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RANDOM MATRICES WITH DIRICHLET DISTRIBUTED ELEMENTS

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We investigate the probability distribution of the product of a Dirichlet distributed random vector and a random matrix with independent Dirichlet distributed rows. We find when the distribution of the product coincides with the distribution of the random vector.

There are many applications of random matrices in multivariate statistical analysis, physics, etc. For references see e.g. Wigner [2], Mehta [1] and Girko [4].

Let us consider a discrete, time-homogeneous, ergodic Markov chain $\xi_0, \xi_1, \dots$, with transition matrix

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

and initial distribution $\boldsymbol{q} = (q_1, q_2, \dots, q_n)$. It is well known that

$$\lim_{n\to\infty} \boldsymbol{q}.\mathbf{Q}^n = \boldsymbol{\pi},$$

where the row vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ does not depend on $\boldsymbol{q}$(see e.g. [5, p. 58–59] and [6, p. 192–196]). The vector $\boldsymbol{\pi}$ is a left eigenvector of the latent root 1 of $\mathbf{Q}$ i.e.

$$\boldsymbol{\pi}.\mathbf{Q} = \boldsymbol{\pi} \tag{1}$$

The natural generalization of (1) from the probabilistic point of view is to randomize the elements of $\boldsymbol{\pi}$ and $\mathbf{Q}$.

By analogy with this non-random phenomenon (1) we will investigate when the product of a Dirichlet distributed random vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$ with parameters $(b_1, b_2, \dots, b_n)$ and the random matrix

$$\mathbf{A} = \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \\ \eta_{n1} & \eta_{n2} & \cdots & \eta_{nn} \end{pmatrix}$$

has the same distribution as $\boldsymbol{\xi}$ under the assumption that $\boldsymbol{\xi}$ and $\mathbf{A}$ are independent and the rows of $\mathbf{A}$ are independent and Dirichlet distributed.

Recall that $(\xi_1, \xi_2, \ldots, \xi_n)$ has a Dirichlet distribution with parameters $(b_1, b_2, \ldots, b_n)$ (briefly $\boldsymbol{\xi} \in D(b_1, b_2, \ldots, b_n)$) if $\xi_1 + \xi_2 + \cdots + \xi_n = 1$ with probability 1 and the random vector $(\xi_1, \xi_2, \ldots, \xi_{n-1})$ has a density with respect to the Lebesgue measure in $(n-1)$-dimensional Euclidean space of the form

$$\frac{\Gamma(b_1+b_2+\cdots+b_n)}{\prod_{i=1}^{n}\Gamma(b_i)} x_1^{b_1-1} x_2^{b_2-1} \ldots x_{n-1}^{b_{n-1}-1}(1-x_1-x_2-\cdots-x_{n-1})^{b_n-1} \text{ for } x_i \geq 0, \ \sum_{i=1}^{n-1} x_i \leq 1$$

and zero otherwise.

Theorem 1. *If*

(i) $(\xi_1, \xi_2, \ldots, \xi_n) \in D(b_1, b_2, \ldots, b_n)$ *and* b_i *for* $i = 1, 2, \ldots, n$ *are positive integers;*

(ii) $(\eta_{i1}, \eta_{i2}, \ldots, \eta_{in}) \in D(a_{i1}, a_{i2}, \ldots, a_{in})$, $i = 1, 2, \ldots, n$ *and* a_{ij} *for* $i, j = 1, 2, \ldots, n$ *are positive integers;*

(iii) $\sum_{j=1}^{n} a_{ij} = b_i$, $i = 1, 2, \ldots, n$ *and* $\sum_{i=1}^{n} a_{ij} = b_j$, $j = 1, 2, \ldots, n$*;*

(iv) the random vectors $(\xi_1, \xi_2, \ldots, \xi_n)$, $(\eta_{11}, \eta_{12}, \ldots, \eta_{1n})$, $(\eta_{21}, \eta_{22}, \ldots, \eta_{2n})$, $\ldots$, $(\eta_{n1}, \eta_{n2}, \ldots, \eta_{nn})$ *are independent,*

then the random vector $\boldsymbol{\xi}.\mathbf{A}$ *has the same Dirichlet distribution as* $\boldsymbol{\xi}$ *i. e.* $\boldsymbol{\xi}.\mathbf{A} \in D(b_1, b_2, \ldots, b_n)$.

