

VARIETIES OF BICOMMUTATIVE ALGEBRAS*

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ABSTRACT. Bicommutative algebras are nonassociative algebras satisfying the polynomial identities of right- and left-commutativity $(x_1x_2)x_3 = (x_1x_3)x_2$ and $x_1(x_2x_3) = x_2(x_1x_3)$. Let \mathfrak{B} be the variety of all bicommutative algebras over a field K of characteristic 0 and let $F(\mathfrak{B})$ be the free algebra of countable rank in \mathfrak{B} . We prove that if \mathfrak{V} is a subvariety of \mathfrak{B} satisfying a polynomial identity $f = 0$ of degree k , where $0 \neq f \in F(\mathfrak{B})$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \dots$, is bounded by a polynomial in n of degree $k - 1$. Since $c_n(\mathfrak{B}) = 2^n - 2$ for $n \geq 2$, and $\exp(\mathfrak{B}) = 2$, this gives that $\exp(\mathfrak{V}) \leq 1$, i.e., \mathfrak{B} is minimal with respect to the codimension growth. When the field K is algebraically closed there are only three pairwise nonisomorphic two-dimensional bicommutative algebras A which are nonassociative. They are one-generated and with the property $\dim A^2 = 1$. We present bases of their polynomial identities and show that one of these algebras generates the whole variety \mathfrak{B} .

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1. Introduction. Bicommutative algebras are nonassociative algebras over a field K satisfying the polynomial identities of right- and left-commutativity

$$(x_1x_2)x_3 = (x_1x_3)x_2, \quad x_1(x_2x_3) = x_2(x_1x_3).$$

In the sequel we consider algebras over a field K of characteristic 0 only. One-sided commutative algebras appeared first in the paper by Cayley [7] in 1857. In the modern language this is the right-symmetric Witt algebra W_1^{rsym} in one variable. Maybe the most important examples of one-side commutative algebras are Novikov algebras which are left-commutative and right-symmetric. The latter means that the algebras satisfy the polynomial identity $(x_1, x_2, x_3) = (x_1, x_3, x_2)$ for the associator $(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$. The motivation to study Novikov algebras comes from the needs of the Hamiltonian operator in mechanics and the equations of hydrodynamics, see Dzhumadil'daev, Ismailov, and Tulenbaev [13] and Drensky and Zhakhayev [12] for details. By Kaygorodov and Volkov [22] when the base field K is algebraically closed of arbitrary characteristic up to isomorphism there are only seven two-dimensional bicommutative algebras A with nontrivial multiplication. Four of them are associative-commutative. Changing the notation and the bases of the algebras in [22] the three nonassociative two-dimensional bicommutative algebras

$$A_{\pi, \varrho}, \quad (\pi, \varrho) = (0, 1), (1, 0), (1, -1),$$

are generated by one element r and satisfy the condition $\dim A_{\pi, \varrho}^2 = 1$. Their multiplication rules are

$$(1) \quad rr^2 = \pi r^2, r^2r = \varrho r^2, r^2r^2 = \pi \varrho r^2.$$

It is easy to see that the same holds over an arbitrary field K of characteristic 0: Up to isomorphism the three algebras $A_{0,1}, A_{1,0}, A_{1,-1}$, are the only one-generated nonassociative two-dimensional bicommutative algebras A with $\dim A^2 = 1$.

The structure of the free bicommutative algebra and the most important numerical invariants of the T-ideal of the polynomial identities were described by Dzhumadil'daev, Ismailov, and Tulenbaev [13], see also the announcement [14]. In [12], jointly with Zhakhayev, we proved that finitely generated bicommutative algebras are weakly noetherian, i.e., satisfy the ascending chain condition for two-sided ideals, and answer into affirmative the finite basis problem for varieties of bicommutative algebras over a field of arbitrary characteristic.

One of the most important measures for the complexity of the polynomial identities of a variety \mathfrak{V} of K -algebras is the codimension sequence $c_n(\mathfrak{V})$,

$n = 1, 2, \dots$, where $c_n(\mathfrak{V})$ is the dimension of the multilinear polynomials of degree n in the free algebra $F_n(\mathfrak{V})$ of rank n . As a first approximation to the more precise estimate of the growth of the codimensions one studies the behaviour of $\sqrt[n]{c_n(\mathfrak{V})}$. In the special case when

$$\exp(\mathfrak{V}) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n(\mathfrak{V})}$$

exists it is called the *exponent* of \mathfrak{V} , see Giambruno and Zaicev [16, 17] who proved that for associative PI-algebras the exponent always exists and is an integer. Following [18] the variety \mathfrak{V} is *minimal of a given exponent* if $\exp(\mathfrak{V}) < \exp(\mathfrak{W})$ for all proper subvarieties \mathfrak{W} of \mathfrak{V} . (In [10] we called such varieties *extremal*.)

It was shown in [13] that for the variety \mathfrak{B} of all bicommutative algebras

$$c_1(\mathfrak{B}) = 1 \text{ and } c_n(\mathfrak{B}) = 2^n - 2, \quad n = 2, 3, \dots$$

Hence $\exp(\mathfrak{B}) = 2$. Our first main result is that the variety \mathfrak{B} is minimal of exponent 2. More precisely we show that if \mathfrak{V} is a subvariety of \mathfrak{B} satisfying a polynomial identity $f = 0$ of degree k , where $0 \neq f \in F(\mathfrak{V}) = F_\infty(\mathfrak{V})$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \dots$, is bounded by a polynomial in n of degree $k - 1$. The results of [13] give that the variety \mathfrak{B} is generated by the free algebra $F_2(\mathfrak{B})$ of rank 2. We slightly improve this and show that \mathfrak{B} is generated by the free algebra $F_1(\mathfrak{B})$ of rank 1. As a byproduct of our approach, starting with the basis of $F(\mathfrak{B})$ in [13] we give a new proof of the description of the cocharacter sequence $\chi_n(\mathfrak{B})$, $n = 1, 2, \dots$. Finally we study the polynomial identities of the two-dimensional algebras $A_{\pi, \varrho}$ with multiplication defined by (1). We show that the algebra $A_{1, -1}$ generates the whole variety \mathfrak{B} . The varieties $\text{var}(A_{0,1})$ and $\text{var}(A_{1,0})$ generated respectively by the algebras $A_{0,1}$ and $A_{1,0}$ are defined as subvarieties of \mathfrak{B} by the polynomial identities $x_1(x_2x_3) = 0$ and $(x_1x_2)x_3 = 0$, i.e., they are equal, respectively, to the varieties of left-nilpotent and right-nilpotent of class 3 bicommutative algebras.

2. Preliminaries. We fix a field K of characteristic 0. All algebras, vector spaces, and tensor products will be over K . Traditionally, one states the results on polynomial identities and cocharacter sequences in the language of representation theory of the symmetric group S_n . Instead, we shall work with representation theory of the general linear group $\text{GL}_d = \text{GL}_d(K)$. Then using the approach developed by Berele [6] and the author [9] we shall translate easily the results in terms of representations of S_n . We start with the necessary background on representation theory of GL_d acting canonically on the d -dimensional vector space KX_d with basis $X_d = \{x_1, \dots, x_d\}$. We refer, e.g., to the books

by Macdonald [24] for general facts and by the author [11] for applications in the spirit of the problems considered here. All GL_d -modules which appear in this paper are completely reducible and are direct sums of irreducible polynomial modules. The irreducible polynomial representations of GL_d are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_d)$, $\lambda_1 \geq \dots \geq \lambda_d \geq 0$. We denote by $W(\lambda) = W_d(\lambda)$ the corresponding irreducible GL_d -module. The action of GL_d on KX_d is extended diagonally on the tensor algebra of KX_d and, up to isomorphism, all $W(\lambda)$ can be found there. The tensor algebra of KX_d is isomorphic, also as a GL_d -module, to the free associative algebra $K\langle X_d \rangle = K\langle x_1, \dots, x_d \rangle$. Since the diagonal action of GL_d on the tensor algebra is not affected by the parentheses, we may work also in the absolutely free algebra $K\{X_d\}$ and in the relatively free algebra $F_d(\mathfrak{V})$ of any variety \mathfrak{V} .

