**Vesselin Petkov**

(on his 65th birthday)

Vesselin Petkov was born on 29 September 1942 in Burgas. In 1967 he graduated from the Faculty of Mathematics, Sofia University. In the autumn of 1969 Vesselin started his PhD thesis with Olga Oleinik and in 1972 he obtained his PhD degree in mathematics at the Faculty of Mathematics and Mechanics, Moscow University. He came back to Sofia in 1972, where he was appointed a Research Fellow at the Institute of Mathematics, Bulgarian Academy of Sciences. In 1977 Vesselin was promoted to Associate Professor in Mathematics at Sofia University. In 1986 he obtained the degree of a Doctor of Science. In 1988 he was appointed a Professor at the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences. Since 1991 he has been a Professor at the University of Bordeaux I, France.

One of the main achievements of Vesselin is that he created a strong group of young mathematicians in the field of Partial Differential Equations and Dynamical Systems. He was a PhD supervisor of Georgi Popov, Vladimir Georgiev, Plamen Stefanov, Georgi Vodev (in Bulgaria) and also of N. Filonov, A. Ishaan, Laurent Michel, F. Catalano, Frederic Naud (in France). All of his students have a successful scientific carrier. Vesselin is a grate motivator, he generously shares his ideas, knowledge, and mathematical experience with his students and collaborators. He enjoys doing, advertising and teaching mathematics.

It is a pleasure to do mathematics with Vesselin. He is always full of ideas, enthusiasm and energy. Not surprisingly he has a long list of collaborations, among them Victor Ivrii, Nikolai Kutev, Georgi Popov, Luchezar Stoyanov, Vladimir Georgiev, Tsviatko Rangelov, Georgi Vodev, Fernando Cardoso, Didier Robert, Pham de Lai, Pierre Vogel, Alain Bachelot, Maciej Zworski, Vincent Bruneau, Mouez Dimassi, Jean-Franois Bony.

We turn now to a brief presentation of the scientific work of Vesselin Petkov. We do not aim at a complete description of his research, rather we would like to present several of his works in different domains of mathematics and mathematical physics. Apart from this we try to avoid technical details. Instead we try to give the flavor of Vesselin's main research achievements.

**I. The Cauchy Problem for nonstrictly hyperbolic equations and propagation of the singularities.** In a series of papers Vesselin studies the Cauchy problem for nonstrictly hyperbolic equations and systems of equations and the propagation of the singularities of the solutions. He became interested in the well-posedness of the Cauchy problem for hyperbolic equations and systems of equations as a PhD student at Moscow University. His results in this domain together with those of V. Ivrii were a starting point for many contemporary investigations on the Cauchy problem for nonstricly hyperbolic equations and systems of equations.

The article V. Ivrii, V. Petkov (*Russian Math. Surveys*, 1974) is one of the pioneering works on the Cauchy problem for nonstricly hyperbolic equations and systems of equations. Nonstrictly hyperbolic means that the characteristic roots are all real but not simple. This is a unified treatment of the necessary conditions for the correctness of the noncharacteristic Cauchy problem in a weak or strong form. General necessary conditions on the low order terms, as the Levi type conditions in the case of constant multiplicity are proved for scalar equations. On the other hand, completely regular hyperbolic operators are described in terms of the fundamental matrix of the principal symbol.

The article V. Petkov (*Trans. Seminar Petrovski*, 1975, *Amer. Math. Soc. Trans.*, 1982) is concerned with necessary conditions for the well-posedness of the Cauchy problem. For the first time Levi conditions on lower order terms for nonsymmetrizable systems are introduced. They are invariant with respect to a change of variables and under canonical transformations. These conditions are essential for the investigation of the well-posedness of the Cauchy problem for hyperbolic systems with multiple characteristics.

In V. Petkov (*Trans. Moscow Math. Soc.*, 1978) the problem of existence and uniqueness of solutions of the Cauchy problem is investigated. An important role plays the Levi condition for nonsymmetrizable hyperbolic systems. A global paramertix is constructed in terms of Fourier Integral Operators.

**II. Spectral and scattering theory of partial differential operators.** The main contributions of Vesselin in this domain are related to the high frequency asymptotics and the semiclassical behavior of different objects of mathematical physics. He studies the high frequency asymptotics of the scattering amplitude for non-convex bodies, the asymptotic behavior of the scattering phase for non-trapping obstacles (with G. Popov), semiclassical spectral asymptotics for quantum Hamiltonians (with D. Robert), the scattering theory for moving obstacles, the singularities of the scattering kernel for moving obstacles, the semi-classical trace formula and the clustering of the eigenvalues for Schrödinger operators (with G. Popov). We would mention the following results in this direction.

