Painleve ODEs and we can study it from the point of view developed in [8]. In [4], using the symbolic computation in MAPLE, the authors produce some solutions of the Calapso equation. In [19] a vector analogue of the classical Calapso equation governing isothermic surfaces in $\mathbb{R}^{n+2}$ is introduced. It is shown that this vector Calapso system admits a (nonlocal) scalar lax pair based on the classical Moutard equation.

Given a holomorphic function $f(z)$ on the complex plane $\mathbb{C}$, define the *Schwarzian derivative* $S(f)$ to be the function

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

This expression is ubiquitous and tends to appear in seemingly unrelated fields of mathematics: classical complex analysis, differential equations, and one-dimensional dynamics, as well as, more recently, Teichmüller theory, integrable systems, and conformal field theory. In [20], the author applying a theorem of Ghys on Schwarzian derivatives gives a new proof of the four-vertex theorem for closed convex curves in the hyperbolic plane. In [14] Duval and Ovsienko, together with L. Guieu, related the Schwarzian derivative to the geometry of Lorentz surfaces.

In this paper motivated by the papers [2] and [17], we introduce the class of radial inverse mean curvature surfaces (RIMC-surfaces), as being the surfaces M such that the set of spheres of centers $p + \dfrac{1}{H(p)}N(p)$ and radius $\dfrac{1}{H(p)}$ are tangent to a fixed oriented plane, where $p \in M$ and H is the mean curvature of M, we note that these surfaces are not Laguerre minimal surfaces. We show that there exist a correspondence between RIMC-surfaces and Bryant surfaces in the hyperbolic space $\mathbb{H}^3$, therefore, the RIMC-surfaces are isothermic. We obtain a Weierstrass type representation for such surfaces which depends on a meromorphic function and a holomorphic function and we characterize a class of RIMC-surfaces parametrized by lines of curvature. Also, in this work motivated by [3] where for each isothermic surface parametrized by lines of curvature associates a solution of the Calapso equation, we show that we can associate these surfaces another solution to the Calapso equation. Moreover, we give explicit solutions of the Calapso equation that depend on holomorphic functions.

2. Preliminiminaries. In this paper the inner product $\langle\cdot,\cdot\rangle : \mathbb{C}\times\mathbb{C}\to\mathbb{R}$ is defined by

$$\langle f,g\rangle = f_1g_1 + f_2g_2, \text{ where } f = f_1 + if_2,\ g = g_1 + ig_2,$$

are holomorphic functions. In the computation we use the following properties: If $f,g,h:\mathbb{C}\to\mathbb{C}$, $z = u_1 + iu_2 \in \mathbb{C}$ are holomorphic functions then

$$\begin{gathered}\langle f,g\rangle_{,1} = \langle f',g\rangle + \langle f,g'\rangle,\ \ \langle f,g\rangle_{,2} = \langle if',g\rangle + \langle f,ig'\rangle,\\ \langle fg,h\rangle = \langle g,\ \overline{f}h\rangle,\ \ \triangle\langle f,g\rangle = 4\langle f',g'\rangle,\\ \langle f,g\rangle + i\langle f,ig\rangle = f\overline{g},\ \ \langle 1,f\rangle\langle 1,if\rangle = \frac{1}{2}\langle 1,if^2\rangle,\ \ \langle 1,f\rangle^2 - \langle 1,if\rangle^2 = \langle 1,f^2\rangle.\end{gathered} \tag{2.1}$$

Here $\langle f,g\rangle_{,i}$ denotes the derivative of $\langle f,g\rangle$ with respect to u_i, $i = 1,2$.

In [3] the author proposes the following problem: Find an isothermic surface $X(u_1,u_2)$ whose first and second fundamental forms are given by

$$I = e^{2\varphi}(du_1^2 + du_2^2),\ \ II = \widetilde{e}du_1^2 + \widetilde{g}du_2^2 \tag{2.2}$$

Using (2.2) and the expressions of the Gaussian curvature and the Codazzi equations we obtain the following system

$$\begin{aligned}\widetilde{e}_{,2} &= (\widetilde{e}+\widetilde{g})\varphi_{,2},\\ \widetilde{g}_{,1} &= (\widetilde{e}+\widetilde{g})\varphi_{,1},\\ \frac{\widetilde{e}\,\widetilde{g}}{e^{2\varphi}} &= -\triangle\varphi.\end{aligned} \tag{2.3}$$

To integrate the system (2.3) we make the following substitution

$$\widetilde{e} = \frac{1}{\sqrt{2}}(\omega+\Omega)e^{\varphi},\quad \widetilde{g} = \frac{1}{\sqrt{2}}(\omega-\Omega)e^{\varphi}. \tag{2.4}$$

Thus, by (2.4) the system (2.3) can be written as

$$\Omega_{,1} \ = \ \omega_{,1} - (\omega+\Omega)\varphi_{,1},$$

$$\begin{aligned}\Omega_{,2} &= -\omega_{,2} + (\omega - \Omega)\varphi_{,2}, \\ \triangle\varphi &= -\frac{1}{2}(\omega^2 - \Omega^2).\end{aligned} \tag{2.5}$$

Therefore, considering (2.5) as a system of three equations in the two unknown functions Ω and φ we must find ω for the system to be consistent.
By successive derivation we obtain that the system (2.5) is integrable if, and only if, ω satisfies the equation

$$\left(\frac{\omega_{,u_1u_2}}{\omega}\right)_{,u_1u_1} + \left(\frac{\omega_{,u_1u_2}}{\omega}\right)_{,u_2u_2} + (\omega^2)_{,u_1u_2} = 0. \tag{2.6}$$

The equation (2.6) is called *Calapso equation* and describes isothermic surfaces in $\mathbb{R}^3$. This equation is very difficult to solve and is strongly connected to the Painleve ODEs, some authors have obtained some solutions using symbolic computation. Also, some authors have found solutions of this equation associated with constant mean curvature surfaces.

As a consequence, each isothermic surface parametrized by lines of curvature gives us an explicit solution of the Calapso equation (2.6), more specifically, using equation (2.4), we obtain:

Remark 2.1. Let $X(u_1, u_2)$ be an isothermic surface parametrized by lines of curvature with first fundamental form given by

$$I = e^{2\varphi}(du_1^2 + du_2^2).$$

Then the function $\omega = \sqrt{2}e^{\varphi}H$ is a solution of the Calapso equation, where H is the mean curvature of X.