The module $W(\lambda) \subset K\{X_d\}$ is generated by a unique, up to a multiplicative constant, multihomogeneous element w_λ of degree $\lambda = (\lambda_1, \dots, \lambda_d)$, i.e., homogeneous of degree λ_k with respect to each variable x_k , called the *highest weight vector* of $W(\lambda)$. In order to state the characterization of the highest weight vectors we recall that for an algebra R the linear operator $\delta : R \rightarrow R$ is a derivation if $\delta(r_1 r_2) = \delta(r_1)r_2 + r_1\delta(r_2)$ for all $r_1, r_2 \in R$. If $\delta : KX_d \rightarrow KX_d$ is any linear operator of the d -dimensional vector space, then δ can be extended in a unique way to a derivation of $K\langle X_d \rangle, K\{X_d\}$, and of any relatively algebra $F_d(\mathfrak{V})$. The following lemma is a partial case of a result by De Concini, Eisenbud, and Procesi [8], see also Almkvist, Dicks, and Formanek [2]. In the version which we need, the first part of the lemma was established by Koshlukov [23].

Lemma 2.1 (see, e.g., Benanti and Drensky [5]). *Let $1 \leq i < j \leq d$ and let $\Delta_{x_j \rightarrow x_i}$ be the derivation of $K\{X_d\}$ defined by $\Delta_{x_j \rightarrow x_i}(x_j) = x_i$, $\Delta_{x_j \rightarrow x_i}(x_k) = 0$, $k \neq j$. If $w(X_d) = w(x_1, \dots, x_d) \in K\{X_d\}$ is multihomogeneous of degree λ_k with respect to x_k , then $w(X_d)$ is a highest weight vector for some $W(\lambda)$ if and only if $\Delta_{x_j \rightarrow x_i}(w(X_d)) = 0$ for all $i < j$. Equivalently, $w(X_d)$ is a highest weight vector for $W(\lambda)$ if and only if*

$$g_{ij}(w(X_d)) = w(X_d), \quad 1 \leq i < j \leq d,$$

where g_{ij} is the linear operator of the KX_d which sends x_j to $x_i + x_j$ and fixes the other x_k .

If W_i , $i = 1, \dots, m$, are m isomorphic copies of the GL_d -module $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1 \oplus \dots \oplus W_m$ has the form $\xi_1 w_1 + \dots + \xi_m w_m$ for some $\xi_i \in K$. Any m linearly independent highest weight vectors can serve

as a set of generators of the GL_d -module $W_1 \oplus \cdots \oplus W_m$. The algebra $F_d(\mathfrak{V})$ decomposes as a GL_d -module as

$$(2) \quad F_d(\mathfrak{V}) = \bigoplus_{\lambda} m_{\lambda}(\mathfrak{V})W(\lambda),$$

where the summation runs on all partitions λ in not more than d parts and the nonnegative integer $m_{\lambda}(\mathfrak{V})$ is the multiplicity of $W(\lambda)$ in the decomposition of $F_d(\mathfrak{V})$. The canonical multigrading of $F_d(\mathfrak{V})$ which counts the degree of each variable in X_d agrees with the action of GL_d in the following way. Let

$$\begin{aligned} H(F_d(\mathfrak{V}), T_d) &= H(F_d(\mathfrak{V}), t_1, \dots, t_d) \\ &= \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) T_d^n = \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) t_1^{n_1} \cdots t_d^{n_d} \end{aligned}$$

be the Hilbert series of $F_d(\mathfrak{V})$ as a multigraded vector space, where $F_d^{(n)}(\mathfrak{V})$ is the multihomogeneous component of $F_d(\mathfrak{V})$ of degree $n = (n_1, \dots, n_d)$. Then

$$H(F_d(\mathfrak{V}), T_d) = \sum_{\lambda} m_{\lambda}(\mathfrak{V}) s_{\lambda}(T_d) = \sum_{\lambda} m_{\lambda}(\mathfrak{V}) s_{\lambda}(t_1, \dots, t_d),$$

where $s_{\lambda}(T_d)$ is the Schur function corresponding to the partition λ .

There is another group action which is important for the theory of algebras with polynomial identities. The symmetric group S_n acts on the vector space $P_n(\mathfrak{V})$ of the multilinear polynomials of degree n in $F_n(\mathfrak{V})$ by

$$\sigma(f(x_1, \dots, x_n)) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \sigma \in S_n, f \in P_n(\mathfrak{V}).$$

The S_n -character of $P_n(\mathfrak{V})$ is called the S_n -cocharacter of \mathfrak{V} . It is known that the decomposition of the n -th cocharacter

$$(3) \quad \chi_n(\mathfrak{V}) = \sum_{\lambda \vdash n} m_{\lambda}(\mathfrak{V}) \chi_{\lambda},$$

where χ_{λ} is the irreducible S_n -character indexed with the partition λ of n , is determined by the Hilbert series of $F_n(\mathfrak{V})$. The multiplicities $m_{\lambda}(\mathfrak{V})$ are the same for $F_n(\mathfrak{V})$ in (2) and for $\chi_n(\mathfrak{V})$ in (3). Finally, we recall a special case of the Young rule (and of the Littlewood-Richardson rule) for the product of two Schur functions $s_{(p)}(T_d)$ and $s_{(q)}(T_d)$ (and also for the tensor product $W(p) \otimes W(q)$)

of the GL_d -modules $W(p)$ and $W(q)$. We assume that $p \geq q$. The case $p < q$ is similar.

$$(4) \quad \begin{aligned} s_{(p)}(T_d)s_{(q)}(T_d) &= \sum_{k=0}^q s_{(p+q-k,k)}(T_d), \\ W(p) \otimes W(q) &\cong \bigoplus_{k=0}^q W(p+q-k, k). \end{aligned}$$

We shall need also estimates for the degree of the irreducible S_n -characters.