The paper V. Petkov, G. Popov (*Ann. Inst. Fourier*, 1982) is concerned with a conjecture of Majda and Ralston concerning the asymptotic behavior of the scattering phase $s(\lambda)$ as $\lambda \to \infty$ for nontrapping obstacles with Dirichlet or Neumann boundary conditions. A complete asymptotic expansion of $s(\lambda)$ is obtained for such obstacles.

V. Petkov, D. Robert (*Comm. Partial Differential Equations*, 1985): Consider a selfadjoint Weyl $h$-pseudo-differential operator $A_h$ with principal symbol $a_0$. Assuming that the measure of the periodic trajectories on a compact regular level surface $\Sigma := \{a_0 = E\}$ is zero, then a two-term asymptotic Weyl formula is obtained for the number $N_h(\lambda)$ of the eigenvalues of $A_h$ close to $E$.

To obtain a two-term asymptotic Weyl formula for semiclassical operators one often requires that the measure of the periodic trajectories on a given compact regular level surface of his principal symbol is zero. This question is treated in V. Petkov, G. Popov, (*Math. Zeitschrift*, 1995). More precisely, let $(\Sigma, \sigma)$ be a contact manifold of dimension $2n-1$, $n \geq 2$, with contact 1-form $\sigma$ and a Reeb vector field $X$. Denote by $\exp(tX)\rho$, $\rho \in \Sigma$, $t \in \mathbb{R}$, the flow of $X$. The manifold is said to have property (Z) if the measure of the set of periodic points of this flow is zero. It has property (P) if the flow is totally periodic, i.e. there is a common nonzero period $T$ of $\exp(tX)\rho$ for all $\rho \in \Sigma$. The main result of the paper is that any analytic connected and complete contact manifold has either property (Z) or property (P).

The paper V. Petkov, G. Popov (*Ann. H. Poincaré*, 1998) is devoted to a systematic study of the semiclassical behavior of the number of eigenvalues $N_{E+rh,c}(h)$ of semiclassical operators $A_h$, $0 < h \ll 1$, in an interval $[E + rh - ch, E + rh + ch]$, where $E$ is a regular value of the principal $a_0$ and the energy shift parameter $r$ and the size constant $c > 0$ are both bounded. For small $h$ the behavior of this quantity depends on an oscillating term $Q(h, r)$ which is determined by the periodic trajectories of the Hamiltonian vector field of $a_0$ on the energy surface $a_0 = E$. This term may be uniformly continuous in $r$, and for such $r$ a semiclassical asymptotic formula of the counting function is found. The values of $r$ for which the oscillating term is discontinuous may provoke a sort of clustering of eigenvalues around $E$. Such jumps of $Q$ in $r$ are described in terms of suitable quantization conditions. A new semi-classical trace formula is obtained which can be considered as an analog of the Gutzwiller semiclassical trace formula without the usual condition of nondegeneracy of the periodic trajectories.

The paper V. Petkov, V. Georgiev (*Ann. Inst. H. Poincaré Phys. Théor.*, 1989) is concerned with an important application of the RAGE (Ruelle-Amrein-Georgescu-Enss) theorem for power bounded operators $V$ in a Hilbert space $H$ ($\sup_{m\in N}\|V^m\| < \infty$) to several basic problems in the scattering theory for periodically moving obstacles.

Consider the wave equation $\Box u = 0$ in a domain $Q$ with Dirichlet boundary condition, where $Q = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : x \in \mathbb{R}^n \setminus K(t)\}$, and the obstacle $K(t)$ moves periodically with period $T$. The monodromy operator $V = U(T,0)$ is power bounded if the global energy is uniformly bounded in $t$. Hence, the RAGE theorem with $V = U(T,0)$ can be applied to the corresponding scattering problem. Under some additional assumptions, the decay of local energy and the existence of wave and scattering operators are established. Moreover, the structure of the range of the wave operator is also studied in details. The spectral properties of the monodromy operator $U(T,0)$ are also studied without assuming that the global energy is bounded uniformly in $t$.