Definition 2.2. *A* congruence of spheres *in* $\mathbb{R}^3$ *is a two-parameter family of spheres with a differentiable* radius function *, whose centers lie on a regular surface.*

Definition 2.3. *An* envelope *of a congruence of spheres is a surface* M *of* $\mathbb{R}^3$ *such that each point of* M *is tangent to a sphere of the congruence.*

Lemma 2.4. *Let* M *be a surface in* $\mathbb{R}^3$ *with Gauss map* N *and* Π *a plane with unit normal vector* η*. Then there exists a congruence of spheres where* M *is an envelope of the congruence and the other envelope is contained in* Π*. Moreover, the radius function* $h : M \to \mathbb{R}$ *is given by*

$$h(p) = \frac{\langle p, \eta\rangle - d(0, \Pi)}{1 - \langle N(p), \eta\rangle}$$

where $d(0, \Pi)$ *denotes the distance from the origin to the plane* Π *and* $N(p) \neq \eta$*.*

Proof. Let $Q(p)$ the point of contact of the congruence of spheres with the plane Π, then

$$p + h(p)N(p) = Q(p) + h(p)\eta$$

hence, making the inner product with η, we obtain

$$h(p) = \frac{\langle p, \eta\rangle - \langle Q(p), \eta\rangle}{1 - \langle N(p), \eta\rangle},$$

and since $\langle Q(p), \eta\rangle$ measures the distance from the origin to the plane Π, follows the result. □

Remark 2.5. We note that fixed a plane Π in $\mathbb{R}^3$ for each surface M the radius function h introduced in the previous lemma is a geometric invariant, analogous to the support function, mean and Gaussian curvatures, which does not depend on the local parametrization of M.

Definition 2.6. *A surface $M \subset \mathbb{R}^3$ is called* radial inverse mean curvature surface *(RIMC-surface) if the congruence of spheres of centers $p + \frac{1}{H(p)}N(p)$, $p \in M$ and radius $\frac{1}{H(p)}$ determines two envelopes which one of them is contained in a plane, H is the mean curvature of M.*

The following theorem can be found in [11].

Theorem 2.7. *Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Consider $X(U)$, as a surface in $(\mathbb{R}^3, \langle , \rangle)$ with Euclidean metric, let N be the normal Gauss map, k_i the principal curvatures, H and K the mean and Gaussian curvatures, respectively. Analogously, consider $X(U)$ like a surface in $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_g)$, with a metric conformal to the Euclidean metric, with the conformal factor F^{-2}, let $\bar{k}_i$ be the principal curvature, $\bar{H}$, and $\bar{K}_E$ the mean and the extrinsic curvatures, respectively. Then*

$$\bar{k}_i = Fk_i + \langle N, \operatorname{grad} F\rangle$$

$$\bar{H} = FH + \langle N, \operatorname{grad} F\rangle$$

$$\bar{K}_E = F^2K + 2HF\langle N, \operatorname{grad} F\rangle + \langle N, \operatorname{grad} F\rangle^2$$

The following proposition shows that there is a correspondence between RIMC-surfaces and Bryant surfaces.

Proposition 2.8. *Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Consider $X(U)$, as a surface in $(\mathbb{R}^3, \langle , \rangle)$ with Euclidean metric, let N be the normal Gauss map, H the mean curvature. Analogously, consider $X(U)$ like a surface in hyperbolic space as the upper half-space model $\mathbb{H}^3 = \{(u_1, u_2, u_3) \in$*

$\mathbb{R}^3/u_3 > 0\}$, *let* $\bar{H}$ *the mean curvature. Moreover, consider* $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3/u_3 = 0\}$. *Then* $X(U)$ *is a RIMC-surface in* $\mathbb{R}^3$ *with respect to* Π *if, and only if,* $X(U)$ *is a Bryant surface in* $\mathbb{H}^3$.

Proof. Let $X = (X_1, X_2, X_3)$ and $N = (N_1, N_2, N_3)$, if $X(U)$ is a RIMC-surface in $\mathbb{R}^3$ with respect to Π, from Lemma 2.4 and definition 2.6, we get

$$\frac{X_3}{1 - N_3} = h = \frac{1}{H}.$$

On the other hand, if $X(U)$ is a surface in hyperbolic space $\mathbb{H}^3$, using Theorem 2.7, with conformal factor $F(u_1, u_2, u_3) = u_3$ we obtain

$$\bar{H} = X_3 H + N_3 = 1$$

hence, the result follows. □

Remark 2.9. As there is a correspondence between RIMC-surfaces and Bryant surfaces, some geometrical properties are preserved, for example the property of being isothermic, the class of rotation surfaces around the axis u_3, but in general both classes of surfaces have their geometric properties and particular applications to be studied.

Definition 2.10. *A surface* $M \subset \mathbb{R}^3$ *is called* isothermic *if it admits parametrization by lines of curvature and the first fundamental form is conformal.*

The following Theorem obtained in [9] characterize locally the surfaces M in $\mathbb{R}^3$ which are the envelopes of a congruence of spheres, whose other envelope is contained in a plane.

Theorem 2.11. *A surface* M *in* $\mathbb{R}^3$ *is the envelope of a congruence of spheres, whose other envelope is contained in the plane* $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$ *if, and only if, there exist an orthogonal local parametrization of* Π, $Y : U \subset \mathbb{R}^2 \to \Pi$ *and a differentiable function* $h : U \subset \mathbb{R}^2 \to \mathbb{R}$, *such that* $X : U \subset \mathbb{R}^2 \to M$, *given by*

$$X(u) = Y(u) - \frac{2h(u)}{S}\left(\frac{h_{,1}}{L_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{L_{22}}(u)Y_{,2}(u) - e_3\right) \tag{2.7}$$

is a parametrization of M, *with* $e_3 = (0,0,1)$, $L_{ii} = \langle Y_{,i}, Y_{,i}\rangle,\ 1 \le i \le 2$ *and*

$$S = \frac{(h_{,1})^2}{L_{11}} + \frac{(h_{,2})^2}{L_{22}} + 1. \tag{2.8}$$

Moreover, the Gauss map is given by

$$N(u) = e_3 + \frac{2}{S}\left(\frac{h_{,1}}{L_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{L_{22}}(u)Y_{,2}(u) - e_3\right). \tag{2.9}$$

The Weingarten matrix is given by

$$W = \frac{2}{P}(SV - 2h\ \det(V)I), \tag{2.10}$$

where the matrix $V = (V_{ij})$ is given by

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \sum_{l=1}^{2} \Gamma_{ij}^{l} h_{,l}\right), \quad 1 \le i, j \le 2, \tag{2.11}$$

$$P = S^2 - 2hS\,tr(V) + 4h^2 \det(V) \neq 0, \tag{2.12}$$

$$\Gamma_{ij}^{j} = \frac{L_{ii,j}}{2L_{ii}}, \text{ for all } i, j \text{ and } \Gamma_{ii}^{j} = -\frac{L_{ii}}{L_{jj}}\Gamma_{ij}^{i},\ 1 \le i \neq j \le 2. \tag{2.13}$$

Also,

$$g_{ij} = \langle X_{,i}, X_{,j}\rangle = L_{ij} - \frac{2h}{S}V_{ji}L_{ii} - \frac{2h}{S}V_{ij}L_{jj} + \left(\frac{2h}{S}\right)^2 \sum_{l=1}^{2} V_{il}V_{jl}L_{ll}. \tag{2.14}$$

The following result is a direct consequence of Theorem 2.11.