Lemma 2.2. *The degree d_λ of the irreducible S_n -character χ_λ , $\lambda = (\lambda_1, \lambda_2) \vdash n$, is a polynomial in n of degree λ_2 .*

Proof. By the hook formula

$$d_\lambda = \frac{n!}{\prod h_{ij}},$$

where h_{ij} is the length of the hook at the (i, j) -position of the Young diagram of λ . For $\lambda = (\lambda_1, \lambda_2) \vdash n$ the lengths of the hooks of the first row are equal, reading them from right to left, to

$$1, 2, \dots, n - 2\lambda_2, n - 2\lambda_2 + 2, \dots, n - \lambda_2 + 1$$

and those of the second row are $1, 2, \dots, \lambda_2$. Hence

$$d_\lambda = \frac{n(n-1) \cdots (n - \lambda_2 + 1)(n - 2\lambda_2 + 1)}{\lambda_2!},$$

which is a polynomial of degree λ_2 in n . \square

Let \mathfrak{B} be the variety of all bicommutative algebras. We assume that the free bicommutative algebras $F = F(\mathfrak{B})$ and $F_d = F_d(\mathfrak{B})$ are freely generated, respectively, by the sets $X = \{x_1, x_2, \dots\}$ and $X_d = \{x_1, \dots, x_d\}$. By [13] the basis of the square F_d^2 of the algebra F_d as a K -vector space consists of the following polynomials:

$$(5) \quad u_{i,j} = x_{i_1}(\cdots(x_{i_{p-1}}(\cdots((x_{i_p}x_{j_1})x_{j_2})\cdots)x_{j_q})),$$

where $p, q \geq 1$, $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq i_p \leq d$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_q \leq d$. For any permutations $\sigma \in S_p$ and $\tau \in S_q$ the element $u_{i,j}$ from (5) satisfy the equality

$$(6) \quad u_{i,j} = x_{i_{\sigma(1)}}(\cdots(x_{i_{\sigma(p-1)}}(\cdots((x_{i_{\sigma(p)}}x_{j_{\tau(1)}})x_{j_{\tau(2)}})\cdots)x_{j_{\tau(q)}})),$$

The properties and the multiplication rules of $F_d(\mathfrak{B})$ from [13] are restated in [12] in the following way. We consider the polynomial algebra

$$K[Y_d, Z_d] = K[y_1, \dots, y_d, z_1, \dots, z_d]$$

in $2d$ commutative and associative variables. We associate to each monomial $u_{i,j}$ in (5) the monomial

$$\psi(u_{i,j}) = y_{i_1} \cdots y_{i_{p-1}} y_{i_p} z_{j_1} z_{j_2} \cdots z_{j_q} \in K[Y_d, Z_d]$$

and extend ψ by linearity to a linear map $\psi : F_d^2 \rightarrow K[Y_d, Z_d]$. The image $\psi(F_d^2)$ is spanned by all monomials

$$Y_d^\alpha Z_d^\beta = y_1^{\alpha_1} \cdots y_d^{\alpha_d} z_1^{\beta_1} \cdots z_d^{\beta_d}, \quad |\alpha| = \sum_{i=1}^d \alpha_i > 0, |\beta| = \sum_{j=1}^d \beta_j > 0.$$

Then we define an algebra G_d generated by X_d with basis

$$X_d \cup \{Y_d^\alpha Z_d^\beta \mid |\alpha|, |\beta| > 0\}$$

and multiplication rules

$$x_i x_j = y_i z_j,$$

$$x_i (Y_d^\alpha Z_d^\beta) = y_i Y_d^\alpha Z_d^\beta,$$

$$(Y_d^\alpha Z_d^\beta) x_j = Y_d^\alpha Z_d^\beta z_j,$$

$$(Y_d^\alpha Z_d^\beta)(Y_d^\gamma Z_d^\delta) = Y_d^{\alpha+\gamma} Z_d^{\beta+\delta}.$$

The algebras F_d and G_d are isomorphic both as algebras and as multigraded vector spaces with isomorphism $\psi : F_d \rightarrow G_d$ which sends $x_i \in F_d$ to $x_i \in G_d$ and acts on F_d^2 in the same way as the linear map $\psi : F_d^2 \rightarrow K[Y_d, Z_d]$ defined above.

3. Free bicommutative algebras. In this section we give an alternative proof of the formula for the cocharacter sequence of \mathfrak{B} given in [13] and describe the highest weight vectors of the irreducible GL_d -submodules of $F_d = F_d(\mathfrak{B})$. The action of GL_d on the d -dimensional vector space KX_d induces a similar action on KY_d and KZ_d which is extended diagonally on the polynomial algebras $K[Y_d]$ and $K[Z_d]$ and on the square G_d^2 of the algebra G_d . In the sequel we equip $K[Y_d]$, $K[Z_d]$, and G_d^2 with this action of GL_d .

Lemma 3.1. *As a multigraded vector space the square F_d^2 of the free bi-commutative algebra F_d is isomorphic to the tensor product $\omega(K[Y_d]) \otimes \omega(K[Z_d])$, where ω is the augmentation ideal of the polynomial algebra, i.e., the ideal of polynomials without constant term. As a GL_d -module F_d^2 is isomorphic to the direct sum of tensor products*

$$\bigoplus_{p,q \geq 1} W(p) \otimes W(q).$$

Proof. We identify the monomial $Y_d^\alpha Z_d^\beta \in K[Y_d, Z_d]$ with $Y_d^\alpha \otimes Z_d^\beta \in K[Y_d] \otimes K[Z_d]$. Then the first part of the lemma is simply a restatement of the fact that the image of F_d^2 under the action of ψ has a basis $\{Y_d^\alpha Z_d^\beta \mid |\alpha|, |\beta| > 0\}$. The second part of the lemma holds because the GL_d -module $K[Y_d]^{(p)}$ of the homogeneous polynomials of degree p in $K[Y_d]$ is isomorphic to $W(p)$ and similarly for $K[Z_d]^{(q)}$. \square

Proposition 3.2 ([13]). *The cocharacter sequence of the variety \mathfrak{B} of all bicommutative algebras is*

$$\chi_n(\mathfrak{B}) = \sum_{(\lambda_1, \lambda_2) \vdash n} m_{(\lambda_1, \lambda_2)}(\mathfrak{B}) \chi_{(\lambda_1, \lambda_2)},$$

where

$$\begin{aligned} m_{(1)}(\mathfrak{B}) &= 1, \\ m_{(n)}(\mathfrak{B}) &= n - 1, \quad n > 1, \\ m_{(\lambda_1, \lambda_2)}(\mathfrak{B}) &= n - 2\lambda_2 + 1, \quad \lambda_2 > 0. \end{aligned}$$

Proof. The multiplicities of the irreducible S_n -characters in the cocharacter sequence (3) and of the irreducible GL_d -modules of the homogeneous component $F_d^{(n)}$ of degree n of the free algebra F_d in (2) are the same for $d \geq n$. Hence we may work in F_d instead of with $\chi_n(\mathfrak{B})$. Since the case $n = 1$ is trivial, we shall assume that $n > 1$. By Lemma 3.1 and the Young rule (4) we derive that the only nontrivial multiplicities $m_\lambda(\mathfrak{B})$ are for $\lambda = (\lambda_1, \lambda_2)$. Then $m_\lambda(\mathfrak{B})$ is equal to the number of tensor products $W(p) \otimes W(q)$ which contain an isomorphic copy of $W(\lambda)$ as a submodule. For $\lambda = (n)$ there are $n - 1$ possibilities

$$W(1) \otimes W(n-1), W(2) \otimes W(n-2), \dots, W(n-1) \otimes W(1),$$

i.e., $m_{(n)}(\mathfrak{B}) = n - 1$. For $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 > 0$ the possibilities are

$$W(\lambda_2) \otimes W(n - \lambda_2), W(\lambda_2 + 1) \otimes W(n - \lambda_2 - 1), \dots, W(n - \lambda_2) \otimes W(\lambda_2),$$

which gives $m_{(\lambda_1, \lambda_2)}(\mathfrak{B}) = n - 2\lambda_2 + 1$. \square