The monograph V. Petkov (Scattering theory for hyperbolic operators, North-Holland, 1989) is concerned with important problems of the scattering theory of the wave equation with moving obstacles and time dependent potentials such as the existence of wave and scattering operators, the behavior of local and global energy, the links between scattering kernels and wave profiles of outgoing (incoming) solutions, inverse scattering problems and symmetric first-order systems.

**III. Resonances, spectral shift function and trace formula.** Vesselin studies the following problems: Breit-Wigner approximation and the distribution of resonances (with M. Zworski), meromorphic continuation of the spectral shift function (with V. Bruneau), resonances for non semi-bounded and Stark Hamiltonians (with M. Dimassi), resonances for non-trapping time periodic perturbations (with J. F. Bony).

The Breit-Wigner approximation of the spectral shift function $s(\lambda)$ plays an important role in the study of the resonances in the quantum physics and chemistry.

In V. Petkov, M. Zworski (*Commun. Math. Physics*, 1999) a Breit-Wigner approximation is obtained for the scattering phase $s(\lambda)$ for compactly supported perturbations without any conditions on the geometry of the trapping set and on the multiplicities of the resonances. In particular, a clustering of resonances is obtained if the union of the periodic trajectories is of a positive Lebesgue measure. It is proved that for $0 < \varepsilon < 1$ and $0 < \delta \leq \varepsilon/2$,

$$s(\lambda+\delta) - s(\lambda-\delta) = \sum \int_{\lambda-\delta}^{\lambda+\delta} \frac{|\mathrm{Im}\,\omega|}{\pi|t-\omega|^2}\,dt + O_\varepsilon(\delta)\lambda^{n-1}, \quad \lambda \geq 1,$$

where the summation is taken aver all the resonances $\omega$ counted with multiplicities and such that $|\omega - \lambda| \leq \varepsilon$.

In V. Petkov, M. Zworski (*Annales Henri Poincaré*, 2001) the semiclassical behavior of the determinant $\det S(\lambda)$ of the scattering matrix is investigated in the case of "black box" scattering. A semiclassical trace formula and a Breit-Wigner approximation is obtained without any conditions on the multiplicities and the location of the resonances.

In V. Bruneau, V. Petkov (*Duke Math. J.*, 2003) a representation of the derivative of the spectral shift function $s(\lambda)$ is obtained for long range perturbations of $-h^2\Delta$, where a scattering determinant in not available. A semiclassical Weyl formula for the asymptotics of the scattering phase is proved in a very general setup. This representation gives also applications to the Breit-Wigner approximation.

## IV. Dynamical systems and related areas.

**IV.1. Generic properties of reflecting rays.** In a series of papers (*Bull. AMS*, 1986; *Amer. J. Math.*, 1987; *Math. Z.*, 1987; *Ergodic Th. Dyn. Sys.*, 1988) Petkov (jointly with Stoyanov) studies *generic properties* of billiard trajectories in $\mathbb{R}^n$ reflecting at a smooth $(n-1)$-dimensional submanifold $X$ of $\mathbb{R}^n$ $(n \geq 2)$. This study is naturally motivated by certain problems in spectral and scattering theory (see e.g. part A and Sect. IV.2 below).

Here *generic* means properties satisfied by almost all smooth $C^k$ perturbations of $X$ (in the space of all $C^k$ embeddings of $X$ into $\mathbb{R}^n$), $k \geq 1$.

In the papers mentioned above a certain technique is developed based on the Multijet Transversality Theorem from differential topology which is then used to prove that billiard trajectories (reflecting rays) in generic bounded domains $\Omega$ (with $X = \partial\Omega$) have the following properties:

**A. For generic bounded domains $\Omega$:**

*A.1.* Distinct periodic trajectories have rationally independent lengths.

*A.2.* Every periodic trajectory has no tangencies to the boundary $\partial\Omega$ and is non-degenerate (i.e. its linear Poincaré map does not have eigenvalues that are roots of 1).

*A.3.* The set of periodic billiard trajectories is countable.

The latter implies in particular that for generic domains $\Omega$, the set of periodic points of the billiard ball map has Lebesgue measure zero*. (For strictly convex domains in the plane this also follows from results of Lazutkin.) Combining this with a result of Ivrii 1980 yields that for generic bounded domains $\Omega$ in $\mathbb{R}^n$ the counting function $N(\lambda)$ for the spectrum of the Laplacian in $\Omega$ has a second term asymptotic as $\lambda \to \infty$.