Theorem 2.12. *Let $M \subset \mathbb{R}^3$ be a connected orientable Riemann. Then $X : M \to \mathbb{R}^3$ is an immersion if, and only if, there exist a holomorphic function $g : M \to \mathbb{C}_\infty$ and a differentiable function $h : M \to \mathbb{R}^+_\infty$, such that $X(M)$ is locally parametrized by*

$$X(z) = (g, 0) - \frac{2h}{S}\left(\frac{g'}{|g'|^2}(h_{,1} + ih_{,2}), -1\right), \tag{2.15}$$

where $z = u_1 + iu_2 \in \mathbb{C}$, $\mathbb{R}^+_\infty = \mathbb{R}^+ \cup \{+\infty\}$, $g'(z) \neq 0\ \forall z \in \mathbb{C}$,

$$S = \frac{(h_{,1})^2 + (h_{,2})^2}{|g'|^2} + 1. \tag{2.16}$$

Moreover, the Gauss map is given by

$$N = e_3 + \frac{2}{S}\left(\frac{g'}{|g'|^2}(h_{,1} + ih_{,2}), -1\right). \tag{2.17}$$

The Weingarten matrix is given by (2.10) *with*

$$P = -S^2 + 2S(1-\gamma) + 4h^2 det(V) \text{ and } \gamma = \frac{h \triangle h}{|g'|^2} - S + 1. \tag{2.18}$$

The matrix $V = (V_{ij})$ is defined by

$$V_{ij} = \frac{1}{|g'|^2}\left(h_{,ij} - \sum_{l=1}^{2} \Gamma_{ij}^{l} h_{,l}\right), \quad 1 \le i, j \le 2, \tag{2.19}$$

where

$$\Gamma_{ii}^{i} = \frac{|g'|_{,i}}{|g'|} \quad \text{and} \quad \Gamma_{ij}^{i} = \frac{|g'|_{,j}}{|g'|} = -\Gamma_{ii}^{j}, \quad 1 \le i \neq j \le 2. \tag{2.20}$$

Proof. Note that $X : M \to \mathbb{R}^3$ is an immersion if, and only if, $X(M)$ is an envelope of a congruence of spheres where an envelope is contained in $\mathbb{C}_\infty$. From Theorem 2.11 there exist a function $h : M \to \mathbb{R}^+_\infty$ and an orthogonal parametrization of the plane. In fact, considering the holomorphic function $g : M \to \mathbb{C}_\infty$ and $Y(z) = (g(z), 0)$ we have $Y_{,1}(z) = (g', 0)$ and $Y_{,2}(z) = (ig', 0)$. From the Cauchy-Riemann equations, we get $L_{11} = L_{22} = |g'|^2$ and $L_{12} = 0$, thus, the parametrization Y is orthogonal. By using (2.7)–(2.10), we obtain (2.15)–(2.18). Also, utilizing (2.11) and (2.13) one has (2.19) and (2.20). This completes the proof of Theorem. □

3. Weierstrass type representation for RIMC-surfaces in $\mathbb{R}^3$.

In this section we present the main result which provides a Weierstrass type representation for RIMC-surfaces in $\mathbb{R}^3$ in terms of a meromorphic function and a holomorphic function.

Theorem 3.1. *Let $M \subset \mathbb{R}^3$ be a connected orientable Riemann. Then M is a RIMC-surface if, and only if, there exist a holomorphic and a meromorphic function $g, f : M \to \mathbb{C}_\infty$, respectively, such that $X(M)$ is locally parametrized by*

$$X(z) = (g, 0) - \frac{|g'|(1+|f|^2)}{|f'|S} \left(\frac{g'}{|g'|} \left(\frac{(1+|f|^2)}{2|f'|} \overline{\left(\frac{F'}{F}\right)} + \frac{f\overline{f'}}{|f'|} \right), -1 \right), \tag{3.1}$$

where

$$S = \left| \frac{(1+|f|^2)}{2|f'|} \left(\frac{F'}{F}\right) + \frac{\overline{f} f'}{|f'|} \right|^2 + 1, \; F = \frac{g'}{f'}. \tag{3.2}$$

The regularity condition is given by

$$P = -4 \left(\frac{1+|f|^2}{2|f'|} \right)^4 |A|^2 \neq 0, \quad A = \frac{F''}{F} - \frac{3}{2} \left(\frac{F'}{F}\right)^2 - \frac{F' f''}{F f'}. \tag{3.3}$$

Moreover, the first and the second fundamental form of X are given by

$$\widetilde{E} = \widetilde{G} = -\frac{|g'|^2 P}{S^2}, \; \widetilde{F} = 0, \tag{3.4}$$

$$\begin{aligned} \widetilde{e} &= -\frac{|g'|^2}{S^2} \left(-S(V_{11} - V_{22}) + \frac{2|f'|}{|g'|(1+|f|^2)} P \right), \\ \widetilde{f} &= \frac{2|g'|^2 V_{12}}{S}, \\ \widetilde{g} &= -\frac{|g'|^2}{S^2} \left(S(V_{11} - V_{22}) + \frac{2|f'|}{|g'|(1+|f|^2)} P \right), \end{aligned} \tag{3.5}$$

where

$$V_{12} = \frac{(1+|f|^2)}{2|f'||g'|} \langle 1, iA \rangle, \quad V_{11} - V_{22} = \frac{(1+|f|^2)}{|f'||g'|} \langle 1, A \rangle. \tag{3.6}$$

To prove the Theorem 3.1 we need the followings Lemmas.