Lemma 3.3. *The following polynomials $w_\lambda^{(k)}$ form a maximal linearly independent system of highest weight vectors of the GL_d -submodules $W(\lambda)$ in G_d^2 :*

$$(7) \quad \begin{aligned} w_{(n)}^{(j)} &= y_1^j z_1^{n-j}, \quad j = 1, 2, \dots, n-1, \\ w_\lambda^{(j)} &= y_1^j (y_1 z_2 - y_2 z_1)^{\lambda_2} z_1^{\lambda_1 - \lambda_2 - j}, \quad j = 0, 1, \dots, \lambda_1 - \lambda_2, \text{ if } \lambda_2 > 0. \end{aligned}$$

Proof. For a fixed λ the elements (7) are linearly independent because are nonzero and of pairwise different degree in y_1 . They are of degree λ_1 with respect to y_1, z_1 and of degree λ_2 with respect to y_2, z_2 . By Proposition 3.2 the multiplicities of $W(n)$ and $W(\lambda)$, $\lambda = (\lambda_1, \lambda_2) \vdash n$, in G_d^2 are, respectively,

$$m_{(n)}(\mathfrak{B}) = n - 1 \text{ and } m_{(\lambda_1, \lambda_2)}(\mathfrak{B}) = n - 2\lambda_2 + 1 = \lambda_1 - \lambda_2 + 1.$$

Hence their number coincides with the number of polynomials in (7). Now, it is sufficient to show that all $w_\lambda^{(j)}$ are highest weight vectors. Applying Lemma 2.1, this is obvious for $w_{(n)}^{(j)}$. Let $\lambda_2 > 0$. The analogue $\Delta_{y_2 \rightarrow y_1, z_2 \rightarrow z_1}$ of the derivation $\Delta_{x_2 \rightarrow x_1}$ acting on $K[Y_d, Z_d]$ sends y_1, z_1 to 0 and y_2, z_2 to y_1, z_1 , respectively. Obviously

$$\Delta_{y_2 \rightarrow y_1, z_2 \rightarrow z_1}(w_\lambda^{(j)}) = \lambda_2 y_1^j (y_1 z_2 - y_2 z_1)^{\lambda_2 - 1} z_1^{\lambda_1 - \lambda_2 - j} \Delta_{y_2 \rightarrow y_1, z_2 \rightarrow z_1}(y_1 z_2 - y_2 z_1) = 0$$

and all $w_\lambda^{(j)}$ are highest weight vectors. \square

4. Subvarieties. In this section we assume that \mathfrak{V} is a proper subvariety of \mathfrak{B} and \mathfrak{V} satisfies a nontrivial polynomial identity $f = 0$ of degree k , where $0 \neq f(X_d) \in F = F(\mathfrak{B})$. Since the case $k = 1$ is trivial we shall assume that $k \geq 2$. In the sequel we shall work mainly in the isomorphic copies G and G_d of the algebras F and F_d instead of in F and F_d . Identifying F and F_d with their isomorphic copies, we shall denote the corresponding elements with the same symbols. In particular, if $f(X_d) \in F_d^2$ we shall write $f(Y_d, Z_d) \in G_d^2$ and vice versa. Since the GL_d -module generated by f contains an irreducible submodule $W(\lambda)$, there exists a highest weight vector w_λ such that the polynomial identity $w_\lambda = 0$ follows from the polynomial identity $f = 0$. Hence we may assume that \mathfrak{V} satisfies some polynomial identity $w_\lambda(x_1, x_2) = 0$ for $\lambda \vdash k$. Then $w_\lambda(Y_2, Z_2) \in G_2$ is a linear combination of the highest weight vectors in (7) and for some $\xi_j \in K$

$$(8) \quad \begin{aligned} w_{(k)} &= \sum_{j=1}^{k-1} \xi_j y_1^j z_1^{k-j}, \text{ for } \lambda = (k), \\ w_{(k)} &= (y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j=0}^{\lambda_1 - \lambda_2} \xi_j y_1^j z_1^{\lambda_1 - \lambda_2 - j}, \text{ for } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0. \end{aligned}$$

If $f(X_d) \in F_d$ is multihomogeneous then its partial linearization $\text{lin}_{x_i} f(X_d)$ in x_i is the component of degree $\deg_{x_i} - 1$ with respect to x_i of the polynomial $f(x_1, \dots, x_i + x_{d+1}, \dots, x_d) \in F_{d+1}$. If $\Delta_{x_i \rightarrow x_{d+1}}$ is the derivation of F_{d+1} which sends x_i to x_{d+1} and the other x_j to 0, then

$$\text{lin}_{x_i} f(X_d) = (\text{lin}_{x_i} f)(X_{d+1}) = \Delta_{x_i \rightarrow x_{d+1}}(f(X_d)).$$

If $u \in F$ then $(\text{lin}_{x_i} f)(x_1, \dots, x_d, u)$ can be expressed in terms of derivations as

$$(\text{lin}_{x_i} f)(x_1, \dots, x_d, u) = \Delta_{x_i \rightarrow u}(f(X_d)),$$

where $\Delta_{x_i \rightarrow u}$ is the derivation of F sending x_i to u and all other generators x_j to 0. The action of the analogue of $\Delta_{x_i \rightarrow x_{d+1}}$ on G^2 is clear: It sends y_i and z_i , respectively, to y_{d+1} and z_{d+1} and all other variables y_j and z_j to 0. We denote this derivation by $\Delta_{y_i \rightarrow y_{d+1}, z_i \rightarrow z_{d+1}}$. Now we shall translate the action of $\delta_{x_i \rightarrow u}$ on F^2 , $u \in F^2$, in the language of G and the usual partial derivatives.

Lemma 4.1. *Let $u \in K[Y, Z]$ be in the image $G^2 \subset K[Y, Z]$ of F^2 . Let $\Delta_{y_i, z_i \rightarrow u}$ be the derivation of $K[Y, Z]$ which sends the variables y_i, z_i to u and the other variables to 0. If $f(X_d) \in F^2$ is multihomogeneous, then the image of $(\text{lin}_{x_i} f)(x_1, \dots, x_d, u)$ in G^2 is*

$$\Delta_{y_i, z_i \rightarrow u}(f) = \left(\frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial z_i} \right) u.$$

Proof. It is sufficient to consider the case when f and u are monomials and $i = 1$:

$$\begin{aligned} f &= (y_1^{\alpha_1} z_1^{\beta_1}) v_1 v_2, v_1 = y_2^{\alpha_2} \cdots y_d^{\alpha_d}, v_2 = z_2^{\beta_2} \cdots z_d^{\beta_d}, \\ \alpha_1 + \beta_1 &\geq 1, |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d > 0, |\beta| = \beta_1 + \beta_2 + \cdots + \beta_d > 0, \\ u &= Y_d^\gamma Z_d^\delta, |\gamma| > 0, |\delta| > 0. \end{aligned}$$

Then

$$\begin{aligned} \Delta_{y_1 \rightarrow y_{d+1}, z_1 \rightarrow z_{d+1}}(f) &= (\alpha_1 y_1^{\alpha_1 - 1} y_{d+1} z_1^{\beta_1} + \beta_1 y_1^{\alpha_1} z_1^{\beta_1 - 1} z_{d+1}) v_1 v_2 \\ &= \frac{\partial f}{\partial y_1} y_{d+1} + \frac{\partial f}{\partial z_1} z_{d+1}, \end{aligned}$$

In virtue of (6) we may assume that the preimage $\psi^{-1} \left(\frac{\partial f}{\partial y_1} y_{d+1} \right)$ in F_{d+1}^2 is of the form $\alpha_1(\cdots (x_{d+1} x_{j_1}) \cdots)$, where the dots before and after $(x_{d+1} x_{j_1})$ correspond

to the beginning and the end of the element in the form (5). Since $u \in F^2$ we obtain that

$$\psi(\alpha_1(\cdots (ux_{j_1}) \cdots)) = \alpha_1(\cdots (Y_d^\gamma Z_d^\delta z_{j_1}) \cdots) = \frac{\partial f}{\partial y_1} u.$$

