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SET-THEORETIC SOLUTIONS OF THE YANG-BAXTER EQUATION

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The paper considers some most recent results on a class of solutions of the now famous Yang-Baxter equation, the so-called set-theoretic solutions. Our approach is algebraic. We discuss also our conjecture on the close relation between the nondegenerate involutive solutions of the set-theoretic Yang-Baxter equation and a class of standard finitely presented semigroups called binomial skew-polynomial semigroups.

1. Introduction. The Yang-Baxter equation appeared in 1967 [17] in Statistical Mechanics and turned out to be one of the basic equations in Mathematical Physics, and more precisely for introducing the Theory of Quantum Groups. At present the study of Quantum Groups, and, in particular, the solutions of the Yang-Baxter equation attracts the attention of a broad circle of scientists and mathematicians. For example, only on the web site xxx.lanl.gov.math.QA, more than 60 mathematical preprints dealing explicitly with the Yang-Baxter equation appeared during the last few years.

Let V be a vector space over a field k. A linear automorphism R of $V \otimes V$ is *a solution of the Yang-Baxter equation* if

$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R) \tag{1.1}$$

which holds in the automorphism group of $V \otimes V \otimes V$. R is a solution of the *quantum Yang-Baxter equation* (QYBE) if

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{1.2}$$

where R^{ij} means R acting on the i-th and j-th component.

Finding all solutions of the Yang-Baxter equation is a difficult task far from its final resolution. Nevertheless many solutions of these equations have been found during the last 16 years and the related algebraic structures (Hopf algebras) have been studied (for example see [11]). Most of these solutions were "deformations" of the identity solution. In 1990 V. Drinfeld [2] posed the question of studying a class of solutions that are obtained in a different way – the so called *set-theoretic solutions*.

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Definition 1.1. *Let X be a nonempty set. Let $r : X \times X \longrightarrow X \times X$ be a one-to-one map of the Cartesian product $X \times X$ onto itself. The map r is called* a set-theoretic solution of the Yang-Baxter equation *if*

$$r_1 r_2 r_1 = r_2 r_1 r_2, \tag{1.3}$$

where r_1 and r_2 are maps on $X \times X \times X$ defined as $r_1(x, y, z) = (r(x, y), z)$ and $r_2(x, y, z) = (x, r(y, z))$.

Each set-theoretic solution r of the Yang-Baxter equation induces an operator R on $V \otimes V$ for the vector space V spanned by X, which is, clearly, a solution of 1.1. Several works dealing with set-theoretic solutions appeared recently, cf. [16, 9, 7, 3, 15, 13].

In [3] P. Etingof, T. Schedler and A. Soloviev study the class of nondegenerate involutive set-theoretic solutions of the Yang-Baxter equation. To each set-theoretic solution r of the Yang-Baxter equation they associate a group G_X called its structure group, generated by X and defined by a set of relations induced by r. They study its natural action on X, proving that the group G_X is solvable. The authors present different methods for constructing nondegenerate symmetric solutions such as affine, multipermutational, twisted unions, etc., discuss the geometric and algebraic interpretations of such solutions, and give their classification in terms of group theory.

Soloviev [15], continues the study of the structure groups G_X and A_X associated with set-theoretic solutions r of the Yang-Baxter equation. Also linear and affine solutions of the Yang-Baxter equation are studied.

Lu, Yan and Zhu [13], propose a general construction of set-theoretic solutions of the Yang-Baxter equation and study its properties. It is shown that their construction includes the earlier ones given by Weinstein-Xu and Etingof-Schedler-Soloviev.