Lemma 3.2. *Let g be a holomorphic function of $z = u_1 + iu_2$, such that the function $h(u_1, u_2)$ satisfies*

$$h\triangle h = |g'|^2 S, \tag{3.7}$$

where S is given by (2.16). *Then there exist a meromorphic function f such that*

$$h = \frac{|g'|(1+|f|^2)}{2|f'|}. \tag{3.8}$$

Proof. Considering $h = e^\phi$ and differentiating this expression we obtain that $\triangle h = h(\triangle\phi + |\nabla\phi|^2)$, hence, $h\triangle h = h^2(\triangle\phi + |\nabla\phi|^2)$. Using this expression and the fact that $h\triangle h = |g'|^2 S$, it follows that

$$\triangle\phi = |g'|^2 e^{-2\phi}. \tag{3.9}$$

Since, g is a holomorphic function we can rewrite (3.9) as

$$\triangle(\phi - \ln|g'|) = e^{-2(\phi - \ln|g'|)}. \tag{3.10}$$

Denoting $\nu = \phi - ln|g'|$ in (3.10) we obtain the *Liouville equation* $\triangle\nu = e^{-2\nu}$, whose solution is given by

$$\nu = \ln\left(\frac{1+|f|^2}{2|f'|}\right),$$

where f is a meromorphic function. Consequently,

$$\phi = \nu + \ln|g'| = \ln\left(\frac{|g'|(1+|f|^2)}{2|f'|}\right),$$

thus, we get (3.8) and the proof of Lemma is complete. □

Lemma 3.3. *If f and g are holomorphic functions and $h = \frac{1}{2}|F|(1+|f|^2)$, where $F = \frac{g'}{f'}$. Then the coefficients of matrix $V = (V_{ij})$ defined in* (2.19) *are given by*

$$V_{11} = \frac{1}{|g'|^2}\left(\frac{h|F'|^2}{|F|^2} + \frac{h\langle F, F''\rangle}{|F|^2} - \frac{2h\langle F, F'\rangle^2}{|F|^4} + \frac{\langle F, F'\rangle\langle f, f'\rangle}{|F|} + |F||f'|^2 \right.$$
$$\left. + \frac{h}{|F|^4}\langle F, iF'\rangle^2 + \frac{1}{|F|}\langle F, iF'\rangle\langle f, if'\rangle - \frac{h}{|f'|^2|F|^2}\langle Ff', F'f''\rangle\right), \tag{3.11}$$

$$V_{12} = \frac{h}{|g'|^2}\left\langle 1, i\left(\frac{F''}{F} - \frac{3}{2}\left(\frac{F'}{F}\right)^2 - \frac{F'f''}{Ff'}\right)\right\rangle, \tag{3.12}$$

$$V_{22}=\frac{1}{|g'|^2}\left(\frac{h|F'|^2}{|F|^2}-\frac{h\langle F,F''\rangle}{|F|^2}-\frac{2h\langle F,iF'\rangle^2}{|F|^4}+\frac{\langle F,iF'\rangle\langle f,if'\rangle}{|F|}+|F||f'|^2\right.$$

$$\left.+\frac{h}{|F|^4}\langle F,F'\rangle^2+\frac{1}{|F|}\langle F,F'\rangle\langle f,f'\rangle+\frac{h}{|f'|^2|F|^2}\langle Ff',F'f''\rangle\right). \tag{3.13}$$

Moreover,

$$V_{11}-V_{22} = \frac{2h}{|g'|^2}\left\langle 1,\frac{F''}{F}-\frac{3}{2}\left(\frac{F'}{F}\right)^2-\frac{F'f''}{Ff'}\right\rangle. \tag{3.14}$$

P r o o f. Differentiating h and using the properties given in (2.1) we obtain

$$h_{,1}=\frac{\langle F,F'\rangle}{|F|^2}h+|F|\langle f,f'\rangle,\quad h_{,2}=\frac{\langle F,iF'\rangle}{|F|^2}h+|F|\langle f,if'\rangle, \tag{3.15}$$

$$\begin{aligned}
h_{,11} &= \frac{|F'|^2h+\langle F,F''\rangle h+\langle F,F'\rangle h_{,1}}{|F|^2}-\frac{2h\langle F,F'\rangle^2}{|F|^4}+\frac{\langle F,F'\rangle\langle f,f'\rangle}{|F|}\\
&\quad +|F||f'|^2+|F|\langle f,f''\rangle,\\
h_{,22} &= \frac{|F'|^2h-\langle F,F''\rangle h+\langle F,iF'\rangle h_{,2}}{|F|^2}-\frac{2h\langle F,iF'\rangle^2}{|F|^4}+\frac{\langle F,iF'\rangle\langle f,if'\rangle}{|F|}\\
&\quad +|F||f'|^2-|F|\langle f,f''\rangle,\\
h_{,12} &= \frac{\langle F,iF''\rangle h+\langle F,F'\rangle h_{,2}}{|F|^2}-\frac{2h\langle F,F'\rangle\langle F,iF'\rangle}{|F|^4}+\frac{\langle F,iF'\rangle\langle f,f'\rangle}{|F|}\\
&\quad +|F|\langle f,if''\rangle,
\end{aligned} \tag{3.16}$$

$$h_{,1}+ih_{,2} = \frac{h}{|F|^2}F\overline{F'}+|F|f\overline{f'}, \tag{3.17}$$

$$h_{,11}-h_{,22}+2ih_{,12} = \frac{2hF\overline{F''}}{|F|^2}-\frac{h(F\overline{F'})^2}{|F|^4}+\frac{2F\overline{F'}f\overline{f'}}{|F|}+2|F|f\overline{f''}. \tag{3.18}$$

In addition, from (2.19) and (2.20), the coefficients of matrix V are given by

$$\begin{aligned}
V_{11} &= \frac{1}{|g'|^2}(h_{,11}-\langle B+iD,1\rangle),\\
V_{12} &= \frac{1}{|g'|^2}(h_{,12}-\langle B+iD,i\rangle),\\
V_{22} &= \frac{1}{|g'|^2}(h_{,22}+\langle B+iD,1\rangle),
\end{aligned} \tag{3.19}$$

where

$$B=\frac{\langle |g'|_{,1}+i|g'|_{,2},h_{,1}-ih_{,2}\rangle}{|g'|}\quad\text{and}\quad D=\frac{\langle |g'|_{,1}+i|g'|_{,2},i(h_{,1}-ih_{,2})\rangle}{|g'|}.$$

From expressions (2.1) we have

$$|g'|_{,1} + i|g'|_{,2} = \frac{g'\overline{g''}}{|g'|}, \quad B + iD = \frac{g'\overline{g''}}{|g'|^2}(h_{,1} + ih_{,2}), \tag{3.20}$$

and since $g' = Ff'$ it follows that,

$$\frac{g'\overline{g''}}{|g'|^2} = \frac{F\overline{F'}|f'|^2 + |F|^2 f'\overline{f''}}{|F|^2|f'|^2}. \tag{3.21}$$

So, by using (3.17), (3.20) and (3.21) we get

$$B + iD \;=\; \frac{h(F\overline{F'})^2}{|F|^4} + \frac{F\overline{F'}f\overline{f'}}{|F|} + \frac{hF\overline{F'}f'\overline{f''}}{|f'|^2|F|^2} + |F|f\overline{f''}. \tag{3.22}$$

Therefore, (3.11)–(3.13) it follows from (3.15), (3.16), (3.19) and (3.22). Finally, utilizing (3.11) and (3.13) after simplification we obtain

$$V_{11} - V_{22} \;=\; \frac{2h}{|g'|^2}\left(\left\langle 1, \frac{F''}{F}\right\rangle - \frac{3}{2}\left(\left\langle 1, \frac{F'}{F}\right\rangle^2 - \left\langle 1, \frac{iF'}{F}\right\rangle^2\right) - \left\langle 1, \frac{F'f''}{Ff'}\right\rangle\right),$$

note that the last expression is equivalent to (3.14), this completes the proof. □

We will now prove Theorem 3.1.

