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MEASURABILITY OF SETS OF PAIRS OF NONPARALLEL POINTS IN THE GALILEAN PLANE

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The measurable sets of pairs of nonparallel points and the corresponding invariant densities with respect to the group of the general similitudes and its subgroups are described.

1. Introduction. In the affine version, the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y–axis of the basic affine coordinate system Oxy [7], [8], [10], [11]. The affine transformations leaving invariant the special direction Oy can be written in the form

$$\begin{aligned} x' &= a_1 + a_2 x, \\ y' &= a_3 + a_4 x + a_5 y, \end{aligned} \tag{1}$$

where $a_1, \dots, a_5 \in \mathbb{R}$ and $a_2 a_5 \neq 0$.

It is easy to verify that the transformations (1) map a line segment and an angle of Γ_2 into a proportional line segment and a proportional angle with the coefficients of proportionality $|a_2|$ and $|a_2^{-1} a_5|$, respectively. Thus they form the group H_5 of the general similitudes of Γ_2. The infinitesimal operators of H_5 are

$$X_1 = \frac{\partial}{\partial x},\ X_2 = x\frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = x\frac{\partial}{\partial y},\ X_5 = y\frac{\partial}{\partial y}.$$

In [1], [2] we proved the following results:

I. The four–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_4^1 = (X_1, X_2, X_3, X_4),\ H_4^2 = (X_1, X_2, X_3, X_5),\ H_4^3 = (X_2, X_3, X_4, X_5),$$
$$H_4^4 = (X_1, X_3, X_4, \alpha X_2 + X_5).$$

II. The three–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_3^1 = (X_1, X_2, X_3),\ H_3^2 = (X_1, X_2, X_5),\ H_3^3 = (X_1, X_3, X_4),\ H_3^4 = (X_2, X_3, X_4),$$
$$H_3^5 = (X_2, X_3, X_5),\ H_3^6 = (X_2, X_4, X_5),\ H_3^7 = (X_1, X_3, \alpha X_2 + \beta X_4 + X_5),$$
$$H_3^8 = (X_3, X_4, \alpha X_1 + X_5),\ H_3^9 = (X_3, X_4, \alpha X_2 + X_5 | \alpha \neq 0),$$
$$H_3^{10} = (X_3, X_2 + 2X_5, \alpha X_1 + X_4 | \alpha \neq 0).$$

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III. The two–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_2^1 = (X_1, X_2),\ H_2^2 = (X_2, X_3),\ H_2^3 = (X_2, X_4),\ H_2^4 = (X_2, X_5),$$
$$H_2^5 = (X_1, \alpha X_2 + X_3),\ H_2^6 = (X_1, \alpha X_2 + X_5),\ H_2^7 = (X_3, \alpha X_1 + X_4 | \alpha \neq 0),$$
$$H_2^8 = (X_3, \alpha X_1 + X_5),\ H_2^9 = (X_3, \alpha X_2 + \beta X_4 + X_5 | \alpha \neq 0),\ H_2^{10} = (X_4, \alpha X_2 + X_3),$$
$$H_2^{11} = (X_4, \alpha X_2 + X_5),\ H_2^{12} = (X_2 + 2X_5, \alpha X_1 + X_4 | \alpha \neq 0).$$

IV. The one–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_1^1 = (X_1),\ H_1^2 = (X_2),\ H_1^3 = (X_3),\ H_1^4 = (X_4),\ H_1^5 = (X_5),$$
$$H_1^6 = (\alpha X_1 + X_4 | \alpha \neq 0),\ H_1^7 = (X_1 + X_5),\ H_1^8 = (\alpha X_2 + X_3 | \alpha \neq 0),$$
$$H_1^9 = (\alpha X_2 + X_5 | \alpha \neq 0),\ H_1^{10} = (\alpha X_2 + \beta X_4 + X_5 | \alpha\beta \neq 0).$$

Here and everywhere in the text α and β are real constants.