Proof. Let $\tau_{ij}(s)$, $s = 1, 2, \ldots, a_{ij}$ for $i, j = 1, 2, \ldots, n$ be independent and identically, exponentially distributed random variables with mean $E\tau_{ij}(s) = 1$. It is well known that the joint distribution of

$$\frac{\tau_{ij}(s)}{\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \quad s = 1, 2, \ldots, a_{ij} \ \text{ for } \ i, j = 1, 2, \ldots, n$$

is Dirichlet one with all parameters equal to 1 (see e.g. Wilks [6] § 7.7).

Taking into account (iii) and the additive property of Dirichlet distribution (if $(\xi_1, \xi_2, \ldots, \xi_n) \in D(c_1, c_2, \ldots, c_n)$ then $(\xi_1 + \xi_2, \xi_3, \ldots, \xi_n) \in D(c_1 + c_2, c_3, \ldots, c_n)$) we can conclude that the random vectors

$$\left(\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{1j}} \tau_{1j}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{nj}} \tau_{nj}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right) \quad \text{and}$$

$$\left(\frac{\sum_{i=1}^{n}\sum_{s=1}^{a_{i1}} \tau_{i1}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{i=1}^{n}\sum_{s=1}^{a_{in}} \tau_{in}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right)$$

have the same Dirichlet distribution which coincides with the distribution of $\boldsymbol{\xi} \in D(b_1, b_2, \ldots, b_n)$. We will briefly denote this coincidence by

$$\boldsymbol{\xi} \sim \left(\frac{\sum_{j=1}^{n} \sum_{s=1}^{a_{1j}} \tau_{1j}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{j=1}^{n} \sum_{s=1}^{a_{nj}} \tau_{nj}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right) \sim \left(\frac{\sum_{i=1}^{n} \sum_{s=1}^{a_{i1}} \tau_{i1}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{i=1}^{n} \sum_{s=1}^{a_{in}} \tau_{in}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right).$$

The independence of the random vector $\left(\frac{\sum_{j=1}^{n} \sum_{s=1}^{a_{1j}} \tau_{1j}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{j=1}^{n} \sum_{s=1}^{a_{nj}} \tau_{nj}(s)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right)$ and the random variable $\sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)$ follows from the properties of the gamma distribution (see e.g. Wilks [6, p. 202]).

By analogy using the independence of $\tau_{ij}(s)$, $s = 1, 2, \ldots, a_{ij}$ for $i, j = 1, 2, \ldots, n$ we have:

(a) the row vectors of the matrix

$$\left(\frac{\sum_{s=1}^{a_{ik}} \tau_{ik}(s)}{\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right)_{1 \leq i,k \leq n}$$

are independent;

(b) $\left(\frac{\sum_{s=1}^{a_{i1}} \tau_{i1}(s)}{\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{s=1}^{a_{in}} \tau_{in}(s)}{\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right) \in D(a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, 2, \ldots, n;$

(c) the random vector $\left(\frac{\sum_{s=1}^{a_{i1}} \tau_{i1}(s)}{\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)}, \ldots, \frac{\sum_{s=1}^{a_{in}} \tau_{in}(s)}{\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)} \right)$ is independent of the random variable $\sum_{j=1}^{n} \sum_{s=1}^{a_{ij}} \tau_{ij}(s)$ for each $i = 1, 2, \ldots, n$. We may interpret (a) and (b) by writing

$$\left(\frac{\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right)_{1\le i,k\le n} \sim \mathbf{A}.$$

From (c) we conclude that the matrix $\left(\frac{\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right)_{1\le i,k\le n}$ and the random vector $\left(\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{1j}}\tau_{1j}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)},\dots,\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{nj}}\tau_{nj}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right)$ are independent.

Consequently

$$\boldsymbol{\xi}.\mathbf{A} \sim \left(\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{1j}}\tau_{1j}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)},\dots,\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{nj}}\tau_{nj}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right)\cdot\left(\frac{\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right)_{1\le i,k\le n} =$$

$$\left(\sum_{i=1}^{n}\left(\frac{\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}.\frac{\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right), k=1,2,\dots,n\right) =$$

$$\left(\sum_{i=1}^{n}\left(\frac{\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}\right), k=1,2,\dots,n\right) =$$

$$\left(\frac{\sum_{i=1}^{n}\sum_{s=1}^{a_{ik}}\tau_{ik}(s)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{a_{ij}}\tau_{ij}(s)}, k=1,2,\dots,n\right) \in D\left(\sum_{i=1}^{n}a_{i1},\sum_{i=1}^{n}a_{i2},\dots,\sum_{i=1}^{n}a_{in}\right) \equiv D(b_1,b_2,\dots,b_n).$$