Proof. Since $F = \dfrac{g'}{f'}$, inserting (3.8) and (3.17) into (2.15) we get (3.1). Note that (2.16) can be written as $S = \dfrac{1}{|g'|^2}|h_{,1} + ih_{,2}|^2 + 1$, hence, substituting (3.17) in this expression it follows (3.2). Since M is a RIMC-surface one has that $\gamma = 1$, thus, from (2.18) we have $(h_{,11} + h_{,22})^2 = \dfrac{|g'|^4S^2}{h^2}$.
Therefore

$$(h_{,11} - h_{,22})^2 = \frac{|g'|^4S^2}{h^2} - 4h_{,11}h_{,22}$$

and

$$|h_{,11} - h_{,22} + 2ih_{,12}|^2 = \frac{|g'|^4S^2}{h^2} - 4h_{,11}h_{,22} + 4h_{,12}^2. \tag{3.23}$$

On the other hand, we obtain through (3.19) that

$$\det(V) = \frac{1}{|g'|^4}\left(h_{,11}h_{,22} - h_{,12}^2 - |B + iD|^2 + \langle B + iD, h_{,11} - h_{,22} + 2ih_{,12}\rangle\right), \tag{3.24}$$

inserting this expression in (2.18) one has

$$\begin{aligned} P \;=\; & -\frac{4h^2}{|g'|^4}|B + iD|^2 - \frac{h^2}{|g'|^4}\left[\frac{|g'|^4S^2}{h^2} - 4h_{,11}h_{,22} + 4h_{,12}^2\right] \\ & + \frac{4h^2}{|g'|^4}\langle B + iD, h_{,11} - h_{,22} + 2ih_{,12}\rangle, \end{aligned} \tag{3.25}$$

and by using (3.23) into (3.25) we get

$$P = -\frac{4h^2}{|g'|^4}\Bigg(|B+iD|^2 - \langle B+iD, h_{,11}-h_{,22}+2ih_{,12}\rangle + \frac{1}{4}|h_{,11}-h_{,22}+2ih_{,12}|^2\Bigg),$$

which can be rewritten as

$$P = -\frac{4h^2}{|g'|^4}\left|B+iD-\frac{1}{2}(h_{,11}-h_{,22}+2ih_{,12})\right|^2. \tag{3.26}$$

Also, (3.18) and (3.22) ensure that

$$B+iD-\frac{1}{2}(h_{,11}-h_{,22}+2ih_{,12}) = h\left(-\frac{\overline{F''}}{\overline{F}}+\frac{3}{2}\left(\frac{\overline{F'}}{\overline{F}}\right)^2+\frac{\overline{F'f''}}{\overline{Ff'}}\right), \tag{3.27}$$

substituting this expression into (3.26) we have that

$$P = -\frac{4h^4}{|g'|^4}\left|\frac{F''}{F}-\frac{3}{2}\left(\frac{F'}{F}\right)^2-\frac{F'f''}{Ff'}\right|^2, \text{ denoting } A = \frac{F''}{F}-\frac{3}{2}\left(\frac{F'}{F}\right)^2-\frac{F'f''}{Ff'}$$

and by using (3.8) it follows (3.3). We observe that the expressions (2.19) and (2.20) ensure that the coefficients of the matrix V satisfy $\triangle h = |g'|^2 tr(V)$.
On the other hand, from (2.18) we get $htr(V) = \gamma + S - 1$, by using this expression together with (2.14), (2.18) and (2.20) one has that the first fundamental form of X is given by

$$\begin{aligned} \widetilde{E} = g_{11} &= \frac{|g'|^2}{S^2}(-P+(1-\gamma)(2S-4hV_{11})), \\ \widetilde{F} = g_{12} &= \frac{-4h|g'|^2V_{12}}{S^2}(1-\gamma), \\ \widetilde{G} = g_{22} &= \frac{|g'|^2}{S^2}(-P+(1-\gamma)(2S-4hV_{22})), \end{aligned} \tag{3.28}$$

by using the fact that $\gamma = 1$ into (3.28) we get (3.4). Similarly, from (2.10), (2.12), (2.18) and (3.28) one has that the second fundamental form of X is given by

$$\begin{aligned} \widetilde{e} &= -\frac{2|g'|^2}{S^2}(V_{11}(2-2\gamma-S)+2hdet(V)), \\ \widetilde{f} &= -\frac{2|g'|^2}{S^2}V_{12}(2-2\gamma-S), \\ \widetilde{g} &= -\frac{2|g'|^2}{S^2}(V_{22}(2-2\gamma-S)+2hdet(V)). \end{aligned} \tag{3.29}$$

Finally, since $\gamma = 1$, substituting (3.8) in the expressions (3.29), (3.12), (3.14) we obtain (3.5) and (3.6). Thus, the proof of Theorem is complete. □

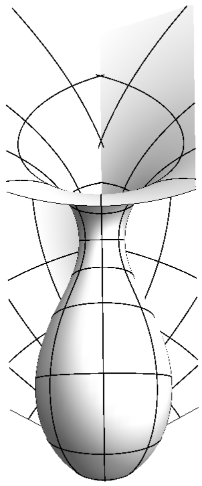

Fig. 1. $f(z) = z$, $g(z) = e^z$

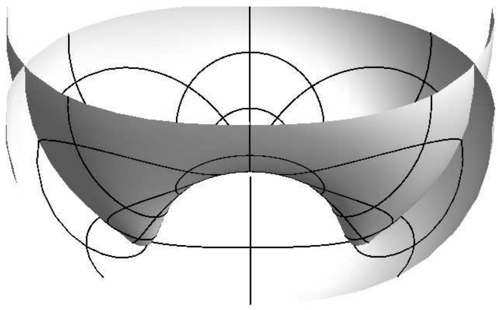

Fig. 2. $f(z) = \sinh z$, $g(z) = \cosh z$

By using the representation (3.1) given in Theorem 3.1 we give the following RIMC-surfaces, see Figures 1 and 2.