Similarly

$$\psi(\beta_1(\cdots (x_{i_p} u) \cdots)) = \beta_1(\cdots (y_{i_p} Y_d^\gamma Z_d^\delta) \cdots) = \frac{\partial f}{\partial z_1} u. \quad \square$$

Lemma 4.2. *If $0 \neq f \in W(\lambda_1, \lambda_2) \subset F(\mathfrak{B})$, then all polynomial identities $w_{(\mu_1, \mu_2)}^{(j)} = 0$ with $\mu_2 \geq \lambda_1$ are consequences of the polynomial identity $f = 0$.*

Proof. As commented in the beginning of the section, we may assume that $f = w_\lambda$ is a highest weight vector in $W(\lambda_1, \lambda_2) \subset F_2$. Hence, working in G_2 instead of in F_2 , w_λ has the form (8), i.e.,

$$w_\lambda = \sum_{j \geq p} \xi_j w_\lambda^{(j)} = (y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j \geq p} \xi_j y_1^j z_1^{\lambda_1 - \lambda_2 - j}, \xi_p \neq 0.$$

First, let $p > 0$, i.e., w_λ is divisible by y_1^p . The partial linearizations of the identity $w_\lambda = 0$ are its consequences. Hence $\Delta_{y_1 \rightarrow y_2, z_1 \rightarrow z_2}(w_\lambda) = 0$ which has the form

$$(y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j \geq p} \xi_j (j y_1^{j-1} y_2 z_1^{\lambda_1 - \lambda_2 - j} + (\lambda_1 - \lambda_2 - j) y_1^j z_1^{\lambda_1 - \lambda_2 - j - 1} z_2) = 0$$

is also a consequence of $w_\lambda = 0$ and the same holds for

$$\begin{aligned} w_{(\lambda_1, \lambda_2 + 1)} &= \Delta_{y_1 \rightarrow y_2, z_1 \rightarrow z_2}(w_\lambda) z_1 - (\lambda_1 - \lambda_2) w_\lambda z_2 \\ &= -(y_1 z_2 - y_2 z_1)^{\lambda_2 + 1} \sum_{j \geq p-1} (j+1) \xi_{j+1} y_1^j z_1^{\lambda_1 - \lambda_2 - j - 1} = 0. \end{aligned}$$

We obtained that $w_{(\lambda_1, \lambda_2 + 1)} = 0$ is a consequence of $w_{(\lambda_1, \lambda_2)} = 0$. It is divisible by y_1^{p-1} but is not divisible by y_1^p . Continuing in this way we shall reach a consequence

$$w_{(\lambda_1, \lambda_2 + p)} = (y_1 z_2 - y_2 z_1)^{\lambda_2 + p} j! \xi_p z_1^{\lambda_1 - \lambda_2 - p} = 0.$$

Now the consequence

$$y_1 \Delta_{y_1 \rightarrow y_2, z_1 \rightarrow z_2}(w_{(\lambda_1, \lambda_2 + p)}) - y_2 (\lambda_1 - \lambda_2 + p) w_{(\lambda_1, \lambda_2 + p)} = 0$$

is of the form $w_{(\lambda_1, \lambda_2+p+1)} = 0$ and is divisible by $z_1^{\lambda_1 - \lambda_2 - p - 1}$ only. Continuing the process we shall obtain as a consequence

$$w_{(\lambda_1, \lambda_1)} = (y_1 z_2 - y_2 z_1)^{\lambda_1} = w_{(\lambda_1, \lambda_1)}^{(0)}.$$

Since all $w_{(\mu_1, \mu_2)}^{(j)}$ with $\mu_2 \geq \lambda_1$ are divisible by $w_{(\lambda_1, \lambda_1)}^{(0)}$ and hence are its consequences, we complete the proof. \square

Corollary 4.3. *If $0 \neq f \in F$ is of degree k then all identities $w_{(\mu_1, \mu_2)}^{(j)} = 0$ with $\mu_2 \geq k$ follow from the identity $f = 0$.*

Proof. The statement follows immediately from Lemma 4.2 because if $(\lambda_1, \lambda_2) \vdash k$, then $\lambda_1 \leq k$. \square

Lemma 4.4. *The polynomial identity $w_{(k, k)}^{(0)} = (y_1 z_2 - y_2 z_1)^k = 0$ has as consequences all identities*

$$(y_1 z_1)^k (y_1 - z_1)^k w_{\mu}^{(j)} = 0$$

for all $\mu = (\mu_1, \mu_2)$ and all $j = 0, 1, \dots, \mu_1 - \mu_2$.

Proof. We apply the derivation $\Delta_{y_2, z_2 \rightarrow y_1 z_1}$ and obtain as a consequence of the identity $w_{(k, k)}^{(0)} = 0$ the identity

$$\begin{aligned} \Delta_{y_2, z_2 \rightarrow y_1 z_1}(w_{(k, k)}^{(0)}) &= y_1 z_1 \left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) (y_1 z_2 - y_2 z_1)^k \\ &= k y_1 z_1 (y_1 z_2 - y_2 z_1)^{k-1} \left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) (y_1 z_2 - y_2 z_1) \\ &= k y_1 z_1 (y_1 - z_1) (y_1 z_2 - y_2 z_1)^{k-1} = 0. \end{aligned}$$

Continuing in this way we obtain

$$\Delta_{y_2, z_2 \rightarrow y_1 z_1}^k(w_{(k, k)}^{(0)}) = k! (y_1 z_1)^k (y_1 - z_1)^k = 0$$

which gives that $(y_1 z_1)^k (y_1 - z_1)^k w_{\mu}^{(j)} = 0$ for all μ and all j . \square

Corollary 4.5. *The variety \mathfrak{B} is generated by its one-generated free algebra $F_1(\mathfrak{B})$.*

Proof. If $\text{var}(F_1(\mathfrak{B})) \neq \mathfrak{B}$, then by Lemma 4.2 the algebra $F_1(\mathfrak{B})$ satisfies some identity $w_{(k,k)}^{(0)}$ and by Lemma 4.4 satisfies the identity $(y_1 z_1)^k (y_1 - z_1)^k = 0$ in one variable. This means that $(y_1 z_1)^k (y_1 - z_1)^k = 0$ in $F_1(\mathfrak{B})$ which is impossible. \square

The following theorem is the first main result of our paper.