In the joint paper of Michel Van den Bergh and the author of this paper [7], are studied the close relations between different mathematical objects such as set-theoretic solutions of Yang-Baxter equations, semigroups of I-type (which appeared recently in the study of Sklyanin algebras) and the semigroups S_0 associated with certain skew-polynomial rings with binomial relations introduced and studied in [4] and [5]. The semigroups S_0, called *skew-polynomial semigroups* are standard finitely presented, more precisely, they are defined in terms of a finite number of generators and quadratic square-free relations, which form a Groebner basis, cf. Definition 2.10. It is proved in [7] that each skew-polynomial semigroup S_0 defines a nondegenerate set-theoretic solution $r = r(S_0)$ of the Yang-Baxter equation. In connection with this result T. Gateva-Ivanova made the conjecture that under the restriction that X is finite and r acts trivialy on $diag(X \times X)$, all nondegenerate involutive solutions can be obtained in this way, cf. Conjecture 2.12.

2. Braided and symmetric sets. In this section we discuss some results of [3] and [15].

For convinience we shall often identify the sets $X \times X$ and X^2, the set of all monomials of length two in the free semigroup $\langle X \rangle$.

Definition 2.1 [3]**.** *Let X be a finite non-empty set, and let $r : X \times X \to X \times X$ be a bijective map. The components of r are the maps $g : X \to X$ and $f : X \to X$ defined* *by the equation*

$$r(x,y) = (g_x(y), f_y(x)).$$

(i) *(X,r) is* nondegenerate *if $g_x(y)$ is a bijective function of y, for each $x \in X$, and $f_y(x)$ is a bijective function of x, for each $y \in X$.*

(ii) *The pair (X,r) is* a braided set *if r satisfies the braid relation:*

$$r_1r_2r_1 = r_2r_1r_2, \tag{2.1}$$

(iii) *(X,r) is* involutive *if*

$$r^2 = id_{X\times X} \tag{2.2}$$

A set (X,r) is symmetric *if it is braided and involutive.*

Clearly, every braided set presents a set-theoretic solution of the Yang-Baxter equation. Nondegenerate symmetric sets are studied in [3]. We shall use the terminology of [3] and shall often call the nondegenerate symmetric sets simply "solutions" meaning nondegenerate solutions of (2.2). The binomial skew-polynomial semigroups, see [4, 5, 7, 10], are also associated with nondegenerate involutive solution of (2.1) and (2.2).

Example 2.2. Let X be a nonempty set and let $r(x,y) = (y,x)$. Then (X,r) is a nondegenerate symmetric set, which is called "the trivial solution"

Example 2.3. Lyubashenko, [2]. Let X be a non-empty set, let f, g be maps $X \to X$ and let $r(x,y) = (g(y), f(x))$. Then (X,r) is nondegenerate if and only if f and g are bijective; (X,r) is braided if and only if $fg = gf$; (X,r) is involutive if and only if $f = g^{-1}$. Clearly, an involutive solution of this type is always braided. In the last case, (X,r) is called *a permutational solution.*

Example 2.4. Let $X = \{x_1, x_2, x_3, x_4\}$ and let r be defined as:

$$r(x_1,x_3) = (x_4,x_2), r(x_4,x_2) = (x_1,x_3), r(x_1,x_4) = (x_3,x_2), r(x_3,x_2) = (x_1,x_4),$$

$$r(x_2,x_3) = (x_4,x_1), r(x_4,x_1) = (x_2,x_3), r(x_2,x_4) = (x_3,x_1), r(x_3,x_1) = (x_2,x_4),$$

$$r(x_1,x_2) = (x_2,x_1), r(x_2,x_1) = (x_1,x_2), r(x_3,x_4) = (x_4,x_3), r(x_4,x_3) = (x_3,x_4),$$

$$r(x_i,x_i) = (x_i,x_i), i = 1, \cdots, 4.$$

Then the set (X,r) is symmetric.

Definition 2.5. *The braid group B_n is the group generated by n generators $b_1, \cdots b_n$ and defining relations*

$$b_ib_j = b_jb_i, \mid i-j \mid > 1; b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}. \tag{2.3}$$

Recall that the symmetric group S_n is isomrphic to the quotient of B_n by the relations $b_i^2 = 1$.

The following proposition is obvious.

Proposition 2.6 [3]. (i) *The assignmet $b_i \to r^{ii+1}$ extends to an action of B_n on X^n if and only if (X,r) is a braided set.*

(ii) The assignmet $b_i \to r^{ii+1}$ extends to an action of S_n on X^n if and only if (X, r) is a symmetric set.

