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NEW PROOFS OF TWO MATRIX INEQUALITIES

Mohamed Salah El-Sayed, Ivan Gantchev Ivanov, Milko Georgiev Petkov

Consider the following matrix inequalities if $A > B > 0$ then $A^{-1} < B^{-1}$ and $\sqrt{A} > \sqrt{B}$ ($\sqrt{B} > 0$), where the matrix inequality $X > Y$ means that the matrix $X - Y$ is positive definite. Some new proofs of these inequalities are proposed.

Introduction.

In this paper some new simple proofs for two matrix inequalities are given. Bellow we use the following notations.

$\mathcal{R}^n$ ($\mathcal{C}^n$) – the set of all real (complex) vectors with n components.

$\mathcal{C}^{m\times n}$ – the set of all $m \times n$ matrices.

$X > Y$ ($X \geq Y$) means that the matrix $X - Y$ is positive definite (semidefinite).

The two matrix inequalities which we want to prove, we will formulate by the next two theorems.

Theorem 1. *If $A > B > 0$ then*

$$A^{-1} < B^{-1}.$$

Theorem 2. *If $A > B > 0$ then*

$$\sqrt{A} > \sqrt{B},$$

where $\sqrt{B} > 0$.

An idea for proving the theorem 1 can be found in [1, p. 125], in the real case. This proof uses the next lemma.

Lemma 1. *If $A > 0, A \in \mathcal{R}^{n\times n}, y$ is any nonzero vector from $\mathcal{R}^n$ then*

$$-(A^{-1}y, y) = \min_{x\in R^n_{x\neq 0}} [(Ax, x)] - 2(x, y)]$$

Another proof of the same theorem [2, p. 85] uses the next lemma

Lemma 2. *If A, B are $n \times n$ Hermitian matrices and $B > 0$ then there exists non-singular matrix T, such that*

$$T^*AT = I$$

$$T^*BT = I,$$

where D is a diagonal matrix.

In next section 2 we will continue with some new proofs of the theorem 1 and in section 3 we will give the new proof of the theorem 2.

Proofs of the Theorem 1.

First Proof. From the condition $A > B > 0$ of theorem 1 we obtain

$$\sqrt{B^{-1}}A\sqrt{B^{-1}} > I$$

$$\sqrt{B}A^{-1}\sqrt{B} < I$$

$$A^{-1} < B^{-1}.$$

Second Proof. We put $A = B + C$ and obtain $C = A - B > 0$. We have

$$\begin{aligned} B &= A - C \\ \sqrt{A^{-1}}B\sqrt{A^{-1}} &= I - \sqrt{A^{-1}}C\sqrt{A^{-1}} \\ \sqrt{A}B^{-1}\sqrt{A} &= \left(I - \sqrt{A^{-1}}C\sqrt{A^{-1}}\right)^{-1} \end{aligned}$$

From $I - \sqrt{A^{-1}}C\sqrt{A^{-1}} > 0$ it follows that all eigenvalues of the matrix $\sqrt{A^{-1}}C\sqrt{A^{-1}}$ belong of the interval $(0, 1)$. That is because of

$$\begin{aligned} \sqrt{A}B^{-1}\sqrt{A} &= I + \sqrt{A^{-1}}C\sqrt{A^{-1}} + \left(\sqrt{A^{-1}}C\sqrt{A^{-1}}\right)^2 + \ldots > I \\ \sqrt{A}B^{-1}\sqrt{A} &> I \\ B^{-1} &> A^{-1}. \end{aligned}$$

Third Proof. The proof is based on the following

Lemma 3. *Let $A \in \mathcal{C}^{n\times n}, U, V^T \in \mathcal{C}^{n\times m} (n \geq m)$. If the matrices A and $I + VA^{-1}U$ are nonsingular then the matrix $A + UV$ is nonsingular and*

$$(A + UV)^{-1} = A^{-1} - A^{-1}U(I + VA^{-1}U)^{-1}VA^{-1} \tag{1}$$

The above formula is called Sherman – Morrison – Woodbury formula.

We are ready to consider the third proof. From $A = B+C$ we receive $A = B+\sqrt{C}\sqrt{C}$. According to (1) we have

$$A^{-1} = B^{-1} - B^{-1}\sqrt{C}(I + \sqrt{C}B^{-1}\sqrt{C})^{-1}\sqrt{C}B^{-1} < B^{-1}$$

$$A^{-1} < B^{-1}.$$

Fourth Proof. We begin with two statements

Lemma 4. ([4, p.113]) *If $A, B > 0$ are $n \times n$ matrices, all the eigenvalues of the AB are positive.*

Lemma 5. *If A, B, C are $n \times n$ positive definite matrices and the matrix equation*

$$AXB = C \tag{2}$$

has the Hermitian solution X, then $X > 0$.