Theorem 4.6. *If \mathfrak{V} is a proper subvariety of the variety \mathfrak{B} of all bicommutative algebras such that \mathfrak{V} satisfies a polynomial identity $f = 0$ of degree k , $0 \neq f \in F(\mathfrak{B})$, then $c_n(\mathfrak{V})$ is bounded by a polynomial of degree $k - 1$.*

Proof. Let

$$(9) \quad \chi_n(\mathfrak{V}) = \sum_{\lambda \vdash n} m_\lambda(\mathfrak{V}) \chi_\lambda, \quad n = 1, 2, \dots,$$

be the cocharacter sequence of \mathfrak{V} . By Proposition 3.2 the summation in (9) runs on $\lambda = (\lambda_1, \lambda_2) \vdash n$. By Corollary 4.3 we obtain that $m_{(\lambda_1, \lambda_2)}(\mathfrak{V}) = 0$ for $\lambda_2 \geq k$. If $\lambda_1 - \lambda_2 \leq 3k - 1$, then

$$m_{(\lambda_1, \lambda_2)}(\mathfrak{V}) \leq m_{(\lambda_1, \lambda_2)}(\mathfrak{B}) \leq \lambda_1 - \lambda_2 + 1 \leq 3k.$$

Now, let $\lambda_1 - \lambda_2 \geq 3k$. By Lemma 4.4, the variety \mathfrak{V} satisfies the identities

$$w_j = (y_1 z_1)^k (y_1 - z_1)^k (y_1 z_2 - y_2 z_1)^{\lambda_2} y_1^j z_1^{\lambda_1 - \lambda_2 - 3k - j} = 0, \quad j = 0, 1, \dots, \lambda_1 - \lambda_2 - 3k.$$

All w_j , $j = 0, 1, \dots, \lambda_1 - \lambda_2 - 3k$, are linearly independent in $F_2(\mathfrak{B})$ and are highest weight vectors for GL_2 -submodules of $F_2(\mathfrak{B})$. Hence the multiplicity $m_\lambda(\mathfrak{V})$ satisfies the inequality

$$m_\lambda(\mathfrak{V}) \leq m_\lambda(\mathfrak{B}) - (\lambda_1 - \lambda_2 - 3k + 1) = (\lambda_1 - \lambda_2 + 1) - (\lambda_1 - \lambda_2 - 3k + 1) = 3k.$$

Hence (9) satisfies the inequality

$$\chi_n(\mathfrak{V}) = \sum_{\substack{(\lambda_1, \lambda_2) \vdash n \\ \lambda_2 < k}} m_\lambda(\mathfrak{V}) \chi_{(\lambda_1, \lambda_2)} \leq \sum_{j=0}^{k-1} 3k \chi_{(n-j, j)}.$$

We obtain that the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \dots$, satisfies

$$c_n(\mathfrak{V}) \leq \sum_{j=0}^{k-1} 3kd_{(n-j, j)}$$

which by Lemma 2.2 is a polynomial of degree $k - 1$. \square

Remark 4.7. We may precise Corollary 4.3: If $\mathfrak{V} \subset \mathfrak{B}$ satisfies an identity $w_\lambda = 0$ of degree k and $\lambda_2 > 0$ in $\lambda = (\lambda_1, \lambda_2) \vdash k$, then $\lambda_1 \leq k - 1$ and \mathfrak{V} satisfies all identities $w_\mu^{(j)} = 0$ for $\mu = (\mu_1, \mu_2)$ and $\mu_2 \geq k - 1$. Hence in this case $c_n(\mathfrak{V})$ is bounded by a polynomial of degree $k - 2$.

Example 4.8. The bound by a polynomial of degree $k - 2$ in Remark 4.7 is sharp. Let \mathfrak{V} be the subvariety of \mathfrak{B} defined by the polynomial identity of right nilpotency $(\cdots((x_1 x_2) x_3) \cdots) x_k = 0$. It is easy to see that the image in G of the T-ideal $T(\mathfrak{V})$ of the identities of \mathfrak{V} is generated as an ordinary two-sided ideal by the products $y_{i_1} z_{i_2} \cdots z_{i_k}$. Hence if $\mu_2 \geq k - 1$, then all $w_{(\mu_1, \mu_2)}^{(j)}$ belong to this T-ideal and

$$w_{(n-k+2, k-2)}^{(n-2k+4)} = y_1^{n-2k+4} (y_1 z_2 - y_2 z_1)^{k-2}$$

does not belong to this ideal. Hence $c_n(\mathfrak{V}) \geq d_{(n-k+2, k-2)}$ which is a polynomial of degree $k - 2$. We do not know whether there exists a variety $\mathfrak{V} \subset \mathfrak{B}$ satisfying a polynomial identity in one variable of degree k such that $c_n(\mathfrak{V})$ grows as a polynomial of degree $k - 1$.

5. Two-dimensional algebras. The classification of all two-dimensional algebras can be traced back to the two-dimensional part of the classification project in the seminal book by B. Peirce [25] published lithographically in 1870 in a small number of copies for distribution among his friends and then reprinted posthumously in 1881 with addenda of his son C. S. Peirce. (See Grattan-Guinness [21] for the contributions of Peirce.) Starting in 2000 with the paper by Petersson [26] (which contains also the history of the classification) and the paper by Anan'in and Mironov [3] there are several papers containing different kinds of classification of two-dimensional algebras – by Goze and Remm [20], Ahmed, Bekbaev, and Rakhimov [1], Rausch de Traubenberg and Slupinski [27], Kaygorodov and Volkov [22]. Concerning the polynomial identities of two-dimensional algebras, Giambruno, Mishchenko, and Zaicev [15] proved that the growth of the codimension sequence $c_n(A)$ of such an algebra A over a field of characteristic 0 is either linear (and bounded by $n + 1$) or grows exponentially as 2^n .

In this section we shall study the polynomial identities of two-dimensional bicommutative algebras over an arbitrary field of characteristic 0. It is well known that if A is an algebra over an infinite field K then the K -algebra A and the E -algebra $E \otimes_K A$ have the same bases of polynomial identities for any extension E of K . More precisely, if $\{f_i\} \subset K\{X\}$ is a basis of the polynomial identities of the K -algebra A , then $\{1 \otimes f_i\} \subset E \otimes_K K\{X\} \cong E\{X\}$ is a basis of the polynomial

identities of the E -algebra $E \otimes_K A$. Hence in the sequel we may assume that the field K is algebraically closed.

The classification of all two-dimensional bicommutative algebras over an arbitrary algebraically closed field of any characteristic is given by Kaygorodov and Volkov in [22]:

Theorem 5.1. *When the base field K is algebraically closed any two-dimensional bicommutative algebra with nontrivial multiplication is isomorphic to one of the seven algebras*

$$(10) \quad \{\mathbf{A}_3, \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0), \mathbf{D}_2(1,1), \mathbf{D}_2(0,0), \mathbf{E}_1(0,0,0,0)\},$$

where the algebras have bases $\{e_1, e_2\}$ and multiplication tables given below:

$$\begin{array}{ll} \mathbf{A}_3 : & e_1e_1 = e_2, \quad e_1e_2 = 0, \quad e_2e_1 = 0, \quad e_2e_2 = 0; \\ \mathbf{B}_2(0) : & e_1e_1 = 0, \quad e_1e_2 = e_1, \quad e_2e_1 = 0, \quad e_2e_2 = 0; \\ \mathbf{B}_2(1) : & e_1e_1 = 0, \quad e_1e_2 = 0, \quad e_2e_1 = e_1, \quad e_2e_2 = 0; \\ \mathbf{D}_1(0,0) : & e_1e_1 = e_1, \quad e_1e_2 = e_1, \quad e_2e_1 = 0, \quad e_2e_2 = 0; \\ \mathbf{D}_2(1,1) : & e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_2e_1 = e_2, \quad e_2e_2 = 0; \\ \mathbf{D}_2(0,0) : & e_1e_1 = e_1, \quad e_1e_2 = 0, \quad e_2e_1 = 0, \quad e_2e_2 = 0; \\ \mathbf{E}_1(0,0,0,0) : & e_1e_1 = e_1, \quad e_1e_2 = 0, \quad e_2e_1 = 0, \quad e_2e_2 = e_2. \end{array}$$