Using some basic concepts of the integral geometry in the sence of M. I. Stoka [9], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6], we find the measurable sets of nonparallel points in Γ_2 with respect to H_5 and its subgroups.

2. Measurability with respect to H_5. In Γ_2 a straight line is said to be special if it is parallel to the special direction. Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are called parallel if the straight line P_1P_2 is special. Then it follows that

$$x_2 - x_1 = 0.$$

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be a pair of nonparallel points, i.e.

$$x_2 - x_1 \neq 0. \tag{2}$$

Under the action of (1) the pair $(P_1, P_2)(x_1, y_1, x_2, y_2)$ is transformed into the pair $(P_1', P_2')(x_1', y_1', x_2', y_2')$ as

$$\begin{aligned} x_i' &= a_1 + a_2 x_i, \qquad a_2 a_5 \neq 0, \quad i = 1, 2. \\ y_i' &= a_3 + a_4 x_i + a_5 y_i, \end{aligned} \tag{3}$$

The transformations (3) form the so–called associated group $\overline{H_5}$ of H_5 [9, p. 34]. The associated group $\overline{H_5}$ is isomorphic to H_5 and the invariant density with respect to H_5 of the pairs (P_1, P_2), if it exists, coincides with the invariant density with respect to $\overline{H_5}$ of the points (x_1, y_1, x_2, y_2) in the set of parameters [9, p. 33]. The infinitesimal operators of $\overline{H_5}$ are

$$Y_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\ Y_2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},$$
$$Y_3 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2},\ Y_4 = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2},\ Y_5 = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

From (2) it follows that the infinitesimal operators Y_3 and Y_4 are arcwise unconnected. On the other hand, we obtain

$$Y_5 = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} Y_3 + \frac{y_2 - y_1}{x_2 - x_1} Y_4$$

and since

$$Y_3\left(\frac{x_2y_1 - x_1y_2}{x_2 - x_1}\right) + Y_4\left(\frac{y_2 - y_1}{x_2 - x_1}\right) \neq 0,$$

we can state the following

Theorem 1. *The sets of pairs of nonparallel points are not measurable with respect to the group H_5 of the general similitudes and have no measurable subsets.*

3. Measurability with the respect to subgroups of $\boldsymbol{H_5}$. The group $\overline{H_4^1} = (Y_1, Y_2, Y_3, Y_4)$, corresponding to the subgroup $H_4^1 = (X_1, X_2, X_3, X_4)$ of H_5, is a simply transitive group and therefore it is measurable. The integral invariant function [9, p. 9] $f = f(x_1, y_1, x_2, y_2)$, satisfying the system or R. Deltheil [3, p. 28], [9, p. 11]

$$Y_1(f) = 0, Y_2(f) + 2f = 0, Y_3(f) = 0, Y_4(f) = 0$$

has the form

$$f = \frac{c}{(x_2 - x_1)^2},$$

where $c = const \neq 0$. Thus we establish:

Theorem 2. *The pairs (P_1, P_2) of nonparallel points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ have the invariant with respect to H_4^1 density*

$$d(P_1, P_2) = \frac{1}{(x_2 - x_1)^2} dP_1 \wedge dP_2,$$

where $dP_i = dx_i \wedge dy_i$, $i = 1, 2$, denote the metric density for points in Γ_2.

By arguments similar to the ones used above we examine the measurability of the set of pairs of nonparallel points with respect to all the rest subgroups of H_5. We collect the results in the following table:

subgroup	measurable set/subset	expression of the density
H_4^1		$(x_2 - x_1)^{-2} dP_1 \wedge dP_2$
H_4^2	$y_2 - y_1 \neq 0$	$(x_2 - x_1)^{-2}(y_2 - y_1)^{-2} dP_1 \wedge dP_2$
H_4^3	it is not measurable and has no measurable subsets	
H_4^4 $\alpha = 0$	it is not measurable and has no measurable subsets	
H_4^4 $\alpha \neq 0$		$(x_2 - x_1)^{-2\frac{\alpha+1}{\alpha}} dP_1 \wedge dP_2$
H_3^1	$y_2 - y_1 = \lambda$	$(x_2 - x_1)^{-2} dP_1 \wedge dx_2$
H_3^2	$y_2 = \lambda y_1,\ \lambda y_1 \neq 0$	$(x_2 - x_1)^{-2} \lvert y_1 \rvert^{-1} dP_1 \wedge dx_2$
H_3^3	$x_2 = x_1 + \lambda,\ \lambda \neq 0$	$dP_1 \wedge dy_2$
H_3^4	$x_2 = \lambda x_1,\ \lambda \neq 1,\ x_1 \neq 0$	$\lvert x_1 \rvert^{-1} dP_1 \wedge dy_2$
H_3^5	$x_2 = \lambda x_1,\ \lambda \neq 1,\ x_1(y_2 - y_1) \neq 0$	$\lvert x_1 \rvert^{-1}(y_2 - y_1)^{-2} dP_1 \wedge dy_2$
H_3^6	$x_2 = \lambda x_1,\ \lambda \neq 1,\ x_1(\lambda y_1 - y_2) \neq 0$	$\lvert x_1 \rvert^{-1}(\lambda y_1 - y_2)^{-2} dP_1 \wedge dy_2$