*A.4.* For generic bounded domains $\Omega$ in $\mathbb{R}^2$ the so called *Poisson relation for manifolds with boundary* (Anderson-Melrose, Invent. Math. 1977) becomes an equality, i.e.

$$\text{sing supp } \sigma(t) = \{0\} \cup \{T_\gamma : \gamma \in \Gamma\},$$

where $\sigma(t)$ is the distribution on $\mathbb{R}$ defined by

$$\sigma(t) = \sum_{j=1}^{\infty} \cos(\lambda_j t), \quad t \in \mathbb{R},$$

$0 \leq \lambda_1^2 \leq \ldots \leq \lambda_j^2 \leq \ldots$ is the spectrum of the Laplacian in $\Omega$, $\Gamma$ is the set of periodic broken geodesics $\gamma$ in $\Omega$, and $T_\gamma$ is the length (period) of $\gamma$. Combining arguments from the above mentioned papers and Stoyanov 1990, one also obtains a similar result for generic bounded strictly convex domains $\Omega$ in $\mathbb{R}^n$ for any $n \geq 3$.

It should be mentioned that when $\Omega$ is convex, each $\gamma \in \Gamma$ is either a closed geodesic on $\partial\Omega$ or a periodic billiard trajectory in $\Omega$.

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*It has been conjectured since the work of Ivrii that the set of periodic points of the billiard ball map (at least in the case of strictly convex compact domains) always has Lebesgue measure zero. However, apart from the above mentioned generic result, almost nothing else has been achieved in this direction so far. In fact, the only known result concerning this problem seems to be that of Rychlik (J. Diff. Geometry 1989) showing that the periodic points of period 3 of the billiard ball map in any strictly convex domain in $\mathbb{R}^2$ with smooth boundary form a set of Lebesgue measure zero. Actually the original proof of Rychlik relied on certain calculations using a computer algebra package. This part of the proof was later replaced by a 'proper mathematical argument' by Stoyanov (J. Diff. Geometry 1991), while Wojtkowski (J. Diff. Geometry 1994) produced independently a different proof of the same fact.

**B. For generic exterior domains** $\Omega$. Now $\Omega$ is an unbounded domain with compact smooth boundary $X = \partial\Omega$. Given $\omega, \theta \in S^{n-1}$, an $(\omega, \theta)$*-ray* in $\Omega$ is a billiard trajectory in $\Omega$ incoming with direction $\omega$ and outgoing with direction $\theta$, after making finitely many reflections at $X = \partial\Omega$.

An $(\omega, \theta)$-ray $\gamma$ is called *ordinary* if it has no tangencies to the boundary. The *sojourn time* $T_\gamma$ is defined as the length of $\gamma$ between two particular cross-sections of the incoming and outgoing directions. Alternatively,

$$T_\gamma = \langle \omega, x_1 \rangle + \sum_{j=1}^{k-1} \|x_j - x_{j+1}\| - \langle \theta, x_k \rangle,$$

where $x_1, x_2, \dots, x_k$ are the successive reflection points of $\gamma$.

For any $x \in \mathbb{R}^n$ close to an incoming point $x_0$ of $\gamma$ define $J_\gamma(x) = \eta \in \mathbb{S}^{n-1}$, where $\varphi_t(x, \omega) = (y, \eta)$ for some large $t > 0$ and $\varphi_t$ denotes the billiard flow acting on $S(\Omega)$. If $\text{rank} dJ_\gamma(x_0) = n - 1$, $\gamma$ is called *non-degenerate*.

The generic properties established for exterior domains include the following:

*B.1.* Different $(\omega, \theta)$-rays have distinct sojourn times.

*B.2.* Every $(\omega, \theta)$-ray is ordinary and non-degenerate.

These were used later in the study of the so called Poisson relation for the scattering kernel and other inverse scattering problems.

**IV.2. Singularities of the scattering kernel and inverse problems.** The results in this area deal with inverse problems in scattering by obstacles in odd dimensional Euclidean spaces. In general, such problems concern the recovering of geometric properties of the obstacle from information related to the scattering amplitude $a(\lambda, \omega, \theta)$ for the wave equation in the exterior of the obstacle with Dirichlet boundary condition. It turns out that all singularities of the Fourier transform of $a(\lambda, \omega, \theta)$, the so called *scattering kernel* $s(t, \omega, \theta)$, are given by sojourn (traveling) times of scattering rays in the exterior of the obstacle. Clearly these sojourn times are a naturally observable data.