Proof. Let X be a positive definite solution of (2). Then

$$X = PCQ, \tag{3}$$

where $P = A^{-1} > 0,\ Q = B^{-1} > 0$. From (3) it follows

$$\sqrt{C}X\sqrt{C} = (\sqrt{C}P\sqrt{C})\ (\sqrt{C}Q\sqrt{C}).$$

From last equality, $(\sqrt{C}X\sqrt{C})^* = \sqrt{C}X\sqrt{C}$ and $\sqrt{C}P\sqrt{C} > 0,\ \sqrt{C}Q\sqrt{C} > 0$ we get $\sqrt{C}X\sqrt{C} > 0$. Hence $X > 0$.

If A, B are given in theorem 1 then from equality

$$A(B^{-1} - A^{-1})B = A - B$$

and Lemma 5 it follows $B^{-1} > A^{-1}$.

Fifth Proof. Since the matrix A, B can be complex we will formulate and use the complex modification of the Lemma 1.

Lemma 6. If $A, \in \mathcal{C}^{n\times n},\ A > 0$ and y is fixed nonzero vector from $\mathcal{C}^n$ then

$$-(A^{-1}y, y) = \min_{x\in\mathcal{C}^n_{x\neq 0}} [(Ax, x) - (x, y) - (y, x)] \tag{4}$$

Proof. In order to prove the equality (4) we will show that the inequality

$$-(A^{-1}y, y) \geq (Ax, x) - (x, y) - (y, x) \tag{5}$$

is fulfilled. The inequality (5) is equality only for $\tilde{x} = A^{-1}y$. Really by puting $x = A^{-1}z$ the inequality (5) receives the form

$$(A^{-1}(y - z), y - z) \geq 0$$

which is obviously fulfilled and $(A^{-1}(y - z), y - z) = 0$ only for $z = y$, i.e. $x = A^{-1}y$.

Assume A, B are matrices from theorem 1 and $y \in \mathcal{C}^n$ is any nonzero vector. Then if nonzero vectors x_1 and x_2 are such ,that

$$-(A^{-1}y, y) = \min_{x\in\mathcal{C}^n_{x\neq 0}} [(Ax, x) - (x, y) - (y, x)] = (Ax_1, x_1) - (x_1, y) - (y, x_1)$$

$$-(B^{-1}y, y) = \min_{x\in\mathcal{C}^n_{x\neq 0}} [(Bx, x) - (x, y) - (y, x)] = (Bx_2, x_2) - (x_2, y) - (y, x_2)$$

then

$$\begin{aligned} -(B^{-1}y, y) &= (Bx_2, x_2) - (x_2, y) - (y, x_2) \\ &< (Bx_1, x_1) - (x_1, y) - (y, x_1) \\ &< (Ax_1, x_1) - (x_1, y) - (y, x_1) = -(A^{-1}y, y) \end{aligned}$$

Consequently $(B^{-1}y, y) > (A^{-1}y, y)$ or $B^{-1} > A^{-1}$.

A Proof of the Theorem 2.

The proof of the theorem 2 we will obtain as use a consequence of the Lyapunoff's theorem.

Theorem 3. *If eigenvalues of the $n \times n$ matrix A have positive real parts and the $n \times n$ matrix $C > 0$ then the matrix equation*

$$AX + XA^* = C$$

has positive definite solution.

Corollary. *If $n \times n$ matrices A, B, C are positive definite and the equation*

$$AX + XA^* = C \tag{6}$$

has Hermitian solution X then $X > 0$.

Really, if $X = X^*$ is a solution of (6) then we have

$$BX + XA = C \tag{7}$$

From (6) and (7) we get

$$(A + B)X + X(A + B) = 2C.$$

From the above equality and theorem 3 we obtain $X > 0$.

Assume, that A, B are the matrices of the Theorem 2. Then from the above corollary and

$$\sqrt{A}(\sqrt{A} - \sqrt{B}) + (\sqrt{A} - \sqrt{B})\sqrt{B} = A - B,$$

we obtain $\sqrt{A} > \sqrt{B}$.

Remark 1. Everywhere above the matrix roots are positive definite.

Remark 2. If the assimptions in theorem 1 and 2 are replaced with $A \geq B > 0$, it follows $A^{-1} \leq B^{-1}$ and $\sqrt{A} \geq \sqrt{B}$.

Remark 3. The theorems 1 and 2 can be obtained as consequence, of the more generalized theorems [3,5].

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Faculty of Mathematics and Informatics
Shoumen University
Shoumen 9712, Bulgaria
`email: i.gantchev@fmi.shu-bg.net`

НОВИ ДОКАЗАТЕЛСТВА НА ДВЕ МАТРИЧНИ НЕРАВЕНСТВА

Мохамед Салах Ел-Саид, Иван Ганчев Иванов, Милко Георгиев Петков

В работата се разглеждат матричните неравенства: ако $A > B > 0$ то $A^{-1}{<}B^{-1}$ и $\sqrt{A} > \sqrt{B}$ ($\sqrt{B} > 0$), където матричното неравенство $X > Y$ означава, че матрицата $X - Y$ е положително определена. Предложени са нови и кратки доказателства за тези неравенства.