Another obvious proposition gives the relation between the braided sets (i.e. the set-theoretic solutions of the Yang-Baxter equation) and the set-theoretic solutions of the quantum Yang-Baxter equation.

Proposition 2.7. *Let $\sigma : X \times X \to X \times X$ be the permutation map defined by $\sigma(x, y) = (y, x)$. Let $R = \sigma \circ r$. (i.e. R is the so called* R-matrix *corresponding to r). Then (X, r) is a braided set if and only R satisfies the quantum Yang-Baxter equation:*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}. \tag{2.4}$$

Furthermore, (X, r) is symmetric if and only if R satisfies (2.4) *and the unitarity condition*

$$R^{21}R = 1. \tag{2.5}$$

There are three basic constructions associated naturally with a braided set (X, r). These are the semigroup $S(r)$, associated with (X, r) (it is defined only when r is involutive, see Definition 2.8), the structure group G_X, see Definition 2.14, and the derived structure group A_X, see [15]. In the case when the set (X, r) is symmetric, the two groups G_X and A_X coincide, so since our attention will be concentrated on symmetric sets, we omit the definition of A_X.

Definition 2.8. *Given a symmetric set (X, r) we consider the semigroup $S = S(r) = \langle X; R\rangle$ with a set of generators X and a set of defining relations*

$$R = \{u = r(u) \mid u \in X^2, u \neq r(u)\}$$

We call the semigroup $S(r)$ the semigroup associated with r.

Let us assume now that $S = \langle X; R\rangle$ is a semigroup with a set of generators X and a set of quadratic binomial defining relations $R = \{xy = zt \mid x, y, z, t \in X\}$, such that each monomial $u \in X^2$, occurs in atmost one relation of R.

Define $r = r(S) : X \times X \to X \times X$ as follows: (i) $r(x, y) = (x, y)$ if xy is a monomial of length 2 which does not occur in any relation of R; and (ii) if $(xy = zt) \in R$, then $r(x, y) = (z, t)$ and $r(z, t) = (x, y)$. We call r the map associated with the semigroup S.

We say that S is a Yang-Baxter semigroup, *or abbreviated* Y-B semigroup, *if the associated map $r = r(S)$ is a set-theoretic solution of the Yang-Baxter equation.*

Note that if r is the map defined by the set of relations of a Y-B semigroup $S = \langle X; R\rangle$, then, clearly, $r^2 = id_{X^2}$, so the set (X, r) is always symmetric.

We give now an example of a Y-B semigroup with 11 generators.

Example 2.9. Let $S = \langle X; R\rangle$, with a set of generators $X = \{1, 2, \cdots, 8, a, b, c\}$ and the set of defining relations:

$$1a = a2, 2a = a1, 2b = b3, 3b = b2, 3a = a4, 4a = a3, 4c = c1, 1c = c4,$$

$$5a = a6, 6a = a5, 6b = b7, 7b = b6, 7a = a8, 8a = a7, 8c = c5, 5c = c8,$$

$$1b = b5, 5b = b1, 2c = c6, 6c = c2, 3c = c7, 7c = c3, 4b = b8, 8b = b4,$$

$$ab = ca, ac = ba, bc = cb, ij = ji, 1 \leq i, j \leq 8.$$

The binomial semigroups of skew-polynomial type were introduced and studied in [4, 5, 7]. These semigroups give a class of nondegenerate symmetric set-theoretic solutions of the Yang-Baxter equation.