In V. Petkov (*Comm. Partial Differential Equations*, 1980), assuming that different $(\omega, \theta)$-rays have distinct sojourn times, Petkov proved that for every ordinary, non-degenerate reflecting $(\omega, \theta)$-ray $\gamma$, we have

$$-T_\gamma \in \text{sing supp } s(t, \omega, \theta),$$

and for $t$ close to $-T_\gamma$ the scattering kernel has the form

$$(1) \quad s(t,\omega,\theta) = \Big(\frac{1}{2\pi i}\Big)^{(n-1)/2}(-1)^{m_\gamma - 1}\exp\Big(i\frac{\pi}{2}\beta_\gamma\Big) \times \Big|\frac{\det dJ_\gamma(u_\gamma)\langle\nu(q_1),\omega\rangle}{\langle\nu(q_m),\theta\rangle}\Big|^{-1/2}\delta^{(n-1)/2}(t+T_\gamma) + \text{lower order singularities.}$$

Here $m_\gamma$ is the number of reflections of $\gamma$, $q_1$ (resp. $q_m$) is the first (resp. the last) reflection point of $\gamma$ and $\beta_\gamma$ is a certain integer related to $\gamma$.

This result proved to be of fundamental importance for a variety of subsequent works dealing with inverse scattering problems (see below for some of these). It generalizes a result of Majda 1977 dealing with the case of a strictly convex obstacle and a ray with one single reflection,

As a consequence of (1) one obtains for example the so called *Poisson relation for the scattering kernel*:

$$(2) \qquad \text{sing supp } s(t,\theta,\omega) \subset \{-T_\gamma : \gamma\}\ ,$$

for any pair of unit vectors $(\omega,\theta)$ with $\omega \neq \theta$, where $\gamma$ runs over the set of all $(\omega,\theta)$-rays in $\Omega$ (Petkov 1980; in final form – Cardoso, Petkov, Stoyanov 1990 and Melrose 1994).

It was shown later (Stoyanov 2003) that if $K$ satisfies a certain rather general geometric condition (see condition ($\mathcal{G}$) below), then (2) becomes an equality for almost all pairs $(\omega,\theta) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$.

In Petkov (*Trans. AMS*, 1989, jointly with Stoyanov) the special but rather important case is considered when $\Omega = \mathbb{R}^n \setminus K$, where the *obstacle* $K$ has the form $K = K_1 \cup \ldots \cup K_s$, $K_j$ being disjoint, compact, strictly convex domains with smooth boundaries satisfying Ikawa's condition

($H$) *The convex hull of every two connected components of $K$ does not have common points with any other connected component of $K$.*

Given a periodic configuration

$$(3) \qquad K_{i_1}, K_{i_2}, \ldots, K_{i_k}, K_{i_1},$$

and $\omega,\theta \in S^{n-1}$, the paper:

– shows the existence for every $q \geq 1$ of a unique $(\omega,\theta)$-ray $\gamma_q$ 'following' (3) $q$ times

– derives an asymptotic for the sojourn time $T_q$ of $\gamma_q$:

$$T_\gamma = q\,d + L + O(\delta^q), \quad q \to \infty$$

for some global constant $\delta \in (0,1)$, $L$ being another constant however dependent on the configuration (3). This asymptotic shows in particular that the length $d$ of the unique periodic billiard trajectory in $\Omega$ following the configuration (3) can be recovered from scattering data.

– derives an asymptotic for the coefficient $c_q$ in front of the main singularity in (1) with $\gamma = \gamma_q$.

In Petkov-Stoyanov (*Ann. Sci. École Norm. Sup.* 1996 and *Inside Out*, Inverse Problems, Publications of MSRI 2003) arbitrary trapping obstacles $K$ in $\mathbb{R}^n$ are considered satisfying the following rather general condition:

($\mathcal{G}$) *If for $(x,\xi) \in T^*(\partial K)$ the normal curvature of $\partial K$ vanishes of infinite order in direction $\xi$, then $\partial K$ is convex at $x$ in direction $\xi$.*

Under this condition it is proved that there exists a sequence $(\omega_m, \theta_m) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and ordinary reflecting non-degenerate $(\omega_m, \theta_m)$-rays with sojourn times $T_m \longrightarrow \infty$.

This has certain consequences for the modified resolvent of the Laplacian – an improved version of these is discussed below.