4. RIMC-surfaces parametrized by lines of curvature. The following result provides a characterization of RIMC-surfaces parametrized by lines of curvature.

Corollary 4.1. *Under the same conditions as in Theorem* 3.1*, the RIMC-surface X given by* (3.1) *is parametrized by lines of curvature if, and only if,*

$$S(g) - S(f) = c, \ c \in \mathbb{R} \setminus \{0\}. \tag{4.1}$$

P r o o f. By Theorem 3.1 we have that a RIMC-surface M is parametrized by lines of curvature if and only if $V_{12} = 0$, hence, through (3.6), this condition is equivalent to

$$A = \frac{F''}{F} - \frac{3}{2}\left(\frac{F'}{F}\right)^2 - \frac{F'f''}{Ff'} = c, \; c \in \mathbb{R} \setminus \{0\}. \tag{4.2}$$

By using the fact that $F = \dfrac{g'}{f'}$ and after elementary calculations we show that (4.2) is equivalent to (4.1). □

Corollary 4.2. *Let f and g holomorphic functions given by*

$$f = z_1 \int B e^{-\int \left(B + \frac{c}{2B}\right) dz} dz + z_2, \quad g = z_3 \int B e^{\int \left(B - \frac{c}{2B}\right) dz} dz + z_4$$

where B is a non-zero holomorphic function. Then the RIMC-surfaces X given by (3.1) *are parametrized by lines by curvature.*

Proof. Putting $\dfrac{f''}{f'} = A - B$, $\dfrac{g''}{g'} = A + B$ and substituting these expressions in (4.1), we get $2B' - 2AB = c$. From this expression we have

$$\frac{f''}{f'} = \frac{B'}{B} - \frac{c}{2B} - B, \quad \frac{g''}{g'} = \frac{B'}{B} - \frac{c}{2B} + B.$$

By integration follows the result. □

Remark 4.3. By the properties of the Schwarzian derivative (see [16]) one has that if $f, g : \mathbb{C} \to \mathbb{C}$ then

i) $S(f) = S(g)$ if and only if $g = \dfrac{z_1 f(z) + z_2}{z_3 f(z) + z_4}$, where $z_j \in \mathbb{C}$, with $z_1 z_4 - z_2 z_3 \neq 0$.

ii) In particular $S(f) = 0$ if and only if $f(z) = \dfrac{z_1 z + z_2}{z_3 z + z_4}$.

Proposition 4.4. *Let f and g meromorphic and holomorphic functions respectively given by*

i) $f(z) = \dfrac{z_1 z + z_2}{z_3 z + z_4}, \; g(z) = \dfrac{z_5 e^{\sqrt{-2c}z} + z_6}{z_7 e^{\sqrt{-2c}z} + z_8}$,

ii) $f(z) = \dfrac{z_5 e^{\sqrt{2c}z} + z_6}{z_7 e^{\sqrt{2c}z} + z_8}, \; g(z) = \dfrac{z_1 z + z_2}{z_3 z + z_4}$,

iii) $f(z) = \dfrac{z_1 e^{w_1 z} + z_2}{z_3 e^{w_1 z} + z_4}, \; g(z) = \dfrac{z_5 e^{w_2 z} + z_6}{z_7 e^{w_2 z} + z_8}, \; w_1^2 - w_2^2 = 2c$,

where $z_1 z_4 - z_2 z_3 \neq 0$, $z_5 z_8 - z_6 z_7 \neq 0$, $c \in \mathbb{R} \setminus \{0\}$, $w_1, w_2, z_j \in \mathbb{C}$, $j = 1, \cdots, 8$. Then the RIMC-surfaces X given by (3.1) *are parameterized by lines of curvature.*

P r o o f. From Corollary 4.1 we have that a RIMC-surface M is parameterized by lines of curvature if and only if the functions f and g satisfies (4.1). Thus, our problem is reduced to find the solutions of (4.1). In order to determine solutions to the equation (4.1) we consider the following cases:
a) If $S(f) = 0$ and $S(g) = c = S(e^{\sqrt{-2c}z})$, then by Remark 4.3, we obtain the solutions are given by i).
b) If $S(f) = -c = S(e^{\sqrt{2c}z})$ and $S(g) = 0$, then by Remark 4.3, we obtain the solutions are given by ii).
c) If $S(f) = -\dfrac{w_1^2}{2} = S(e^{w_1 z})$ and $S(g) = -\dfrac{w_2^2}{2} = S(e^{w_2 z})$, then by Remark 4.3, the solutions are given by iii). □

By using the representation (3.1) given in Theorem 3.1 we get the following examples of RIMC-surfaces parameterized by lines of curvature.

Example 1. Considering $z_1 = z_2 = -i, z_3 = 1, z_4 = -1, z_5 = \dfrac{1}{\sqrt{2}}, z_6 = z_7 = 0, z_8 = 1,\ w_1 = 2, w_2 = \sqrt{2}, c = 1$ in item iii) of Proposition 4.4, we have

$$f(z) = -i\coth z,\ g(z) = \frac{1}{\sqrt{2}}e^{\sqrt{2}z}.$$

Now, using these functions in (3.1) we get a RIMC-surface parametrized by lines of curvature (see Fig. 3).

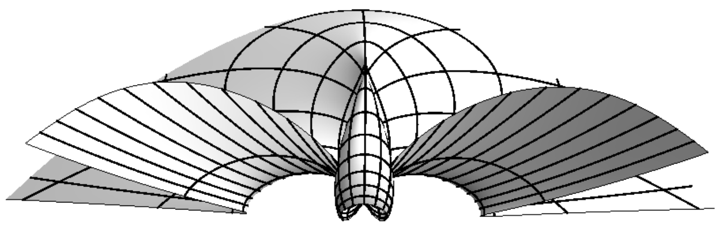

Fig. 3. RIMC-surface parametrized by lines of curvature

Example 2. Considering $z_1 = \dfrac{1}{2}, z_2 = z_3 = 0, z_4 = 1, z_5 = \dfrac{1}{\sqrt{2}}, z_6 = z_7 = 0, z_8 = 1,\ w_1 = 2,\ w_2 = \sqrt{2},\ c = 1$ in item iii) of Proposition 4.4, we have

$$f(z) = \frac{1}{2}e^{2z},\ g(z) = \frac{1}{\sqrt{2}}e^{\sqrt{2}z}.$$

Now, using these functions in (3.1) we get a RIMC-surface parametrized by lines of curvature (see Fig. 4).