Definition 2.10. *We say that the semigroup* S_0 *is of* skew-polynomial type *if it has a standard finite presentation as* $S_0 = \langle X; R_0 \rangle$*, where the set of generators* X *is ordered* $x_1 < x_2 < \cdots < x_n$*, and the set* $R_0 = \{xy = zt \mid x, y, z, t \in X\}$*, contains precisely* $n(n-1)/2$ *quadratic square-free binomial defining relations such that:*

i) for every relation $xy = zt \in R_0$*, one has* $x \neq y$*, and* $z \neq t$*;*

ii) each monomial xy*, with* $x \neq y$ *occurs in exactly one relation of* R_0

iii) if $xy = zt \in R_0$*, then* $x > y$ *implies* $x > z$*, and* $z < t$ *(or equivalently,* $z < t$ *implies* $x > y$ *and* $x > z$*);*

iv) the monomials xyz *with* $x > y > z$*,* $x, y, z \in X$*, do not give rise to new relations in* S_0*, or equivalently,* R_0 *is a Groebner basis with respect to the deg-lex ordering of the free semigroup* $\langle X \rangle$*.*

It is shown in [7], see the Theorem 3.5, that for each binomial skew-polynomial semigroup S_0 the associated map $r = r(S_0)$ is a nondegenerate involutive solution of the set-theoretic Yang-Baxter equation.

As a corollary many "good" algebraic and homological properties of the corresponding semigroup ring kS_0 (over an arbitrary field k) are obtained like being a domain, Koszul, Cohen-Macaulay, regular in the sense of Artin-Schelter, etc. In particular S_0 is cancellative . Hence it is naturally embedded in its group of quotient $gr(S_0)$. A natural question arises.

Question 2.11. *Let* $r : X \times X \longrightarrow X \times X$ *be an involutive set-theoretic solution of the Yang-Baxter equation, let* $S = S(r)$ *be the associated semigroup. What can be said about the set of defining relations* R *of* S*?*

In 1996 Tatiana Ivanova made the following conjecrure reported in her talk at the International Conference in Ring Theory, Miscolc 1996.

Conjecture 2.12. *Let* $r : X \times X \longrightarrow X \times X$ *be an involutive set-theoretic solution of the Yang-Baxter equation. Assume that* $r(x, x) = (x, x)$ *for all* $x \in X$*. and* $r(x, y) \neq (x, y)$ *for all* $x, y \in X, x \neq y$*. Then the set* X *can be ordered so, that the associated semigroup* $S = S(r)$ *is of skew-polynomial type.*

Another conjectures related to question 2.11 were made by Pavel Etingof and Thomas Schedler. In 1999 T. Ivanova proved that each of the three conjectures is equivalent to 2.12. We only formulate the conjectures in this section, and give a sketch of the proof of their equivalence and some results related to them in section 3.

Conjecture 2.13. *Let* $r : X \times X \longrightarrow X \times X$ *be a nondegenerate involutive set-theoretic solution of the Yang-Baxter equation. Assume that*

$$(**) \qquad r(x, x) = (x, x) \text{ for all } x \in X.$$

Then:
1) *(T. Gateva-Ivanova) The set X can be ordered so that:*
$$(r(x,y) = (z,t),\ x > y) \Longrightarrow (x > z, z < t, y < t).$$
2) *(P. Etingof) There is an ordering on X such that:*
$$(r(x,y) = (z,t),\ x > y) \Longrightarrow (z < t).$$
3) *(T. Schedler) The set X is r-decomposable, i.e. can be presented as a union of two disjoint nonempty subsets X_1 and X_2, which are r-invariant, i.e.*
$$(r(x,y) = (z,t),\ x,y \in X_i) \Longrightarrow ((z,t) \in X_i).$$

Definition 2.14 [3]**.** *Given a braided set (X,r)* the structure group G_X of (X,r) *is defined as the group generated by the elements of X, with a set of defining relations:*
$$R = \{u = r(u) \mid u \in X^2, u \neq r(u)\}.$$

Example 2.15 [3]. If (X,r) is the trivial pair ($r(x,y) = yx$) then $G_X = Z^X$ the free abelian group generated by X.

Example 2.16. Let S_0 be a binomial skew polynomial semigroup. Let $r = r(S_0)$ be the map defined by the relations of S_0. Then the set (X,r) is symmetric. Furthermore, since S_0 is a cancellative semigroup, it has a group of quotient $gr(S_0)$, which is a central localization of S_0, see [10]. It is clear, that the groups $gr(S_0)$ and the structure group G_X are isomorphic. In this case the set X is embedded in G_X.