**Remark.** From general results on propagation of singularities (Melrose-Sjöstrand 1978, 1982), if $K$ is non-trapping, there exist $\epsilon > 0$ and $d > 0$ so that the *modified resolvent* $R_\chi(\lambda)$, where $\chi$ is an appropriate cut-off function, has no poles in the domain

$$U_{\epsilon,d} = \{\lambda \in \mathbb{C} : d - \epsilon \log(1 + |\lambda|) \le \operatorname{Im}\lambda < 0\}.$$

Moreover according to a result of Vainberg 1988,

$$\|R_\chi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \le \frac{C}{|\lambda|} e^{C|\operatorname{Im}\lambda|}, \ \forall \lambda \in U_{\epsilon,d}.$$

To describe the next results we need a bit of notation. Let $C$ be a sphere in $\mathbb{R}^n$ containing obstacle $K$. Given $W \subset C \times \mathbb{S}^{n-1}$, denote by $\mathcal{O}(W)$ the set of all pairs of directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that there exists an ordinary reflecting $(\omega, \theta)$-ray issued from a point $(x, \omega) \in W$. A generalized bicharacteristic $\gamma$ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$ is called *weakly non-degenerate* if for every neighborhood $W$ of $(y, \eta)$ the set $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

The paper Petkov-Stoyanov (Preprint, 2006) deals with obstacles $K$ satisfying the condition $(\mathcal{G})$ such that there exists at least one trapping weakly non-degenerate bicharacteristic issued from some point $(y, \eta) \in C \times \mathbb{S}^{n-1}$. It is then proved that there exists a sequence of ordinary reflecting non-degenerate $(\omega_m, \theta_m)$-rays $\gamma_m$ with sojourn times $T_m \longrightarrow \infty$ so that

$$-T_m \in \operatorname{sing\,supp} s(t, \omega_m, \theta_m), \;\; \forall m \in \mathbb{N}.$$

Moreover, assuming in addition that there exist $m \in \mathbb{N}$, $\alpha \geq 0$, $\epsilon > 0$, $d > 0$ and $C > 0$ so that the scattering amplitude $a(\lambda, \theta, \omega)$ is analytic in $U_{\epsilon,d}$ and

$$|a(\lambda, \theta, \omega)| \leq C(1 + |\lambda|)^m e^{\alpha |\operatorname{Im} \lambda|} \tag{4}$$

for all $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and all $\lambda \in U_{\epsilon,d}$, it is shown that there are no trapping weakly non-degenerate rays in $\Omega$.

**Remark.** The analyticity of $R_\chi(\lambda)$ in $U_{\epsilon,d}$ and the estimate

$$\|R_\chi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \leq C'(1 + |\lambda|)^{m'} e^{\alpha' |\operatorname{Im} \lambda|} \tag{5}$$

for all $\lambda \in U_{\epsilon,d}$ with $m' \in \mathbb{N}$, $\alpha' \geq 0$, imply (4) with suitable $m$ and $\alpha$. In particular, if the estimate (5) holds, then the obstacle $K$ has no trapping weakly non-degenerate rays.

The monograph V. Petkov, L. Stoyanov (Geometry of reflecting rays and inverse spectral problems, John Wiley & Sons 1992) contains an elaborate study of various properties of billiard trajectories (reflecting rays) and more generally, generalized bicharacteristics in the sense of Melrose and Sjöstrand (1978, 82), and their relationship with certain spectral and scattering problems. The monograph contains a comprehensive investigation of the Poisson relation for bounded domains and its analogue for the scattering kernel. The special case of scattering by several strictly convex domains satisfying Ikawa's condition (H) is considered in details and some interesting invariants are studied related to singularities of the scattering kernel. All results described in Sect. IV.1 above and most of these in Sect. IV.2 are presented with detailed proofs in the monograph.

**IV.3. Singularities of the dynamical zeta function.** Assume again that $\Omega = \mathbb{R}^n \setminus K$, where $K = K_1 \cup \ldots \cup K_s$ and $K_j$ are disjoint, compact, strictly convex domains with smooth boundaries satisfying Ikawa's condition (H).