Remark 5.2. In the classification of two-dimensional bicommutative algebras given in the preliminary version of [22, Section 7.2] there is one more algebra $\mathbf{D}_1(1,0)$. This algebra is isomorphic to the algebra $\mathbf{D}_1(0,0)$ because in [22, Table 1] the pairs $(0,0)$ and $(1,0)$ belong to the same orbit of the cyclic group of order 2 generated by ϱ , where

$$\varrho(\alpha, \beta) = (1 - \alpha + \beta, \beta), \quad (\alpha, \beta) \in K^2.$$

The algebra \mathbf{A}_3 is commutative and nilpotent of class 3. Hence the T-ideal of its polynomial identities is generated by

$$x_1x_2 = x_2x_1, \quad (x_1x_2)x_3 = 0$$

and the cocharacter sequence of \mathbf{A}_3 is

$$\chi_1(\mathbf{A}_3) = \chi_{(1)}, \quad \chi_2(\mathbf{A}_3) = \chi_{(2)}, \quad \chi_n(\mathbf{A}_3) = 0, \quad n = 3, 4, \dots$$

Similarly, the algebras $\mathbf{D}_2(1,1)$, $\mathbf{D}_2(0,0)$, and $\mathbf{E}_1(0,0,0,0)$ are associative-commutative, have the same bases of polynomial identities consisting of

$$x_1x_2 = x_2x_1, \quad (x_1x_2)x_3 = x_1(x_2x_3)$$

and their cocharacter sequence is

$$c_n(\mathbf{D}_2(1, 1)) = c_n(\mathbf{D}_2(0, 0)) = c_n(\mathbf{E}_1(0, 0, 0, 0)) = \chi_{(n)}, \quad n = 1, 2, \dots$$

Hence to complete the description of the polynomial identities of two-dimensional bicommutative algebras it is sufficient to handle the cases $A = \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0, 0)$. These three algebras satisfy the condition $\dim A^2 = 1$. For our purposes it is more convenient to have another presentation of the algebras. The next proposition shows that over any field of characteristic 0 the two-dimensional bicommutative algebras A with $\dim A^2 = 1$ are in the list of Theorem 5.1.

Proposition 5.3. *Over an arbitrary field K of characteristic 0 there are only five nonisomorphic two-dimensional bicommutative algebras A such that $\dim A^2 = 1$. They are one-generated and isomorphic to the algebras*

$$(11) \quad A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1},$$

where the algebra $A_{\pi,\varrho}$ is with multiplication given in (1). The five algebras in (11) are isomorphic, respectively, to the algebras

$$\mathbf{A}_3, \mathbf{D}_2(0, 0), \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0, 0)$$

from the list in (10).

Proof. Let A have a basis $\{a, b\}$, where $a \in A \setminus A^2$, $b \in A^2$. If $a^2 \neq 0$, then $a^2 = \alpha b$, $0 \neq \alpha \in K$. Hence A is generated by a and has a basis $\{a, a^2\}$.

Now, let $a^2 = 0$. If $ab = \beta b \neq 0$, $0 \neq \beta \in K$, then the identity of right-commutativity gives

$$\beta ba = (ab)a = (aa)b = 0,$$

and hence $ba = 0$. Let $b^2 = \gamma b$, $\gamma \in K$. Then for $\eta \in K$

$$(a + \eta b)^2 = \eta(\beta + \eta\gamma)b$$

and we always can choose η in a way to have $(a + \eta b)^2 \neq 0$. Again A is one generated and has a basis $\{a + \eta b, (a + \eta b)^2\}$. Similarly, if $ba \neq 0$, then $ab = 0$ and again A is one-generated.

Hence we may assume that A has a basis $\{r, r^2\}$. Let

$$rr^2 = \pi r^2, r^2r = \varrho r^2, \quad \pi, \varrho \in K.$$

Then the right-commutativity implies

$$r^2r^2 = (rr)r^2 = (rr^2)r = \pi r^2r = \pi \varrho r^2.$$

Hence the multiplication of the algebra A is as of the algebra $A_{\phi, \varrho}$ in (1). Let $\pi = 0$, $\varrho \neq 0$. If we replace the generator r by $r = \varrho r_1$, then

$$r^2r = \varrho r^2, \quad \varrho^3 r_1^2 r_1 = \varrho \varrho^2 r_1^2, \quad r_1^2 r_1 = r_1^2,$$

i.e., $A_{0, \varrho} \cong A_{0,1}$. Similarly, $A_{\pi, 0} \cong A_{1,0}$. If $\pi = \varrho \neq 0$, then the change of the generator r with $r = \pi r_1$ gives that

$$r^2r = \pi r^2, \quad \pi^3 r_1^2 r_1 = \pi \pi^2 r_1^2, \quad r_1^2 r_1 = r_1^2, \quad r_1 r_1^2 = r_1^2,$$

and $A_{\pi, \pi} \cong A_{1,1}$. Finally, let $\pi \neq \varrho$ be different from 0. We fix solutions ξ and η of the linear system

$$\pi(\xi + \varrho\eta) = 1, \quad \varrho(\xi + \pi\eta) = -1.$$

Then $r_1 = \xi r + \eta r^2$ satisfies the conditions

$$\begin{aligned} r_1^2 &= (\xi + \pi\eta)(\xi + \varrho\eta)r^2 = -\frac{1}{\pi\varrho}r^2, \\ r_1 r_1^2 &= -\frac{1}{\pi\varrho}(\xi r + \eta r^2)r^2 = -\frac{1}{\pi\varrho}\pi(\xi + \eta\varrho)r^2 = \pi(\xi + \eta\varrho)r_1^2 = r_1^2, \\ r_1^2 r_1 &= \varrho(\xi + \pi\eta)r_1^2 = -r_1^2, \end{aligned}$$

i.e., $A_{\pi, \varrho} \cong A_{1,-1}$.

The isomorphisms between the algebras $A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1}$ and, respectively, the algebras $\mathbf{A}_3, \mathbf{D}_2(0,0), \mathbf{B}_2(0), \mathbf{B}_2(1), \mathbf{D}_1(0,0)$ are given as follows:

$$\begin{aligned} A_{0,0} \cong \mathbf{A}_3 : & \quad r \rightarrow e_1, & \quad r^2 \rightarrow e_2; \\ A_{1,1} \cong \mathbf{D}_2(0,0) : & \quad r \rightarrow e_1 + e_2, & \quad r^2 \rightarrow e_1; \\ A_{0,1} \cong \mathbf{B}_2(0) : & \quad r \rightarrow e_1 + e_2, & \quad r^2 \rightarrow e_1; \\ A_{1,0} \cong \mathbf{B}_2(1) : & \quad r \rightarrow e_1 + e_2, & \quad r^2 \rightarrow e_1; \\ A_{1,-1} \cong \mathbf{D}_1(0,0) : & \quad r \rightarrow e_1 - 2e_2, & \quad r^2 \rightarrow -e_1. \end{aligned} \quad \square$$

The next theorem gives bases for the polynomial identities and the cocharacter sequences of the three nonassociative algebras $A_{0,1}, A_{1,0}, A_{1,-1}$.