Proposition 2.17 [3]. *Let (X,r) be nondegenerate. Then (X,r) is braided, if and only if the following three conditions hold:*
(i) *the assignment $x \to f_x$ is a right action of G_X on X;*
(ii) *the assignment $x \to g_x$ is a left action of G_X on X;*
(iii) *the linking relation*
$$f_{g_{f_y(x)}(z)}(g_x(y)) = g_{f_{g_y(z)}(x)}(f_z(y))$$
holds.

We give now a modification of Theorem 1.6 of [15].

Theorem 2.18 [15]. *Let (X,r) be a nondegenerate braided finite set. Let Γ be the intersection of the kernels of the left and right actions from Proposition* 2.17*. Then Γ is a normal abelian subgroup of G_X of finite index.*

Analogous statement is given in [13], cf. Prop. 6.

Using a different argument based on what she calls "cyclic condition" T. Gateva-Ivanova also proves that in the case when $X = \{x_1, \cdots, x_n\}$, and (X,r) is a nondegenerate symmetric set, with r acting trivialy on $diag(X \times X)$, the structure group G_X contains as a normal subgroup of finite index the free abelian group $[x_1^P, \cdots, x_n^P]$, where $P = (n-1)!$, see [6].

Definition 2.19 [3]. *Let (X,r) be a nondegenerate symmetric set.*

(a) *A subset Y of X is* r-invariant *(or abbreviated* invariant*) if $r(Y,Y) \subset (Y,Y)$.*

(b) *The set X is* decomposable *if it is a union of two nonempty disjoint nondegenerate invariant subsets.*

Example 2.20. Let X be a finite set, and let (X,r) be a "permutational solution" (see Example 2.3). That is, $r(x,y) = (p(y), p^{-1}(x))$, where p is a permutation of X. Then X is indecomposable if and only if p is a cyclic permutation.

Proposition 2.21. *Let S_0 be a binomial skew-polynomial semigroup generated by X, let $r = r(S_0)$ be the associated solution of the Yang-Baxter equation. Then the set (X,r) is decomposable.*

The proposition follows straightforward from Theorem 3.6 given in the next section.

Proposition 2.22 [3]. *A nondegenerate symmetric set (X,r) is indecomposable if and only if G_X acts transitively on X.*

The following theorem describes the nondecomposable symmetric sets in the case when X is a finite set of a prime order.

Theorem 2.23 [3]**.** *Let p be a prime number, let X be a finite set of order p. Suppose (X,r) is an indecomposable nondegenerate symmetric set. Then (X,r) is isomorphic to the cyclic permutation solution (X,r_0), where $r_0(x_i,x_j) = (x_{j-1}, x_{i+1})$. (The notation $j-1$, and $i+1$ is taken modulo n, i.e. $x_{j-1} = x_n$, for $j = 1$, and $x_{n+1} = x_1$.)*

The following corollary shows that the Conjecture is true in the case when X is of prime order.

Corollary 2.24. *Let (X,r) be a nondegenerate symmetric set. Let $\mid X \mid= p$, where p is a prime. Suppose $r(x,x) = (x,x)$, for each $x \in X$. Then (X,r) is decomposable.*

Theorem 2.25 [3]**.** *The structure group G_X of a finite nondegenerate symmetric set is solvable.*

In [3] are also studied the quantum algebras associated to a nondegenerate symmetric set by the Faddeev-Reshetikin-Takhtajan-Sklyanin construction, see [14]. The authors present methods for constructing of nondegenerate symmetric sets: linear, affine, multipermutation solutions, twisted unions, and generalized twisted unions, study the properties and present a classification of such solutions.

A. Solovyev [15] continues the study of braided sets. He introduces *the rank* of a finite nondegenerate braided set (X,r) as the rank of its structure group G_X. It is shown in [15], cf. Corollary 1, that if X is a set of order n then the rank of (X,r) is atmost n. Furthermore the rank is precisely n if and only if (X,r) is symmetric. He introdices injective solutions and studies injective linear and affine solutions.