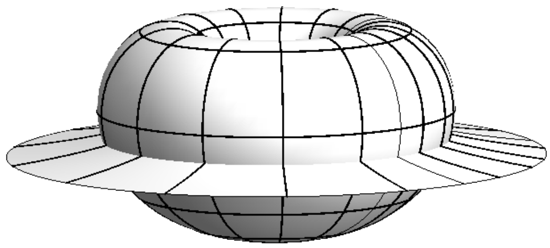

Fig. 4. RIMC-surface parametrized by lines of curvature

Example 3. Considering $z_1 = \frac{1}{2} + i, z_2 = z_3 = 0, z_4 = 1, z_5 = -\sqrt{2}i, z_6 = z_7 = 0, z_8 = 1, c = \frac{1}{4}$ in item i) of Proposition 4.4, we have

$$f(z) = \left(\frac{1}{2} + i\right) z,\ g(z) = -\sqrt{2} i e^{\frac{\sqrt{2}}{2} iz}.$$

Now, using these functions in (3.1) we get a RIMC-surface parametrized by lines of curvature (see Fig. 5).

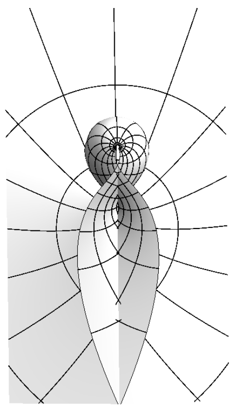

Fig. 5. RIMC-surface parametrized by lines of curvature

Example 4. Considering $z_1 = 1, z_2 = z_3 = 0, z_4 = 1, z_5 = z_6 = -2\sqrt{8i-1}, z_7 = \sqrt{65}, z_8 = -\sqrt{65}, w_1 = 2(1+i), w_2 = \sqrt{8i+1}, c = -\frac{1}{2}$ in item iii) of Proposition 4.4, we have

$$f(z) = e^{2(1+i)z},\ g(z) = -\frac{2}{\sqrt{8i+1}} \coth\left(\frac{\sqrt{8i+1}}{2}\right) z.$$

Now, using these functions in (3.1) we get a RIMC-surface parametrized by lines of curvature (see Fig. 6).

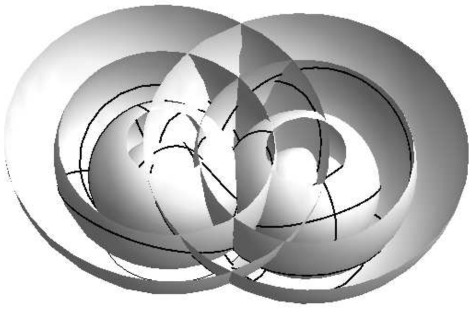

Fig. 6. RIMC-surface parametrized by lines of curvature

Example 5. Considering $z_1 = z_2 = i, z_3 = 1, z_4 = -1, z_5 = z_6 = -4\sqrt{1+2i}, z_7 = \sqrt{5}, z_8 = -\sqrt{5}, w_1 = \dfrac{i-1}{2}, w_2 = \dfrac{\sqrt{1-2i}}{2}, c = -\dfrac{1}{8}$ in item iii) of Proposition 4.4, we have

$$f(z) = i \coth\left(\frac{i-1}{4}\right) z,\ g(z) = -\frac{4}{\sqrt{1-2i}} \coth\left(\frac{\sqrt{1-2i}}{4}\right) z.$$

Now, using these functions in (3.1) we get a RIMC-surface parametrized by lines of curvature (see Fig. 7).

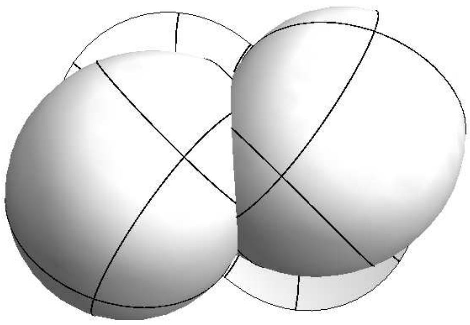

Fig. 7. RIMC-surface parametrized by lines of curvature

5. Solutions of the Calapso equation. The next result provides solutions for the Calapso equation and we will see that these solutions depend on a holomorphic function and a meromorphic function.

Proposition 5.1. *Let X be a RIMC-surface, f and g meromorphic and holomorphic functions respectively satisfying* (4.1). *Then the function*

$$\omega = \frac{\sqrt{2}c(1+|f|^2)}{S|f'|} \quad \textit{where} \quad S = \left|\frac{(1+|f|^2)}{2|f'|}\left(\frac{g''}{g'} - \frac{f''}{f'}\right) + \frac{\overline{f}f'}{|f'|}\right|^2 + 1, \tag{5.1}$$

is a solution for the Calapso equation.

Proof. From Remak 2.1 and Theorem 3.1 we get

$$\widetilde{E} = \widetilde{G} = \frac{4c^2}{S^2|g'|^2}h^4 = e^{2\varphi}.$$

Since, $\omega = \sqrt{2}He^{\varphi}$, where H is the mean curvature of X and $H = \dfrac{1}{h}$, $e^{\varphi} = \dfrac{2c}{S|g'|}h^2$ one has that $\omega = \dfrac{2c\sqrt{2}h}{S|g'|}$, by using (3.8) we obtain (5.1), thus, the proof is complete. □

Proposition 5.2. *Let B be a holomorphic function and f given by*

$$f = z_1 \int Be^{-\int \left(B + \frac{c}{2B}\right) dz} dz + z_2, \quad z_1, z_2 \in \mathbb{C}.$$

Then the function ω given by $\omega = \dfrac{\sqrt{2}c(1+|f|^2)}{S|z_1 Be^{-\int(B+\frac{c}{2B})dz}|}$ *where* $S = \dfrac{|e^{\int(B+\frac{c}{2B})dz}|^2}{|z_1|^2}\left|1+|f|^2 + z_1\bar{f}e^{-\int\left(B+\frac{c}{2B}\right)dz}\right|^2 + 1$ *is a solution for the Calapso equation.*

Proof. The proof follows from Corollary 4.2 and Proposition 5.1. □

The following result provides explicit solutions of the Calapso equation.