subgroup	measurable set/subset	expression of the density
H_3^7 $\alpha \neq 0, 1$	$y_2 = y_1 + \lambda(x_2 - x_1)^{\frac{1}{\alpha}} + \frac{\beta}{\alpha - 1}(x_2 - x_1)$	$\lvert x_2 - x_1\rvert^{-\frac{2\alpha+1}{\alpha}} dP_1 \wedge dx_2$
H_3^7 $\alpha = 0$	$x_2 = x_1 + \lambda,\ \lambda \neq 0,$ $y_2 - y_1 + \beta\lambda \neq 0$	$(y_2 - y_1 + \beta\lambda)^{-2} dP_1 \wedge dy_2$
H_3^7 $\alpha = 1$	$y_2 = y_1 + (x_2 - x_1)(\lambda + \beta \ln \lvert x_2 - x_1\rvert)$	$\lvert x_2 - x_1\rvert^{-3} dP_1 \wedge dx_2$
H_3^8 $\alpha \neq 0$	$x_2 = x_1 + \lambda,\ \lambda \neq 0$	$e^{-\frac{2}{\alpha}x_1} dP_1 \wedge dy_2$
H_3^8 $\alpha = 0$	it is not measurable and has no measurable subsets	
H_3^9	$x_2 = \lambda x_1,\ \lambda \neq 1$	$e^{-\frac{\alpha+2}{\alpha}x_1} dP_1 \wedge dy_2$
H_3^{10}	$y_2 = y_1 + \frac{1}{2\alpha}[x_2^2 - x_1^2 - \lambda(x_2 - x_1)^2]$	$(x_2 - x_1)^{-4} dP_1 \wedge dx_2$
H_2^1	$y_1 = \lambda_1,\ y_2 = \lambda_2$	$(x_2 - x_1)^{-2} dx_1 \wedge dx_2$
H_2^2	$x_1 \neq 0,\ x_2 = \lambda_1 x_1,\ \lambda_1 \neq 1,$ $y_2 = y_1 + \lambda_2$	$\lvert x_1\rvert^{-1} dP_1$
H_2^3	$x_1 \neq 0,\ x_2 = \lambda_1 x_1,\ \lambda_1 \neq 1,$ $y_2 = \lambda_1 y_1 + \lambda_2$	$\lvert x_1\rvert^{-1} dP_1$
H_2^4	$x_1 \neq 0,\ x_2 = \lambda_1 x_1,\ \lambda_1 \neq 1,$ $y_2 = \lambda_2 y_1,\ y_1 \neq 0$	$\lvert x_1 y_1\rvert^{-1} dP_1$
H_2^5 $\alpha \neq 0$	$x_2 = x_1 + \lambda_1 e^{\alpha y_1},\ \lambda_1 > 0,$ $y_2 = y_1 + \lambda_2$	$e^{-\alpha y_1} dP_1$
H_2^5 $\alpha = 0$	$x_2 = x_1 + \lambda_1, \lambda_1 \neq 0,$ $y_2 = y_1 + \lambda_2$	dP_1
H_2^6	$x_2 = x_1 + \lambda_1 y_1^{\alpha},\ \lambda_1 \neq 0,$ $y_2 = \lambda_2 y_1,\ y_1 \neq 0$	$\lvert y_1\rvert^{-(\alpha+1)} dP_1$
H_2^7	$x_2 = x_1 + \lambda_1,\ \lambda_1 \neq 0,$ $y_2 = \frac{1}{2\alpha}(2\lambda_1 x_1 + 2\alpha y_1 + \lambda_1^2 + \lambda_2)$	dP_1
H_2^8 $\alpha \neq 0$	$x_2 = x_1 + \lambda_1,\ \lambda_1 \neq 0,$ $y_2 = \lambda_2 e^{\frac{1}{\alpha}x_1} + y_1$	$e^{-\frac{1}{\alpha}x_1} dP_1$
H_2^8 $\alpha = 0$	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ \lambda_1 \neq \lambda_2,$ $y_2 - y_1 \neq 0$	$(y_2 - y_1)^{-2} dy_1 \wedge dy_2$
H_2^9 $\alpha \neq 1$	$x_1 \neq 0, x_2 = \lambda_1 x_1, \alpha_1 \neq 1,$ $y_2 = \lambda_2 x_1^{\frac{1}{\alpha}} + \frac{\beta(\lambda_1 - 1)}{\alpha - 1} x_1 + y_1$	$\lvert x_1\rvert^{-\frac{\alpha+1}{\alpha}} dP_1$
H_2^9 $\alpha = 1$	$x_1 \neq 0,\ x_2 = \lambda_1 x_1,\ \lambda_1 \neq 1,$ $y_2 = \lambda_2 x_1 + \beta(\lambda_1 \ln \lvert\lambda_1 x_1\rvert -$ $- \ln \lvert x_1\rvert) x_1 + y_1$	$(x_1)^{-2} dP_1$
H_2^{10} $\alpha \neq 0$	$x_1 \neq 0,\ x_2 = \lambda_1 x_1,\ \lambda_1 \neq 1,$ $y_2 = y_1 + \frac{1}{\alpha}[(1 + \lambda_1)(\lambda_2 + \ln \lvert x_1\rvert - \alpha y_1)]$	$(x_1)^{-2} dP_1$
H_2^{10} $\alpha = 0$	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ \lambda_1 \neq \lambda_2$	$dy_1 \wedge dy_2$
H_2^{11} $\alpha \neq 0$	$x_1 \neq 0,\ x_2 = \lambda_1 x_1, \lambda_1 \neq 1,$ $y_2 = \lambda_1 y_1 - \lambda_2 x_1^{\frac{1}{\alpha}}$	$\lvert x_1\rvert^{-\frac{\alpha+1}{\alpha}} dP_1$