3. Yang-Baxter semigroups with square free relations. We keep the notation from the previous sections. The set $X = \{x_1, \cdots, x_n\}$ will be always finite. Given a symmetric set (X,r) by $S(r)$ we denote the associated Yang-Baxter semigroup defined in Definition 2.8. For convinience we shall often identify the sets $X \times X$ and X^2. Recall that $S(r) = \langle X; R\rangle$, where for the set of defining relations R one has $(xy = y'x') \in R$ if and only if $r(xy) = (y'x')$. Throughout the section we shall assume that r acts trivially on $diag(X \times X)$, i.e. for each $x \in X$ one has:

$$r(xx) = xx. \tag{3.1}$$

Lemma 3.1 [6]. *Let (X, r) be a nondegenerate set. Suppose that $r(xx) = xx$ for all $x \in X$. Then for every payr (x, y), with $x \neq y, x, y \in X$, one has $r(xy) = y'x'$, where $y' \neq x$, and $x' \neq y$.*

Proof. It follows from the hypothesis of the lemma that for the function g_x, defined in Definition 2.1 one has $g_x(x) = x$, thus, since (X, r) is nondegenerate $y \neq x$ implyes $g_x(y) \neq g_x(x) = x$. This gives $y' \neq x$. Analogous argument shows that $x' \neq y$. $\square$

Lemma 3.2 [6]. *For a symmetric set (X, r) with r acting trivialy on $diag(X \times X)$ the following three conditions are equivalent:*

1. *(X, r) is nondegenerate;*
2. *$r(xy) \neq xy$ for all $x, y \in X$, with $x \neq y$;*
3. *For every pair $x \neq y, x, y \in X$ one has $r(xy) = y'x'$, with $y' \neq x$, and $x' \neq y$.*

Theorem 3.3 [6]. *Let (X, r) be a symmetric set, let $P = (n-1)!$, let $S = S(r)$ be the associated Yang-Baxter semigroup. Suppose:*

(i) $r(xx) = xx$ for all $x \in X$;

(ii) $r(xy) \neq xy$ for all $x, y \in X$, $x \neq y$.

Then S satisfies the following conditions:

1. *$x \neq y$ implies $(xy = y'x') \in R$, with $x \neq y'$, $y \neq x'$.*
2. *For every pair $t, y \in X, t \neq y$, there exists a pair $z, u \in X, z \neq u$, such that $tty = zuu$ is an equality in S.*
3. *For every pair $t, y \in X, t \neq y$, there exists a pair $z, u \in X, z \neq u$, such that $tyy = zzu$ is an equality in S.*
4. *(*The Cyclic conditions.*) For any pair $t, y \in X$, $t \neq y$, there exist two finite sequences $\sigma'_{t,y} = \{t = t_1, \cdots, t_p\} \subset X$; $\sigma''_{t,y} = \{y = y_1, \cdots, y_q\} \subset X$, such that $p + q \leq n$, $\sigma'_{t,y} \bigcap \sigma''_{t,y} = \emptyset$, and the following equalities hold in S:*

$$t_i y_j = y_{j+1} t_{i+1},\ 1 \leq i < p, 1 \leq j < q,$$

$$t_p y_j = y_{j+1} t_1,\ 1 \leq j < q, \quad t_i y_q = y_1 t_{i+1},\ 1 \leq i < p, \quad t_p y_q = y_1 t_1.$$

The number $p + q$ is called the length of the cycle determined by t and y.