Theorem 5.4. (i) *As subvarieties of the variety \mathfrak{B} of all bicommutative algebras the varieties $\text{var}(A_{0,1})$ and $\text{var}(A_{1,0})$ generated by the algebras $A_{0,1}$ and $A_{1,0}$ are defined by the identities of left-nilpotency $x_1(x_2x_3) = 0$ and right-nilpotency $(x_1x_2)x_3 = 0$, respectively. Their cocharacter and codimension sequences coincide and are*

$$\chi_1(A_{0,1}) = \chi_1(A_{1,0}) = \chi_{(1)}, \chi_n(A_{0,1}) = \chi_{(n)} + \chi_{(n-1,1)}, \quad n = 2, 3, \dots,$$

$$c_n(A_{0,1}) = c_n(A_{1,0}) = n, \quad n = 1, 2, \dots$$

(ii) *The algebra $A_{1,-1}$ generates the whole variety \mathfrak{B} .*

Proof. (i) Clearly the algebra $A_{0,1}$ satisfies the polynomial identity $x_1(x_2x_3) = 0$. The origins in $F = F(\mathfrak{B})$ of the polynomials $w_\lambda^{(j)}$ from (7) have the form

$$w_{(n)}^{(j)}(x_1) = \underbrace{x_1(\cdots(x_1((x_1x_1)\cdots)x_1))}_{j \text{ times}} \cdots,$$

$$w_{(\lambda_1, \lambda_2)}^{(j)}(x_1, x_2) = \underbrace{x_1(\cdots x_1)}_{j \text{ times}} ((\cdots ((x_1x_2 - x_2x_1)^{\lambda_2} \underbrace{x_1 \cdots x_1}_{\lambda_1 - \lambda_2 - j \text{ times}}) \cdots)).$$

Obviously $w_{(\lambda_1, \lambda_2)}^{(j)}$ follows from $x_1(x_2x_3) = 0$ for $\lambda = (n)$, $j = 2, \dots, n-1$, $n \geq 3$, for $\lambda = (n-1, 1)$, $j = 1, \dots, n-2$, and for $\lambda = (\lambda_1, \lambda_2)$, $\lambda_2 \geq 2$. On the other hand $w_{(n)}^{(1)}(r) = r^2 \neq 0$, $w_{(n-1, 1)}^{(0)}(r, r^2) = -r^2 \neq 0$. This shows that the identities of $A_{0,1}$ follow from $x_1(x_2x_3) = 0$, $\chi_1(A_{0,1}) = \chi_{(1)}$, $\chi_n(A_{0,1}) = \chi_{(n)} + \chi_{(n-1, 1)}$, $n = 2, 3, \dots$, and $c_n(A_{0,1}) = n$, $n = 1, 2, \dots$. The proof for $A_{1,0}$ is similar.

(ii) By Corollary 4.5 it is sufficient to show that the algebra $A_{1,-1}$ does not satisfy an identity in one variable. Let

$$w_{(n)}(y_1, z_1) = \sum_{j=1}^{n-1} \xi_j w_{(n)}^{(j)}(y_1, z_1), \quad \xi_j \in K,$$

be a polynomial in G which corresponds to a homogeneous polynomial identity $f(x_1) = 0$ in one variable and of degree $n \geq 2$, $0 \neq f(x_1) \in F(\mathfrak{B})$. We shall evaluate $f(x_1)$ on all $\gamma r + \delta r^2 \in A_{1,-1}$, $\gamma, \delta \in K$. Since

$$(\gamma r + \delta r^2)^2 = (\gamma^2 - \delta^2)r^2,$$

$$(\gamma r + \delta r^2) \cdot (\gamma r + \delta r^2)^2 = (\gamma - \delta)(\gamma^2 - \delta^2)r^2,$$

$$(\gamma r + \delta r^2)^2 \cdot (\gamma r + \delta r^2) = -(\gamma + \delta)(\gamma^2 - \delta^2)r^2,$$

we obtain that the evaluation of the proimage of $w_{(n)}^{(j)}(y_1, z_1)$ on $\gamma r + \delta r^2$ is equal to

$$(-1)^{n-j-1}(\gamma^2 - \delta^2)(\gamma - \delta)^{j-1}(\gamma + \delta)^{n-j-1}r^2 = (-1)^{n-1}(\delta - \gamma)^j(\delta + \gamma)^{n-j}.$$

Hence

$$f(\gamma r + \delta r^2) = (-1)^{n-1}w_{(n)}(\delta - \gamma, \delta + \gamma)r^2 = 0.$$

When γ and δ run on the whole field K the same holds for $\delta - \gamma$ and $\delta + \gamma$. Therefore the polynomial $w_{(n)}(y_1, z_1)$ vanishes evaluated on the infinite field K and hence is identically equal to 0. This means that $A_{1,-1}$ does not satisfy any polynomial identity in one variable and hence generates the whole variety \mathfrak{B} . \square

The following easy lemma gives an upper bound for the codimensions of a finite dimensional algebra. It makes more precise the bound for the codimensions established for graded algebras in [4] and independently in [19].

Lemma 5.5. *Let A be a finite dimensional algebra and let k be a positive integer. Then for all $n \geq k$*

$$c_n(A) \leq \dim(A^k) \dim^n(A).$$

Proof. Let $\dim(A) = p$ and $\dim(A^k) = q$. We fix a basis $\{r_1, \dots, r_q\}$ of A^k and extend it to a basis $\{r_1, \dots, r_p\}$ of A . We consider the multilinear identity

$$f(x_1, \dots, x_n) = \sum_{(\sigma)} \xi_{(\sigma)}(x_{\sigma(1)} \cdots) (\cdots x_{\sigma(n)}) = 0, \quad \xi_{(\sigma)} \in K,$$

where the summation runs on all permutations $\sigma \in S_n$ and all possible bracket decompositions. Clearly, $f(x_1, \dots, x_n) = 0$ is a polynomial identity for A if and only if $f(r_{i_1}, \dots, r_{i_n}) = 0$ for all possible choices of the basis elements r_{i_1}, \dots, r_{i_n} . Since $\deg(f) = n$ and $n \geq k$ the evaluations of $f(x_1, \dots, x_n)$ on R belong to A^k . Let

$$f(r_{i_1}, \dots, r_{i_n}) = \sum_{j=1}^q f_j(r_{i_1}, \dots, r_{i_n}) r_j,$$

where $f_j(r_{i_1}, \dots, r_{i_n}) \in K$ are linear functions in the coefficients $\xi_{(\sigma)}$. Considering $\xi_{(\sigma)}$ as unknowns, we obtain the linear homogeneous system

$$f_j(r_{i_1}, \dots, r_{i_n}) = 0, \quad r_{i_1}, \dots, r_{i_n} \in \{r_1, \dots, r_p\}, j = 1, \dots, q.$$

The system has $n!C_n$ unknowns, where C_n is the n -th Catalan number (equal to the number of the bracket decompositions). Since the codimension $c_n(A)$ is equal to the rank of the system and the system has qp^n equations, its rank is less or equal to qp^n and the same holds for the n -th codimension $c_n(A)$. \square

Remark 5.6. It was shown in [15] that if the two-dimensional algebra A has a one-dimensional nilpotent ideal, then $c_n(A) \leq n + 1$. The algebras $A_{0,1}$ and $A_{1,0}$ satisfy this condition and Theorem 5.4 (i) shows that their codimensions are very close to the upper bound. For the algebra $A_{1,-1}$ the results in [15] give that

$$\frac{2^n}{n^2} \leq c_n(A_{1,-1}) \leq 2^{n+1}.$$

Since $\dim(A_{1,-1}) = 2$ and $\dim(A_{1,-1}^2) = 1$ Lemma 5.5 implies $c_n(A_{1,-1}) \leq 2^n$. By [13] and Theorem 5.4 (ii) we have that $c_n(A_{1,-1}) = c_n(\mathfrak{B}) = 2^n - 2$. Again, this is very close to the upper bound 2^n .

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