subgroup	measurable set/subset	expression of the density
H_2^{11} $\alpha = 0$	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ \lambda_1 \neq \lambda_2,$ $\lambda_2 y_1 - \lambda_1 y_2 \neq 0$	$(\lambda_2 y_1 - \lambda_1 y_2)^{-2} dy_1 \wedge dy_2$
H_2^{12}	$x_2 = \lambda_1(\frac{1}{2}x_1^2 - \alpha y_1)^{\frac{1}{2}} + x_1,\ \lambda_1 \neq 0,$ $y_2 = \frac{1}{2\alpha}\{[\lambda_1(\frac{1}{2}x_1^2 - \alpha y_1)^{\frac{1}{2}} + x_1]^2 - \lambda_2(x_1^2 - 2\alpha y_1)\},$ $x_1^2 - 2\alpha y_1 \neq 0$	$\lvert x_1^2 - 2\alpha y_1\rvert^{-\frac{3}{2}} dP_1$
H_1^1	$x_1 \neq 0,\ y_1 = \lambda_1,\ x_2 = x_1 + \lambda_2,$ $\lambda_2 \neq 0,\ y_2 = \lambda_3$	dx_1
H_1^2	$x_1 \neq 0,\ y_1 = \lambda_1,\ x_2 = \lambda_2 x_1,\ \lambda_2 \neq 0,$ $y_2 = \lambda_3$	$\lvert x_1\rvert^{-1} dx_1$
H_1^3	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ y_2 = y_1 + \lambda_3,$ $\lambda_1 \neq \lambda_2$	dy_1
H_1^4	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ y_2 = \frac{1}{\lambda_1}(\lambda_2 y_1 + \lambda_3)$ $\lambda_1 \neq 0,\ \lambda_1 \neq \lambda_2$	dy_1
H_1^5	$x_1 = \lambda_1,\ x_2 = \lambda_2,\ y_2 = \lambda_3 y_1,$ $y_1 \neq 0,\ \lambda_1 \neq \lambda_2$	$\lvert y_1\rvert^{-1} dy_1$
H_1^6	$y_1 = \frac{1}{2\alpha}(x_1^2 - \lambda_1),\ x_2 = x_1 + \lambda_2,$ $y_2 = \frac{1}{2\alpha}[(x_1 + \lambda_2)^2 - \lambda_3],\ \lambda_2 \neq 0$	dx_1
H_1^7	$y_1 = \frac{1}{\lambda_1}e^{x_1},\ \lambda_1 \neq 0,$ $x_2 = x_1 + \lambda_1,\ \lambda_2 \neq 0,\ y_2 = \frac{\lambda_3}{\lambda_1}e^{x_1}$	dx_1
H_1^8	$x_1 \neq 0,\ y_1 = \frac{1}{\alpha}(\ln\lvert x_1\rvert + \lambda_1),$ $x_2 = \lambda_2 x_1,\ \lambda_2 \neq 1,$ $y_2 = \frac{1}{\alpha}(\ln\lvert x_1\rvert + \lambda_1) + \lambda_3$	$\lvert x_1\rvert^{-1} dx_1$
H_1^9	$x_1 \neq 0,\ y_1 = \lambda_1 x_1^{\frac{1}{\alpha}},\ x_2 = \lambda_2 x_1,$ $\lambda_2 \neq 1, y_2 = \lambda_1 \lambda_3 x_1^{\frac{1}{\alpha}}$	$\lvert x_1\rvert^{-1} dx_1$
H_1^{10} $\alpha \neq 1$	$x_1 \neq 0,\ y_1 = \lambda_1 x_1^{\frac{1}{\alpha}} + \frac{\beta}{\alpha-1}x_1,$ $x_2 = \lambda_2 x_1,\ \lambda_2 \neq 1,$ $y_2 = \lambda_2^{\frac{1}{\alpha}} \lambda_3 x_1^{\frac{1}{\alpha}} + \frac{\beta\lambda_2}{\alpha-1}x_1$	$\lvert x_1\rvert^{-1} dx_1$
H_1^{10} $\alpha = 1$	$x_1 \neq 0,\ y_1 = x_1(\lambda_1 + \beta \ln\lvert x_1\rvert),$ $x_2 = \lambda_2 x_1, \lambda_2 \neq 0, 1,$ $y_2 = \lambda_2 x_1(\lambda_3 + \beta\lvert \ln\lvert\lambda_2 x_1\rvert)$	$\lvert x_1\rvert^{-1} dx_1$

Remark. In the table $\lambda, \lambda_1, \lambda_2, \lambda_3, \in R$.

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ НЕПАРАЛЕЛНИ ТОЧКИ В ГАЛИЛЕЕВАТА РАВНИНА

Адриян Върбанов Борисов

Описани са измеримите множества от двойки непаралелни точки и съответните им инвариантни гъстоти относно групата на общите подобности.