5. *Given $t, y \in X$, there exist unique $a, b \in X$ such that $ta = yb$. Furthermore, $t = y$ implies $t = a = b$.*
6. *The set (X, r) is nondegenerate.*
7. *For $x, y \in X$ there exists a $z \in X$ such that $xy^P = z^P x$.*
8. *The equality $x^P y^P = y^P x^P$ holds for every $x, y \in X$. Furthermore, S contains the free abelian semigroup generated by $x_1^P, \cdots, x_n^P$, and the free abelian group $A = gr[x_1^P, \cdots, x_n^P]$ generated by $x_1^P, \cdots, x_n^P$ is a normal subgroup of finite index in the structure group G_X.*

Proof. We give here a sketch of the proof. Let $t \neq y$ and let $r(ty) = au$. It follows from (ii) that $ty \neq au$ (as monomials). Let $r(ta) = zv$. Consider the "Yang-Baxter diagram"

$$\begin{array}{ccc} tty & \xrightarrow{r_1} & tty \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle r_2} \\ tau & & tau \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ zvu & \xrightarrow{r_2} & zvu \end{array}$$

It follows then that $r(vu) = vu$, which by (ii) and (i) is possible only if $v = u$. This proves $tty = zuu$, so condition 2 holds. We also claim that in the equality $r(ty) = au$ one has $a \neq t$, and $u \neq y$. Indeed, if we assume that $r(ty) = tu$, then the Yang-Baxter diagram

$$\begin{array}{ccc} tty & \xrightarrow{r_1} & tty \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle r_2} \\ ttu & & ttu \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ ttu & \xrightarrow{r_2} & ttu \end{array}$$

gives that $r(tu) = tu$, which is possible only for $t = u$, thus the assumption $r(ty) = tu$ implies $t = u$, which contradicts (i). The inequality $u \neq y$ can be proved analogously. Condition 1 has been proved. Analogous argument proves 3. It follows from Condition 2 that if $ty = y't'$, one has $ty' = y''t'$, so if we fix t, in finitely many steps we obtain a sequence $y_1 = y, y_2 = y', \cdots, y_q$, $q < n$, such that $ty_j = y_{j+1}t'$, for $j < q$ and $ty_q = y_1t'$. Analogous argument proves the existence of the sequence $t_1 = t, t_2 = t', \cdots, t_p$, $p < n$, such that the cyclic conditions 4 are satisfied. It follows from (i) that $t_i \neq y_j$ for all $i, j.$, thus $p + q \leq n$. The cyclic conditions imply 5. Indeed, in the notation of 4. take $a = y_q$, and $b = t_2$. All remaining conditions of the theorem follow from the cyclic conditions. □

The following theorem can be extracted from [7], Theorems 1.3, 1.4.

Theorem 3.4 [7]. *Let $S = \langle X; R\rangle$ be a Yang-Baxter semigroup, let $r = r(R)$ be the map associated with the set of relations R. Suppose for each $x, y \in X$ there exist unique $a, b \in X$, such that $r(xa) = yb$. Let $A = kS$ be the semigroup algebra over a field k. Then*

1. *S is of I-type.*
2. *A has finite global dimension.*
3. *A is Koszul.*
4. *A is Noetherian.*
5. *A satisfies the Auslander condition.*
6. *A is regular in the sense of Artin-Schelter.*
7. *A is Cohen-Macaulay.*

8. *A is a domain, in particular S is cancellative.*

The semigroups of I-type are defined in [7]. For the definition of "Cohen-Macaulay" and the "Auslander condition" see [12]. Regular rings are defined in [1].

If S_0 is a binomial skew-polynomial semigroup, then it is proved in [5] that all the conditions 1, 2, 3, 4, 5, 7, 8 of Theorem 3.3 are satisfied. Furthermore, S_0 is a Yang-Baxter semigroup, as shows the following theorem.

Theorem 3.5 [7]**.** *Let S_0 be a semigroup of skew-polynomial type. Then the associated map $r = r(S_0)$ is a (nondegenerate involutive) solution of the set-theoretic Yang-Baxter equation, that is S_0 is a Yang-Baxter semigroup.*

The following observations are made in [10]. It follows from the condition $xy^p = z^p x$, for each $x, y \in X$ that the group $G = gr(S)$ acts by conjugation on the set $\{x_1^P, \cdots, x_n^P\}$. Then this set is a disjoint union of distinct conjugacy classes, say $C_1, \cdots, C_r$. For each i, $1 \leq i \leq r$, let $z_i = \Pi_{c \in C_i} c$. Then z_i is a central element of S. For an element $z = x_1^{pk_i} \cdots x_n^{pk_n} \in A$, (with each $k_i \geq 0$ denote by $c(z) = \{i \mid k_i > 0\}$, the content of z.