The *dynamical zeta function* is defined by

$$Z_D(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma \left| I - P_\gamma \right|^{-1/2} e^{-s d_\gamma} \,, \tag{6}$$

where $\Xi$ denotes the set of all periodic reflecting rays in $\overline{\Omega}$, $m_\gamma$ is the number of reflections of $\gamma$, $T_\gamma$ is the period (length) of $\gamma$, $d_\gamma$ is the primitive period of $\gamma$, $P_\gamma$ is the linear Poincaré map related to $\gamma$, and $|\det(I-P_\gamma)| = |I-P_\gamma|$. There exists $s_1 \in \mathbb{R}$ (abscissa of absolute convergence) such that (6) is absolutely convergent for $\mathrm{Re}(s) > s_1$,

**Remarks.** (a) $Z_D(s)$ is the Laplace transform of the distribution

$$\sum_{\gamma\in\Xi}(-1)^{m_\gamma}T_\gamma\,|I-P_\gamma|^{-1/2}\,\delta(t-d_\gamma)$$

which in turn is the sum of the principal singularities of

$$u(t) = \sum_{\lambda_j} e^{it\lambda_j},\ t>0\ ,$$

where $\lambda_j \in \mathbb{C}$ are the poles of the scattering matrix $S(z)$ and the summation is over all poles counted with their multiplicities.

(b) By a result of Ikawa 1990, the existence of an analytic singularity of $Z_D(s)$ implies the existence of $\delta > 0$ such that there is an infinite number of poles $\{z_j\}_{j\in\mathbb{N}}$ of the scattering matrix $S(z)$ satisfying $0 < \mathrm{Im}(z_j) \le \delta$ for all $j \in \mathbb{N}$.

(c) It has been conjectured that the singularities of $Z_D(s)$ determine approximatively the poles of the scattering matrix.

In Petkov (*Nonlinearity*, 1999) it is established that the zeta function $Z_D(s)$ admits an analytic continuation in $\mathrm{Re}(s) \ge s_1$, and under some generic conditions about $K$ we have

$$\sup_{t\in\mathbb{R}}|Z_D(s_1+\mathbf{i}\,t)| = \infty$$

and

$$\sup_{t\in\mathbb{R}}\frac{1}{|Z_D(s_1+\mathbf{i}\,t)|} = \infty.$$

Moreover, if $s_2$ is the abscissa of absolute convergence of the series $\Pi_2(s)$ obtained from $Z_D(s)$ summing only over the rays $2\gamma$, $\gamma\in\mathcal{P}$, then $s_2 < s_1$.

A more delicate study involves the zeta function

$$\Pi(s) = \sum_{\gamma\in\mathcal{P}}(-1)^{m_\gamma}T_\gamma\,|I-P_\gamma|^{-1/2}e^{-s\,T_\gamma},\quad \mathrm{Re}(s) > s_1,$$

where $\mathcal{P}$ is the set of all primitive periodic rays in $\Omega$. Denoting by $h_\Pi$ the abscissa of holomorphy of $\Pi(s)$, i.e.

$$h_\Pi = \inf\{t \in \mathbb{R} : \ \Pi(s) \ \text{is analytic for} \ \mathrm{Re}(s) > t\},$$

we have $h_\Pi < s_1$.

It is shown in PETKOV (*Canad. Math. Bull.*, to appear) that at least one of the functions $Z_D(s)$ and $\Pi(s)$ has a singularity at $s = s_2$ and $Z_D(s) - \Pi(s)$ is analytic for $\mathrm{Re}(s) > s_2$. Moreover, if $s_2 \neq h_\Pi$, then for every sufficiently small $\epsilon > 0$, $Z_D(s)$ has a singularity at some $z$ with $\mathrm{Re}(z) > \max\{s_2, h_\Pi\} - \epsilon$.

Finally, let us mention that, under some geometric conditions about the obstacle $K$, in a recent work of PETKOV and STOYANOV (Preprint, 2007) an interesting relationship is established between the analytic properties of the resolvent of the Laplacian in $\Omega$ and those of the zeta function

$$Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}.$$

Here $\lambda_{\gamma,i}$ $(i = 1, \ldots, N-1)$ are the eigenvalues of the linear Poincaré map $P_\gamma$ with $|\lambda_{\gamma,i}| > 1$,

$$\delta_\gamma = -\frac{1}{2} \log(\lambda_{1,\gamma} \ldots \lambda_{N-1,\gamma}),$$

and $r_\gamma = 0$ if $m_\gamma$ is even and $r_\gamma = 1$ if $m_\gamma$ is odd. More details about this result can be found in the article by Stoyanov later on in this issue.