Proposition 5.3. *The functions ω_j given by*

i) $\alpha = z_1z_4 - z_2z_3 \neq 0,\ z_j \in \mathbb{C}$,

$$\omega_1 = \frac{c\sqrt{2}|\alpha|\left(|z_1z+z_2|^2 + |z_3z+z_4|^2\right)}{\left|(|z_1z+z_2|^2+|z_3z+z_4|^2)\left(\frac{\sqrt{-2c}(-z_7e^{\sqrt{-2c}z}+z_8)}{2(z_7e^{\sqrt{-2c}z}+z_8)} + \frac{z_3}{z_3z+z_4}\right) - \alpha\left(\frac{\overline{z_1z+z_2}}{z_3z+z_4}\right)\right|^2 + |\alpha|^2},$$

ii) $\beta = z_5z_8 - z_6z_7 \neq 0,\ z_j \in \mathbb{C},\ A = z_5e^{\sqrt{2c}z} + z_6,\ B = z_7e^{\sqrt{2c}z} + z_8$,

$$\omega_2 = \frac{2c\sqrt{|c|}|\beta e^{\sqrt{2c}z}|\left(|A|^2+|B|^2\right)}{\left|(|A|^2+|B|^2)\left(\frac{\sqrt{2c}(-z_7e^{\sqrt{2c}z}+z_8)}{2B} + \frac{z_3}{z_3z+z_4}\right) + \frac{\sqrt{2c}\beta e^{\sqrt{2c}z}\overline{A}}{B}\right|^2 + 2|c||\beta e^{\sqrt{2c}z}|^2},$$

iii) $\alpha = z_1z_4 - z_2z_3 \neq 0$, z_j, $w_j \in \mathbb{C}$, $A = z_1e^{w_1z} + z_2$, $B = z_3e^{w_1z} + z_4$, $w_1^2 - w_2^2 = 2c$.

$$\omega_3 = \frac{c\sqrt{2}|w_1\alpha e^{w_1z}|\left(|A|^2+|B|^2\right)}{\left|(|A|^2+|B|^2)\left(\frac{w_2(-z_7e^{w_2z}+z_8)}{2(z_7e^{w_2z}+z_8)} - \frac{w_1(-z_3e^{w_1z}+z_4)}{2B}\right) - \frac{w_1\alpha e^{w_1z}\overline{A}}{B}\right|^2 + |w_1\alpha e^{w_1z}|^2}.$$

are solutions for the Calapso equation.

P r o o f. The proof is a direct consequence from Propositions 4.4 and 5.1. □

For the paper [3] we have that for each isothermic surface parametrized by lines of curvature we have a solution of the Calapso equation, in the next we prove that this solution is associated with another solution of the Calapso equation. Before we present the following definition.

Definition 5.4. *Let M be a surface with principal curvatures k_1 and k_2. The* skew curvature *of M (see [15]) is given by*

$$H' = \frac{1}{2}(k_1 - k_2). \tag{5.2}$$

Proposition 5.5. *Let $X(u_1, u_2)$ be an isothermic surface parametrized by lines of curvature with first fundamental form given by*

$$I = e^{2\varphi}(du_1^2 + du_2^2).$$

Then the function $\Omega = \sqrt{2}e^{\varphi}H'$ is a solution of the Calapso equation, where H' is the skew curvature of X.

P r o o f. Differentiating the first equation of (2.5) with respect to variable u_2 and the second equation with respect to u_1, adding and subtracting these expressions we obtain

$$\frac{\omega_{,12}}{\omega} = \varphi_{,12} + \varphi_{,1}\varphi_{,2}\ , \quad \frac{\Omega_{,12}}{\Omega} = -\varphi_{,12} + \varphi_{,1}\varphi_{,2}.$$

Calculating the Laplacian of these expressions and using the third equation of (2.5), we have

$$\triangle\left(\frac{\omega_{,12}}{\omega}\right) - \triangle\left(\frac{\Omega_{,12}}{\Omega}\right) = 2\triangle(\varphi_{,12}) = 2(\triangle\varphi)_{,12} = (\Omega^2 - \omega^2)_{,12}.$$

This equation is equivalent to

$$\triangle\left(\frac{\omega_{,12}}{\omega}\right) + (\omega^2)_{,12} = \triangle\left(\frac{\Omega_{,12}}{\Omega}\right) + (\Omega^2)_{,12}$$

Since ω is solution of the Calapso equation then Ω is also the solution of the Calapso equation, finally using the equations (2.4) and (5.2) we obtain the result. □

Proposition 5.6. *Let f be a holomorphic function. Then the function*

$$\Omega = \frac{2\sqrt{2}|f'|}{1+|f|^2} \tag{5.3}$$

is a solution of the Calapso equation.

Proof. Given a holomorphic function f there exists a function g such that equation (49) is satisfied. From Proposition 5.5 we have that the function $\Omega = \sqrt{2}e^{\varphi}H'$ is a solution of the Calapso equation, using (3.3)–(3.6) and (3.8), we obtain that

$$H' = \frac{c|g'|(1+|f|^2)}{S|f'|e^{2\varphi}}, \quad e^{\varphi} = \frac{2ch^2}{S|g'|}.$$

From these expressions follows the result. □

Remark 5.7. By Proposition 5.6, for each holomorphic function f we obtain a solution of the Calapso equation.

It is known that rigid motions of $\mathbb{R}^3$ and inversions preserve the property of a surface being isothermic, as an application we obtain the following result.

Proposition 5.8. *Let $X(u_1, u_2)$ be an isothermic surface parametrized by lines of curvature with Gauss map $N(u_1, u_2)$ and first fundamental form given by*

$$I = e^{2\varphi}(du_1^2 + du_2^2).$$

Then for all vector $V \in \mathbb{R}^3$ the function

$$\widetilde{\omega} = \sqrt{2}e^{\varphi}\left(H + 2\frac{\langle X+V, N\rangle}{\langle X+V, X+V\rangle}\right)$$

is a solution of the Calapso equation where H is the mean curvature of X.

Proof. Considering an inversion of X: $Y = \dfrac{X+V}{\langle X+V, X+V\rangle}$, we get that the first fundamental form and the mean curvature of Y are given by

$$\tilde{I} = \frac{e^{2\varphi}}{\langle X+V, X+V\rangle^2}(du_1^2 + du_2^2), \quad \widetilde{H} = \langle X+V, X+V\rangle H + 2\langle X+V, N\rangle. \tag{5.4}$$

From Remark 2.1, we obtain that $\tilde{\omega} = \dfrac{\sqrt{2}e^{\varphi}}{\langle X+V, X+V\rangle}\widetilde{H}$ is a solution of the Calapso equation, using (5.4) follows the result. □

Remark 5.9. Using this proposition can be obtained new explicit solutions of the Calapso equation in terms of the solutions already obtained previously. This work can be used to classify RIMC-surfaces with additional geometric properties, can also be used to obtain explicit solutions of the Calapso equation associated with new classes of isothermic surfaces, future work in this directions will be present.

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