For each $i, 1 \leq i \leq r$, let $S_i = \langle x_j \mid j \in c(z_i)\rangle$. The monoids $S_1, \cdots, S_r$ are called *the components* of S_0.

Theorem 3.6 [10]**.** *Let S_0 be a binomial skew-polynomial semigroup. Let the submonoids $S_1, S_2, \cdots, S_r$ be the components of S_0. Then the following conditions hold.*

1. *$S_j S_i = S_i S_j$, for each i, j. In particular, each S_i is a binomial skew polynomial semigroup.*

2. *Every element $s \in S_0$ has a unique representation of the form $s = s_1 \cdots s_r$, with $s_i \in S_i$. So $S_0 = S_1 \cdots S_r$.*

3. *$gr(S_0) = G_1 \cdots G_r$, where $G_i = gr(S_i)$.*

4. *S_0 is cyclic (i.e. $n = 1$) if and only if it has only one component.*

Condition 4 of Theorem 3.6 says that if $n > 1$, then for the number of components r of S_0, one has $r > 1$. It follows then from 1 that the set (X, r) is decomposable which proves proposition 2.21.

We now give a sketch of the proof of the equivalence of the conjectures 2.12 and 2.13.

Proof. Clearly Conjecture 2.12 implies 1 of Conjecture 2.13. Assume now 1 of Conjecture 2.13 is true. By Theorem 3.5 S is of I-type, thus the semigroup algebra kS, over an arbitrary field k, has the same Hilbert series as the commutative polynomial ring $k[x_1, \cdots, x_n]$. This implies that the set of relations R is a Groebner basis with respect to the chosen order on X, hence S is a binomial skew-polynomial semigroup wich gives the equivalence 2.12 $\Longleftrightarrow$ 2.13.1. The implication 2.12 $\Longrightarrow$ 2.13.3. follows from Proposition 2.21. The implication 2.13.3. $\Longrightarrow$ 2.13.1. can be proved by induction on n. The equivalence 2.13.1. $\Longleftrightarrow$ 2.13.2. follows from the Cyclic conditions 4 of Theorem 3.3. $\square$

It has been proved (cf. [6] for 1), 2), 3), and [3] for 4) that the Conjecture is true in the following cases:

1) The number of generators n is atmost 9;

2) The number of generators n is a) at most 23, or b) n arbitrary, with $n \neq 2^m$, and the relations satisfy the condition:

$(***)$ for each $x \neq y, x, y \in X$, either $xy = yx$, or there exists a $z \in X$ such that $xy = zx$, or $xy = yz$. (In other words the maximal length of a cycle is 3.)

3) The number n is arbitrary, and the monomial W, generating the socle of the Koszul dual $A!$ of the semigroup algebra $A = kS$ can be presented as a product of all generators $x_1, \cdots, x_n$, (i.e. $W = x_{i_1} \cdots x_{i_n}$, where $i_1, \cdots, i_n$ is a permutation of $1, \cdots, n$.)

4) The number n is prime.

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ТЕОРЕТИКО-МНОЖЕСТВЕНИ РЕШЕНИЯ НА УРАВНЕНИЕТО НА ЯНГ-БАКСТЕР

Татяна Гатева-Иванова

В статията се разглеждат някои най-нови резултати върху един клас решения, т.н. "теоретико-множествени решения" на вече прочутото уравнение на Янг-Бакстер. Подходът ни е алгебричен. Обсъждаме и една наша хипотеза относно тясната връзка между неизродените решения на теоретика-множественото уравнение на Янг-Бакстер и един клас от стандартно крайно представими полугрупи, наречени "биномни подгрупи от косо-полиномен тип".