Corollary. *Under the assumptions of Theorem 1, if* $\mathbf{A}(1)$, $\mathbf{A}(2)$, $\ldots$ *is a sequence if independent random matrices identically distributed as* $\mathbf{A}$ *and if the random vector*

$\boldsymbol{\xi}(0) \sim \boldsymbol{\xi}$ *is independent of* $\mathbf{A}(1)$*,* $\mathbf{A}(2)$*, … then*

$$\boldsymbol{\xi}(0) \sim \boldsymbol{\xi}(1) \sim \cdots \sim \boldsymbol{\xi}(n) \sim \cdots$$

where $\boldsymbol{\xi}(n) = \boldsymbol{\xi}(0).\mathbf{A}(1).\mathbf{A}(2).\ldots.\mathbf{A}(n)$*.*

Proof. The proof follows by induction after applying consecutively Theorem 1 to the product

$$\boldsymbol{\xi}(n-1).\mathbf{A}(n) = \boldsymbol{\xi}(n), \quad n = 2, 3, \ldots.$$

Denote by $\{p_1 D(c_{11}, c_{12}, \ldots, c_{1n}) + p_2 D(c_{21}, c_{22}, \ldots, c_{2n}) + \cdots + p_l D(c_{l1}, c_{l2}, \ldots, c_{ln})\}$ the mixture of Dirichlet distributions $D(c_{i1}, c_{i2}, \ldots, c_{in})$, $i = 1, 2, \ldots, l$ with weights $p_1, p_2, \ldots, p_l$. Notice that Theorem 1 follows immediately from the next more general result:

Theorem 2. *If*

(i) $(\xi_1, \xi_2, \ldots, \xi_n) \in D(b_1, b_2, \ldots, b_n)$ *and* $b_i, i = 1, 2, \ldots, n$ *are positive integers;*

(ii) $(\eta_{i1}, \eta_{i2}, \ldots, \eta_{in}) \in D(a_{i1}, a_{i2}, \ldots, a_{in})$*,* $i = 1, 2, \ldots, n$ *and* a_{ij} *for* $i, j = 1, 2, \ldots, n$ *are positive integers;*

(iii) $\displaystyle\sum_{j=1}^{n} a_{ij} \le b_i, \quad i = 1, 2, \ldots, n;$

(iv) the random vectors $(\xi_1, \xi_2, \ldots, \xi_n)$*,* $(\eta_{11}, \eta_{12}, \ldots, \eta_{1n})$*,* $(\eta_{21}, \eta_{12}, \ldots, \eta_{1n})$*, …,* $(\eta_{n1}, \eta_{n2}, \ldots, \eta_{nn})$ *are independent,*

then the random vector

$$\boldsymbol{\xi}.\mathbf{A} \in \left\{ \sum_{\substack{l_1 \ge g_1, l_2 \ge g_2, \cdots, l_n \ge g_n \\ l_1 + l_2 + \cdots + l_n = b_1 + b_2 + \cdots + b_n}} \frac{\prod_{i=1}^{n} \begin{pmatrix} l_i - 1 \\ g_i - 1 \end{pmatrix}}{\prod_{i=1}^{n} \begin{pmatrix} b_i - 1 \\ d_i - 1 \end{pmatrix}} D(l_1, l_2, \ldots, l_n) \right\},$$

where $d_i = \displaystyle\sum_{j=1}^{n} a_{ij}$*,* $i = 1, 2, \ldots, n$*;* $g_j = \displaystyle\sum_{i=1}^{n} a_{ij}$*,* $j = 1, 2, \ldots, n$ *and* l_i *for* $i = 1, 2, \ldots, n$ *are positive integers.*

The proof of this fact will be presented elsewhere.

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СЛУЧАЙНИ МАТРИЦИ С ДИРИ™ЛЕ РАЗПРЕДЕЛЕНИ ЕЛЕМЕНТИ

Цветан Игнатов, Наталия Иванова, Татяна Маджарова

Ние изследваме вероятностното разпределение на произведението от Дирихле разпределен случаен вектор и случайна матрица с независими Дирихле разпределени редове. Намираме кога разпределението на произведението съвпада с разпределението на случайния вектор.